



# 993SM – Laboratory of Computational Physics week VI April 3, 2023

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# Metropolis Algorithm

by Metropolis, Rosenbluth, Rosenbluth, Teller and Teller (1953)

- 1) to generate random points with a given distribution
- 2) to calculate averages with importance sampling
- 3) in particular: in the canonical ensemble

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# Metropolis Algorithm

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**Idea: produce a random walk with points  $\{x_i\}$  whose asymptotic probability distribution  $p_N(x)$  of the occupied positions approaches  $p(x)$  after a large number  $N$  of steps**

# Metropolis Algorithm

I) to generate random points with a given distribution  $p(x)$

**Idea: produce a random walk with points  $\{x_i\}$  whose asymptotic probability distribution  $p_N(\mathbf{x})$  of the occupied positions approaches  $p(x)$  after a large number  $N$  of steps**

A **random walk** in general is defined by specifying a **transition probability**  $T(x_i \rightarrow x_j)$  from one value  $x_i$  to another value  $x_j$  and the distribution of points  $x_0, x_1, x_2, \dots$  converges to a certain  $p(x)$

## Comment:

need to consider a RW more general than the 'standard' RW with length and probability fixed for each step.

Remind: a RW with fixed length and  $p_{\text{left}} = p_{\text{right}}$  gives  $P_N(x)$  that for large  $N$  tends to a gaussian distribution with a standard deviation that depends on  $N$ :

$$\sigma^2 = Dt; \quad D = \frac{\ell^2}{2\Delta t}; \quad \Delta t = \frac{t}{N} \implies \sigma^2 = \ell^2 N/2 \quad P(x, N\Delta t) = \sqrt{\frac{2}{\pi N}} e^{-x^2/(2N)}$$

(here  $\ell=1$ ; remind also the factor of 2 due to discretization)

The recipe to obtain a gaussian distribution with given  $\sigma$  from simple RWs was to generate *several* RWs with the same  $N$  and do the histogram of their end-points.

The approach we are going to discuss now is something **different**, the focus being **one RW**.

# Markov chains

Consider a sequence of “configurations”  $C = \{C_1, C_2, \dots, C_N\}$  stochastically generated, i.e.  $C_{k+1}$  is obtained from the previous one,  $C_k$ , by making some random changes on the former.

The sequence is a **Markov chain** if the probability of making a transition from  $C_k$  to  $C_{k+1}$  is not dependent on how we arrived at  $C_k$  (its history), i.e. no memory.

The sequence of points  $x_0, x_1, x_2, \dots$  of a simple RW is a Markov chain.

# The detailed balance

Choose a **transition probability**  $T(x_i \rightarrow x_j)$  from one value  $x_i$  to another value  $x_j$  (from one configuration  $C_i$  to another one  $C_{i+1}$ ) such that the distribution of points  $x_0, x_1, x_2, \dots$  (of configurations) converges to the desired  $p(x)$ .

It is sufficient (not necessary) to satisfy the condition:

$$p(x_i)T(x_i \rightarrow x_j) = p(x_j)T(x_j \rightarrow x_i)$$

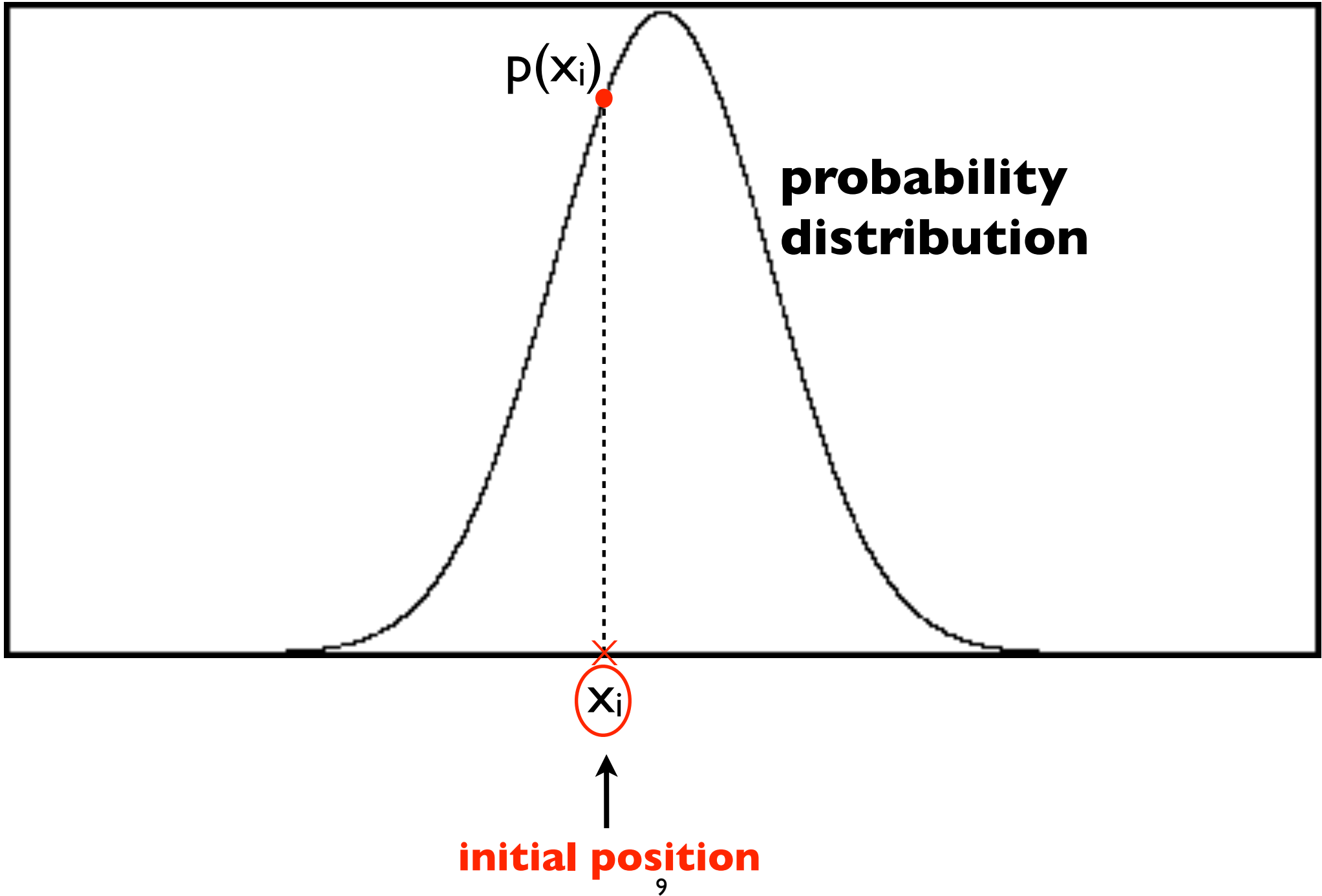
A simple choice (not unique!) is:

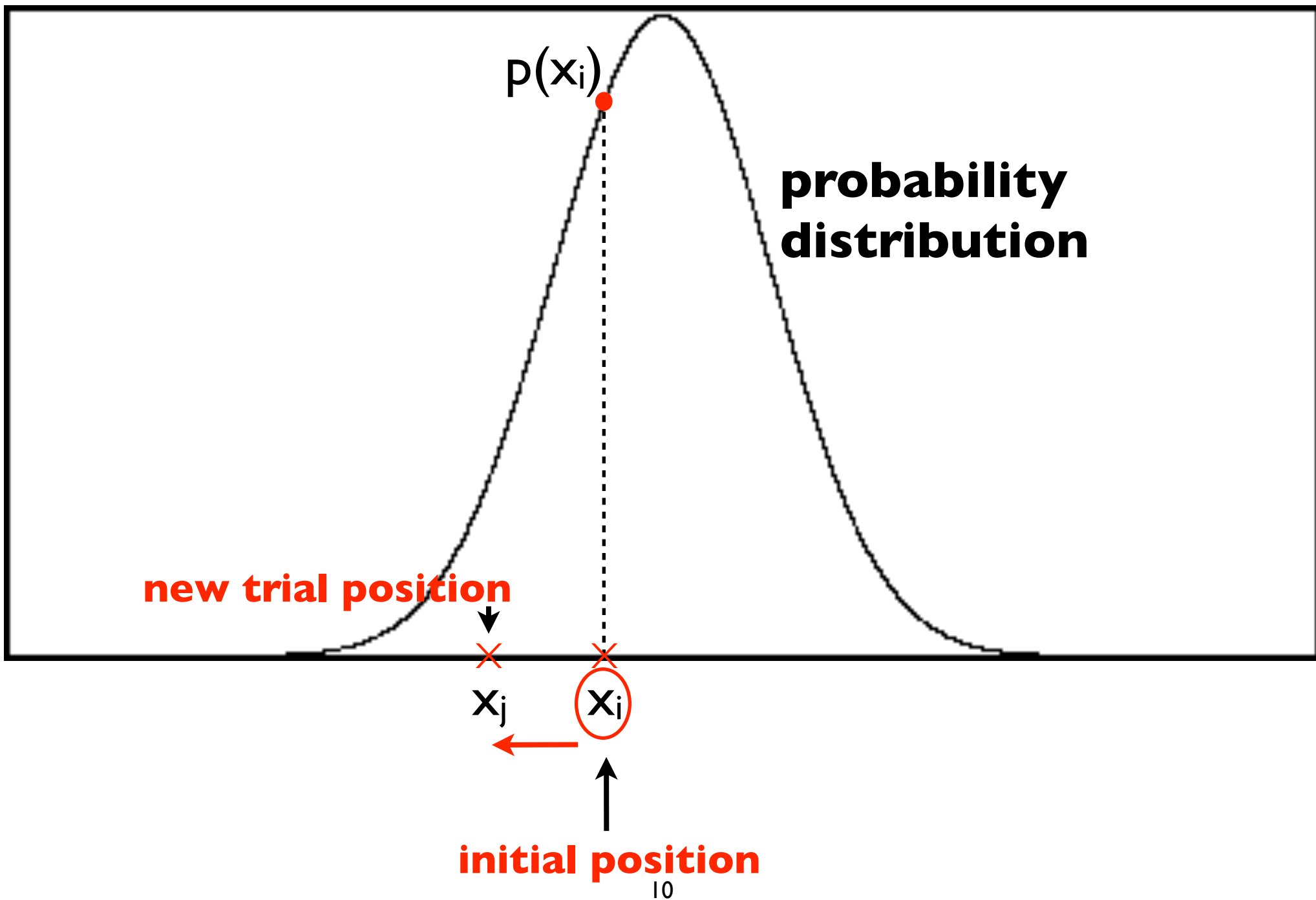
$$T(x_i \rightarrow x_j) = \min \left[ 1, \frac{p(x_j)}{p(x_i)} \right]$$

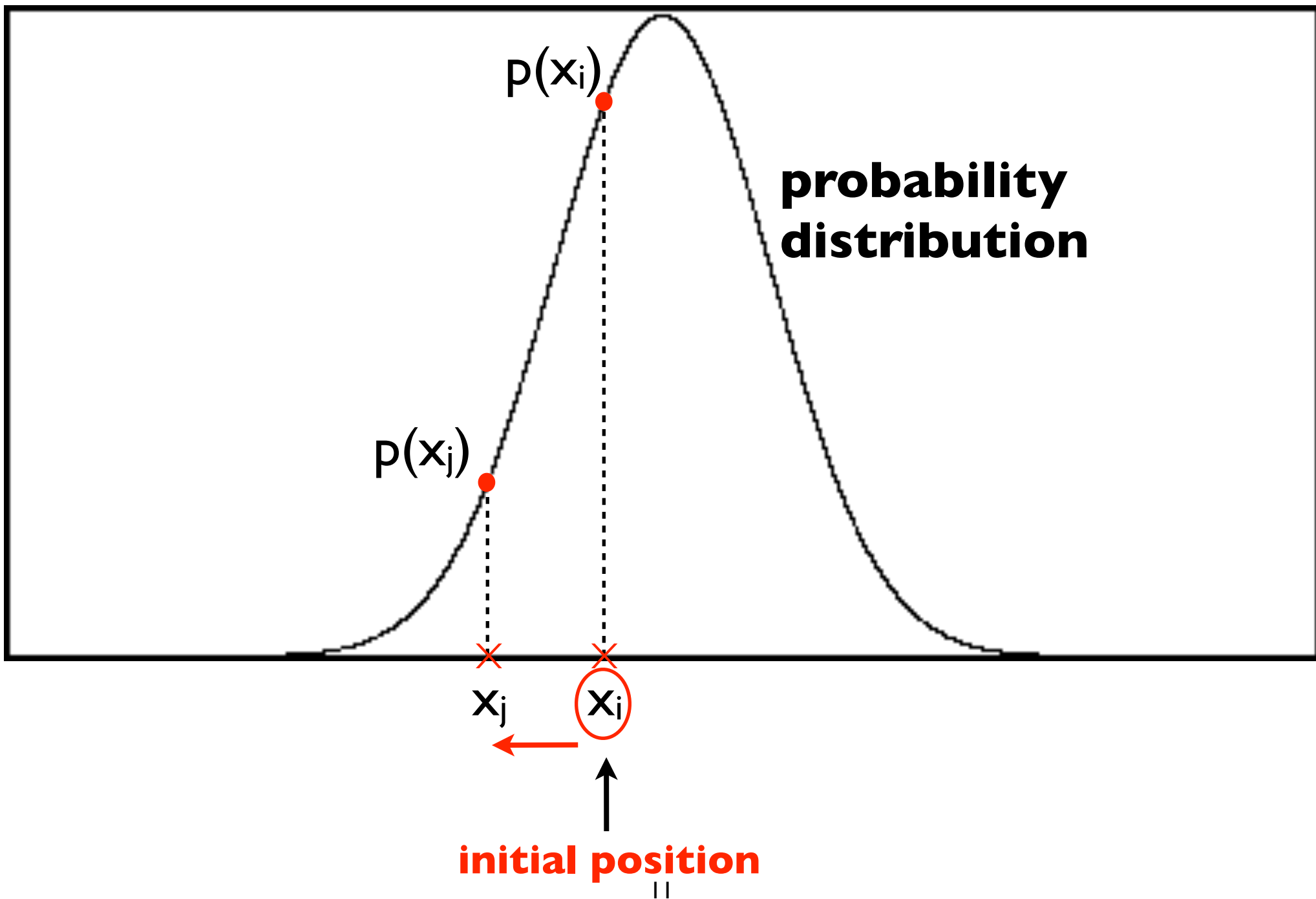
(We can easily verify...)

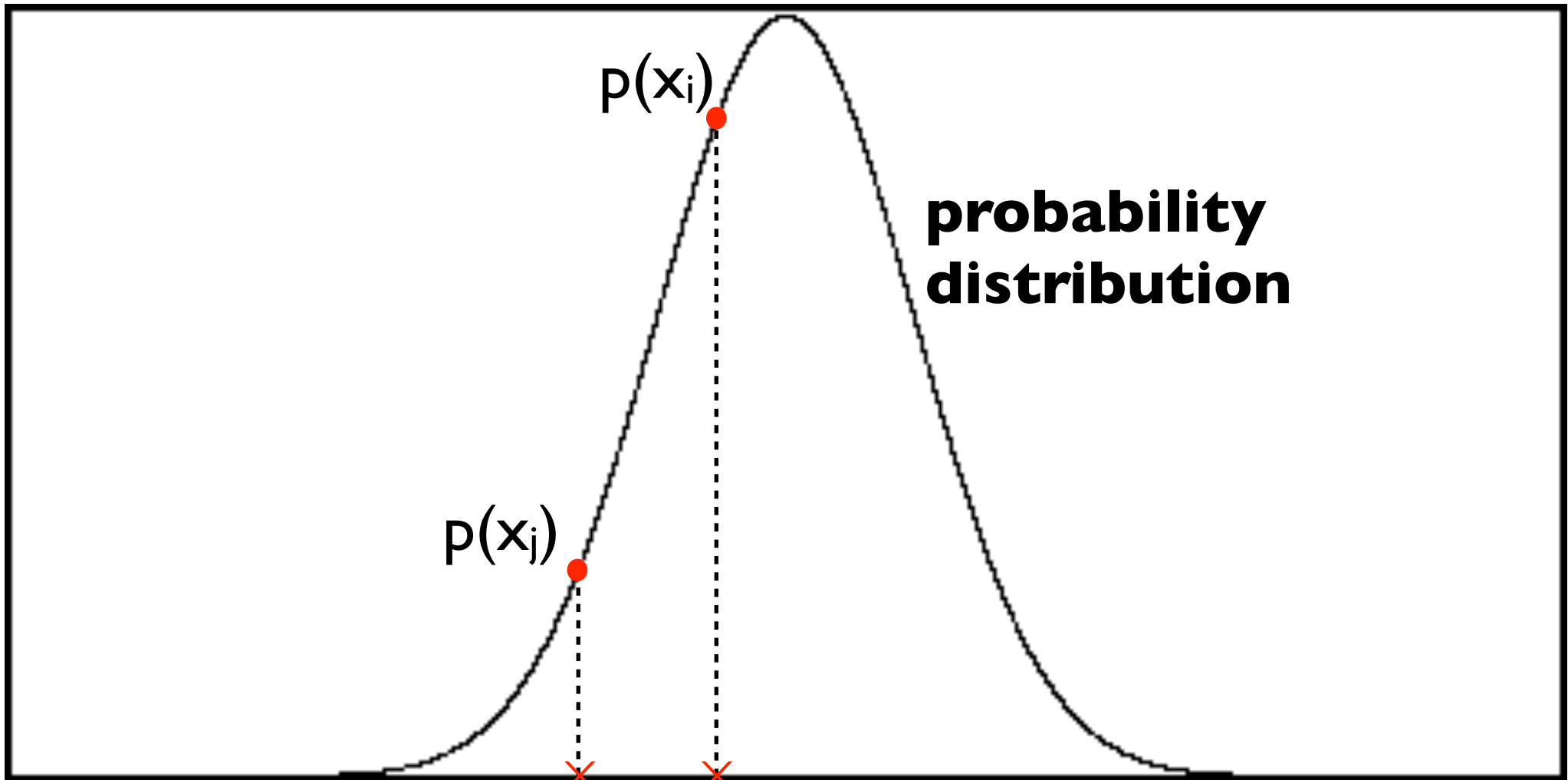


Example:









move with probability  
 $p(x_j)/p(x_i) < 1$

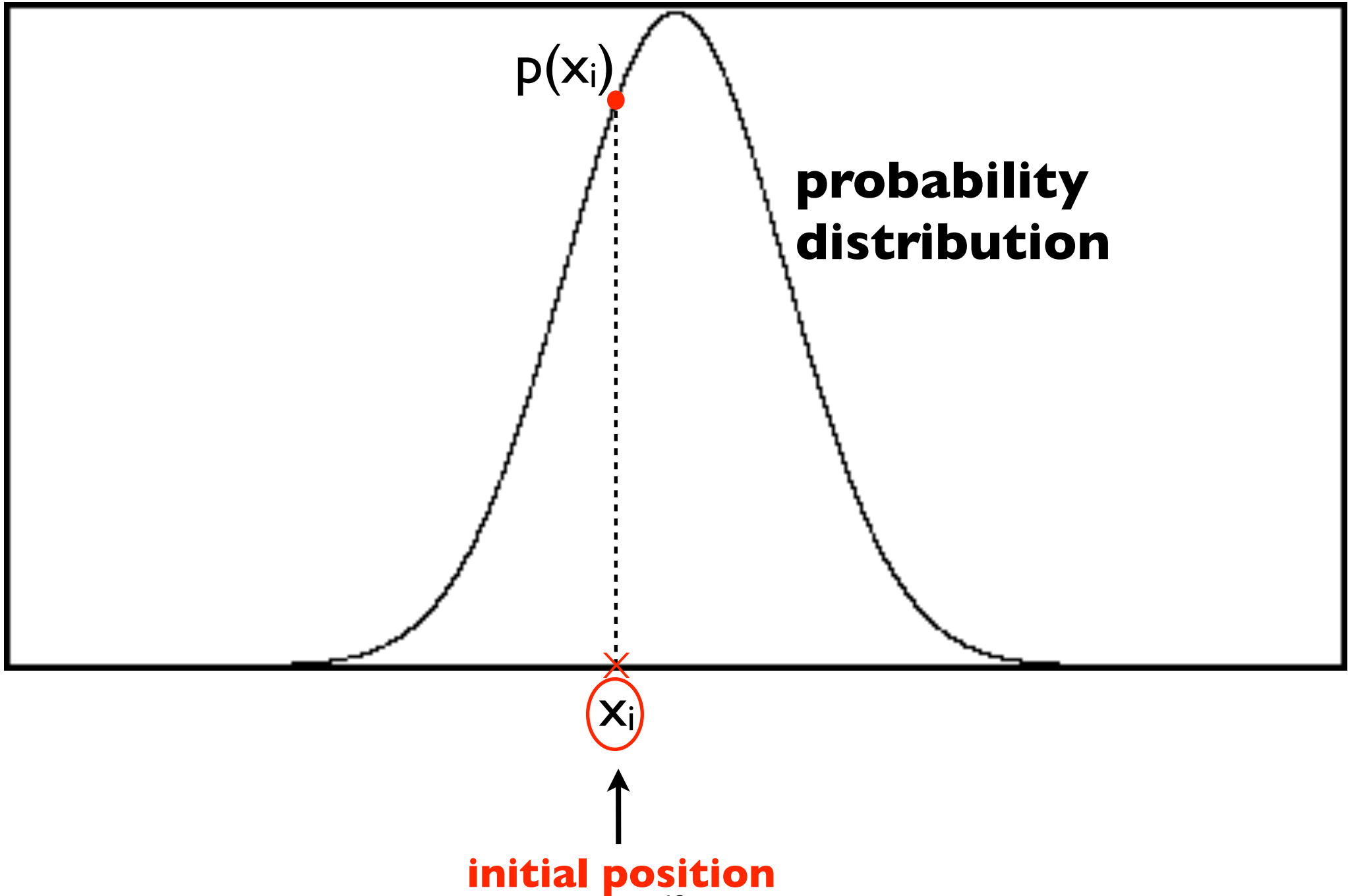
$x_j$

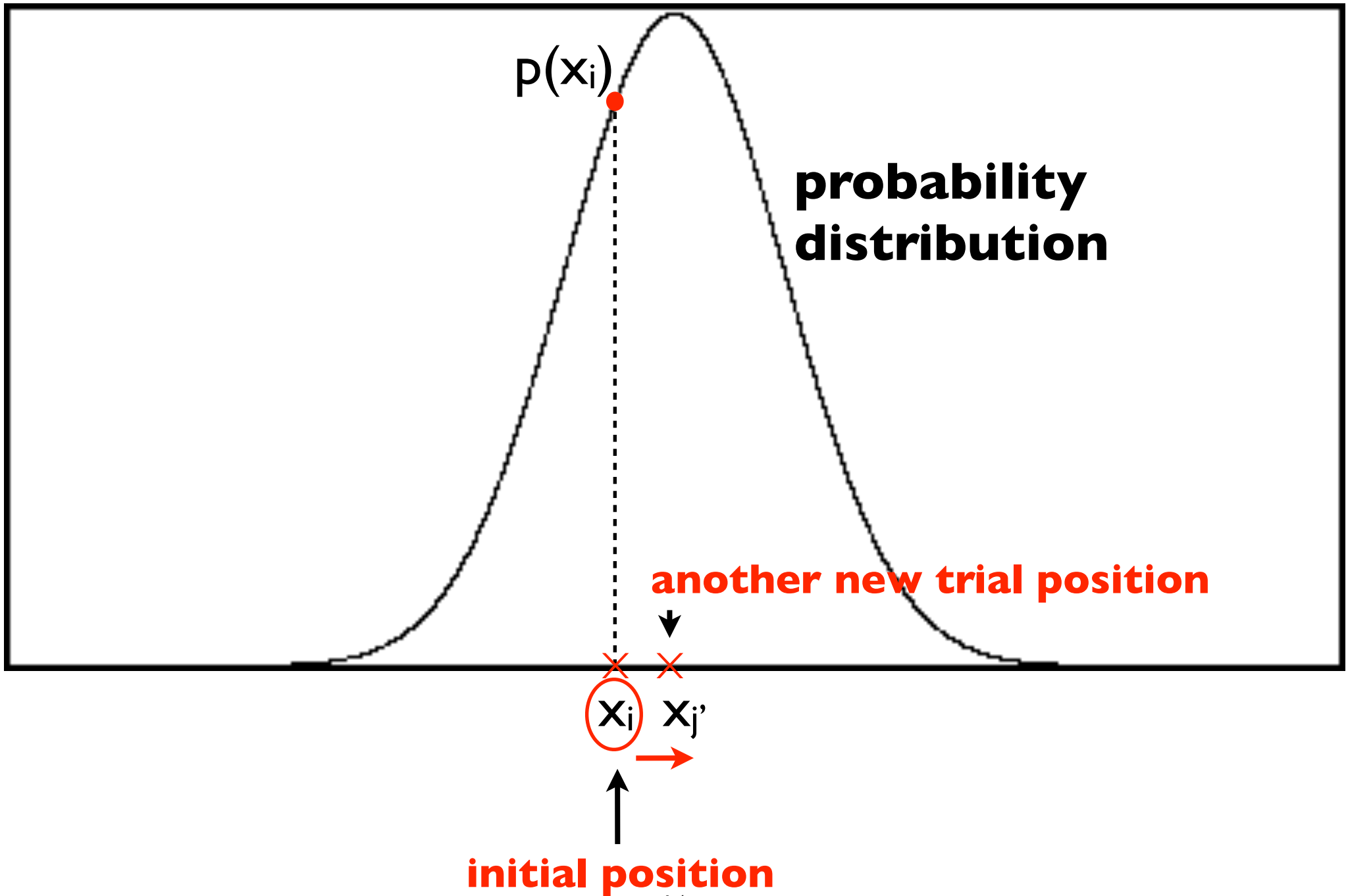
$x_i$

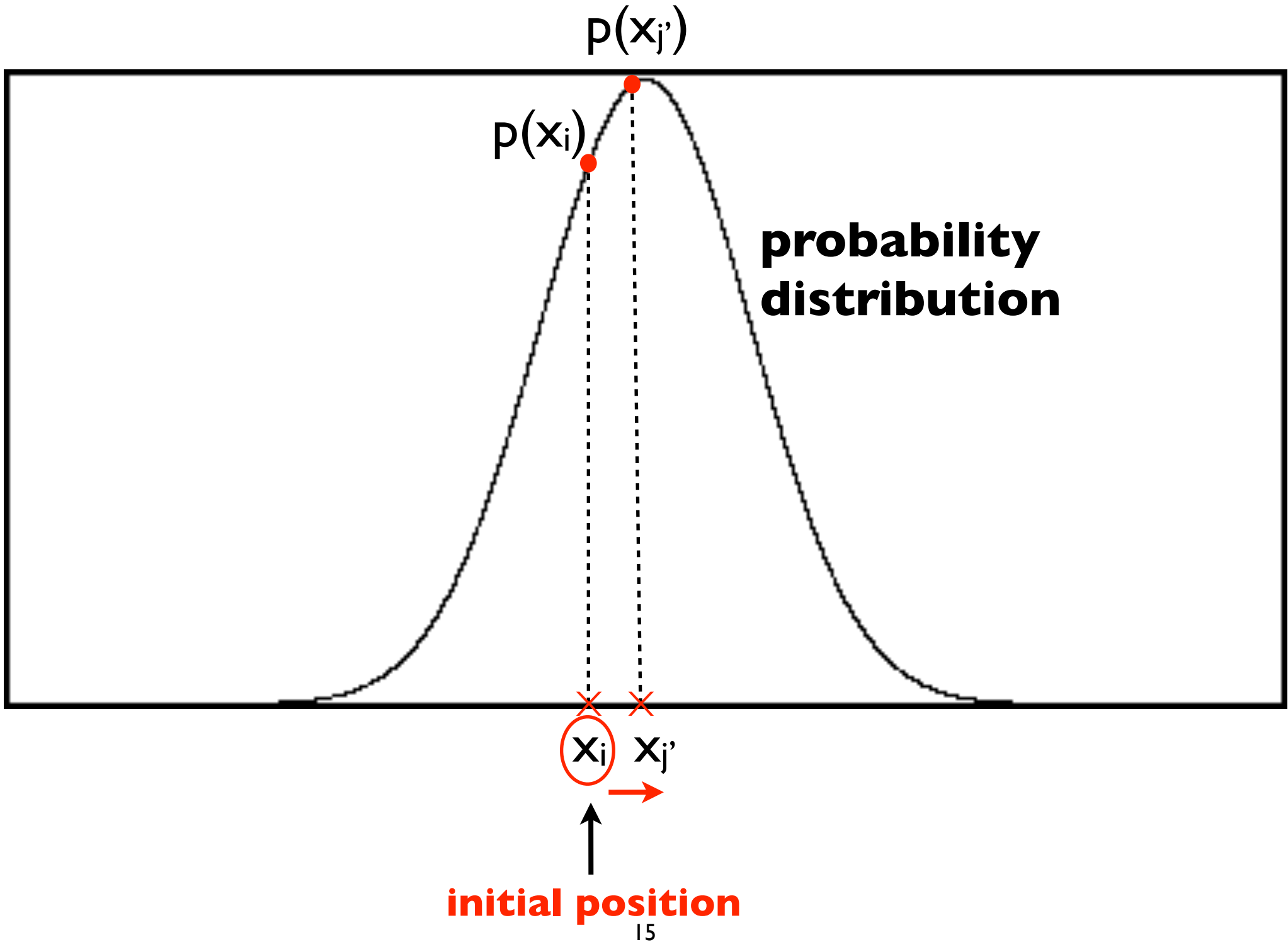
**initial position**

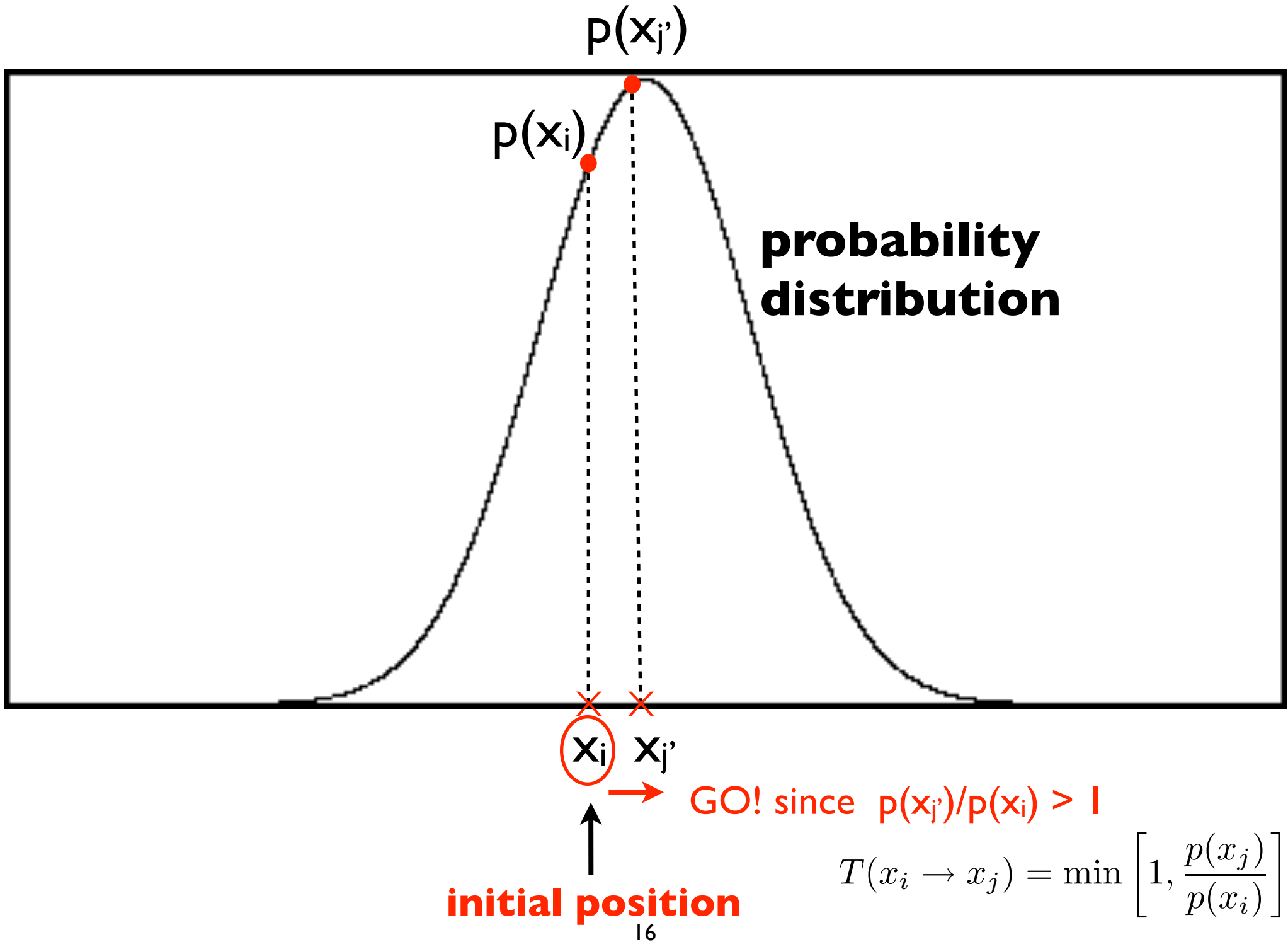
$$T(x_i \rightarrow x_j) = \min \left[ 1, \frac{p(x_j)}{p(x_i)} \right]$$

or:



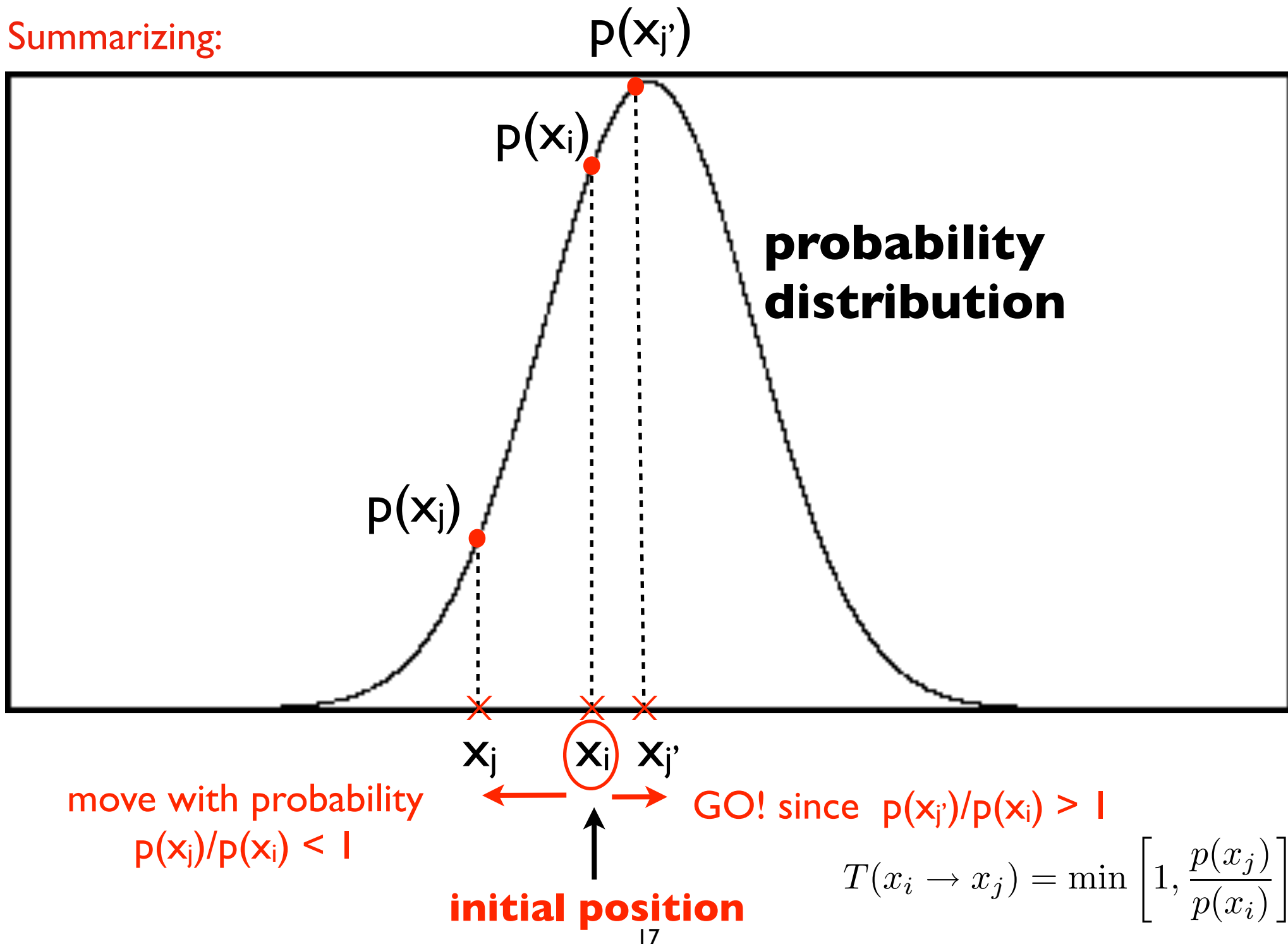








Summarizing:



# The Metropolis algorithm

$p(x)$  is given.

If the “walker” is at position  $x_i$  and we wish to generate  $x_{i+1}$ , we can implement this choice of  $T(x_i \rightarrow x_j)$  by the following steps:

1. Choose a trial position  $x_{\text{trial}} = x_i + \delta_i$ , where  $\delta_i$  is a random number in the interval  $[-\delta, \delta]$ .
2. Calculate  $w = p(x_{\text{trial}})/p(x_i)$ .
3. If  $w \geq 1$ , accept the change and let  $x_{i+1} = x_{\text{trial}}$ .  
else
4. If  $w < 1$ , generate a random number  $r$ .
5. If  $r \leq w$ , accept the change and let  $x_{i+1} = x_{\text{trial}}$ .
6. If the trial change is not accepted, then let  $x_{i+1} = x_i$ .

The algorithm from 1) to 6) has to be repeated until the distribution  $p(x)$  of the points  $\{x_i\}$  is reached.

# note:

it's important **how to handle the rejected attempts** for the generation of the random walk:

in case of a rejected attempt, the walker does not move, and we have to consider again the point where we tried to move from;

in the integration with importance sampling, a point which is unchanged after a rejected attempt does enter again in the average, i.e. its weight in the sum increases

# Questions:

- how to choose  $x_0$  ?
- how to choose  $\delta$  ?  
(if too small, most trial steps accepted, but the walker moves too slowly; if too large, only a few trial steps are accepted...)
- equilibration is necessary (how many steps?)

# Answers:

- **how to choose  $x_0$  ?**  
Convenient to start from a maximum
- **how to choose  $\delta$  ?**  
(if too small, most trial steps accepted, but the walker moves too slowly; if too large, only a few trial steps are accepted...)  
A good compromise is a choice accepting from  $\sim 1/3$  to  $\sim 1/2$  of the trial steps
- **equilibration is necessary (how many steps?)**  
A possible criterion based on error estimate

# Some programs:

in [moodle2.units.it](http://moodle2.units.it):

**gauss\_metropolis.f90**

metropolis\_sampling.f90

direct\_sampling.f90

boltzmann\_metropolis.f90

```

! gauss_metropolis.f90
! METROPOLIS generation of random numbers with a Gaussian distribution
!  $P(x) = \exp(-x^2/(2\sigma^2))/\sqrt{2\pi\sigma^2}$ 
.... start from a given x=x0 .....
do i=1,n

```

```

!cccccccccccccccccccccccccccccccccccccccc
expx = - x**2 / (2*sigma**2)  ← ! the exponent of p(x)
call random_number(rnd)      !
xp = x + delta * (rnd-0.5)    ! the exponent of p(x')
expxp = - xp**2 / (2*sigma**2) ← ! metropolis
w = exp (expxp-expx)          ! algorithm
call random_number(rnd)      ! the ratio p(x')/p(x)
if (w > rnd) then             !
    x = xp                    !
!cccccccccccccccccccccccccccccccccccccccc

endif

```

```

enddo

```

```

! gauss_metropolis.f90
! METROPOLIS generation of random numbers with a Gaussian distribution
!  $P(x) = \exp(-x^2/(2\sigma^2))/\sqrt{2\pi\sigma^2}$ 
.... start from a given x=x0 .....
do i=1,n
  x1 = x1 + x
  x2 = x2 + x**2
  x3 = x3 + x**3
  x4 = x4 + x**4
  !!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
  expx = - x**2 / (2*sigma**2)      !
  call random_number(rnd)           !
  xp = x + delta * (rnd-0.5)        !
  expxp = - xp**2 / (2*sigma**2)    !
  w = exp (expxp-expx)              !
  call random_number(rnd)           !
  if (w > rnd) then                  !
    x = xp                          !
  !!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
  acc=acc+1.
endif
  ibin = nint(x/deltaisto)
  if (abs(ibin) < maxbin/2) istog(ibin) = istog(ibin) + 1
enddo

```

**← calculate some momenta**

**← calculate the acceptance ratio**

**← calculate the histogram**



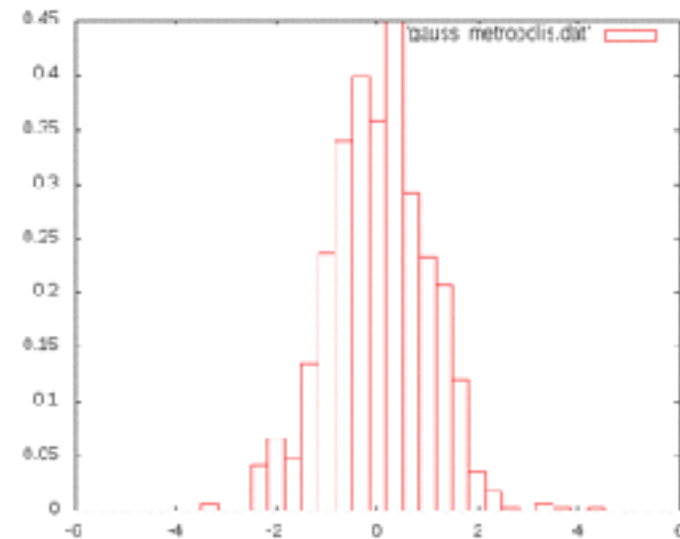
# Metropolis generation of random numbers distribution

I)

let's use the Metropolis method to generate a gaussian distribution

example of application:

( $n=1000$ ,  $x_0=0$ ,  $\delta=5$ ,  $\sigma=1$ )



(with `gauss_metropolis.f90`)

## Answers from numerical experiments:

- **how to choose  $x_0$ ?**  
Convenient to start from a maximum
- **how to choose  $\delta$ ?**  
(if too small, most trial steps accepted, but the walker moves too slowly; if too large, only a few trial steps are accepted...)  
A good compromise is a choice accepting from  $\sim 1/3$  to  $\sim 1/2$  of the trial steps: depends on  $\sigma$
- **equilibration is necessary (how many steps?)**  
A possible criterion based on error estimate:  
consider when  $\langle x^2 \rangle \approx \sigma^2$

# example of application of the Metropolis algorithm

- 1) to generate random points with a given distribution
- 2) to calculate averages with importance sampling
- 3) in particular: in the canonical ensemble

# example of application of the Metropolis algorithm

1) to generate random points with a given distribution

2) to calculate averages with importance sampling

of the form

$$\overset{\text{weighted average mean}}{\downarrow} \langle f \rangle = \frac{\int p(x) f(x) dx}{\int p(x) dx}$$

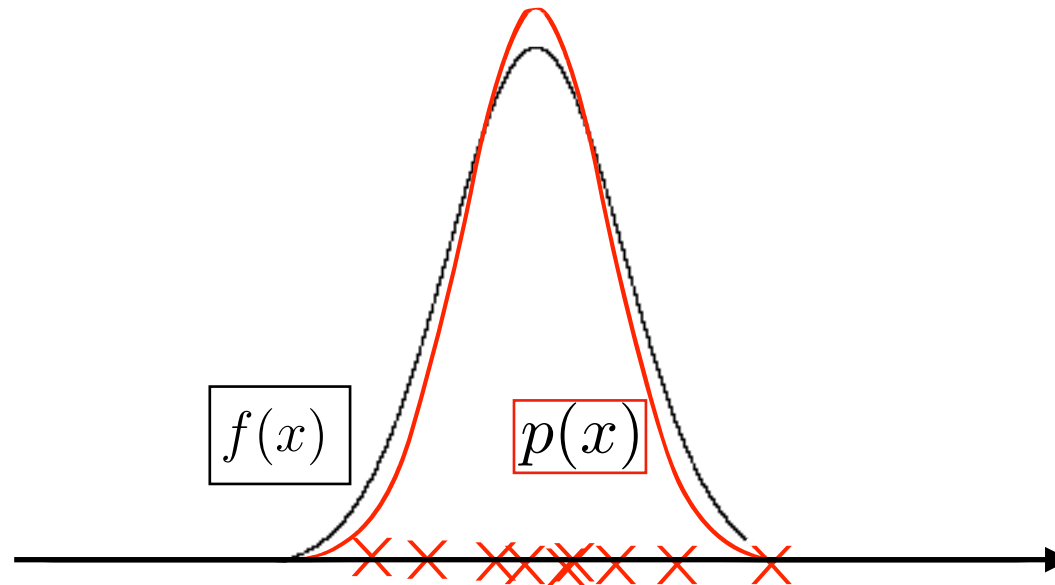
where the probability distribution  $p(x)$  does not need to be normalized

(here: 1D, but generally useful also for multidimensional integrals)

# example of application of the Metropolis algorithm

A special case of **importance sampling** where certain possible sampling attempts are rejected.

- generate points according  $p(x)$
- importance sampling of  $f(x)$  using  $p(x)$



reminder from Lecture V:

# “importance sampling”

consider a **distribution function**  $p(x)$  **easy to integrate**  
**and close to**  $f(x)$  :

weighted average mean



$$\int_a^b f(x) dx = \int_a^b \left[ \frac{f(x)}{p(x)} \right] p(x) dx = \left\langle \frac{f(x)}{p(x)} \right\rangle \int_a^b p(x) dx$$

where  $\left\langle \frac{f(x)}{p(x)} \right\rangle \approx \frac{1}{N} \sum_{i=1}^N \left[ \frac{f(x_i)}{p(x_i)} \right]$

with  $\{x_i\}$  distributed according to  $p(x)$ .

Here : substituting  $\frac{f(x)}{p(x)} \Rightarrow f(x)$  and rewriting, we have :  $\langle f \rangle = \frac{\int p(x) f(x) dx}{\int p(x) dx}$

# Some programs:

in [moodle2.units.it](http://moodle2.units.it):

`gauss_metropolis.f90`

**`metropolis_sampling.f90`**

**`direct_sampling.f90`**

`boltzmann_metropolis.f90`

# Metropolis Sampling

Using a method to generate a distribution  $p(x)$ , we can efficiently sample integrals of the form

$$\langle f \rangle = \frac{\int p(x) f(x) dx}{\int p(x) dx}$$

**example of application:**

See `metropolis_sampling.f90` in the exercises:  
example of estimate of average position, average kinetic, potential and total energies in the ground state of the harmonic oscillator:

$f(x)$  : physical quantity;  $p(x) = |\psi(x)|^2$



# Metropolis Sampling

$$\langle f \rangle = \frac{\int p(x) f(x) dx}{\int p(x) dx}$$

$f(x)$  : physical quantity;  $p(x) = |\psi(x)|^2$

Consider the hamiltonian :  $\mathcal{H} = -\frac{1}{2}\nabla^2 + \frac{1}{2}x^2$

Consider a wavefunction (not necessarily the ground state) :  $\psi(x) = \exp(-x^2/4\sigma^2)$

Interesting physical quantities are related to  $E_{\text{pot}}$ ,  $E_{\text{kin}}$ ,  $E_{\text{tot}}$

The potential energy can be considered as  $f(x)$  (it is a multiplicative operator); kinetic and total energy can not, but their expectation value can be related to the average value of  $x^2$ :

$$\langle E_{\text{pot}} \rangle = \frac{\langle \psi | \frac{1}{2}x^2 | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\int \frac{1}{2}x^2 |\psi(x)|^2 dx}{\int |\psi(x)|^2 dx}$$

$$\langle E_{\text{kin}} \rangle = \frac{\langle \psi | -\frac{1}{2}\nabla^2 | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\int \left( \frac{1}{4\sigma^2} - \frac{x^2}{8\sigma^4} \right) |\psi(x)|^2 dx}{\int |\psi(x)|^2 dx}$$

$\sigma$  here is a given  
input parameter



↑  
4 is ok

# Metropolis Sampling

$$\langle f \rangle = \frac{\int p(x) f(x) dx}{\int p(x) dx}$$

$f(x)$  : physical quantity;  $p(x) = |\psi(x)|^2$

Consider the hamiltonian :  $\mathcal{H} = -\frac{1}{2}\nabla^2 + \frac{1}{2}x^2$

Consider a wavefunction (not necessarily the ground state) :  $\psi(x) = \exp(-x^2/4\sigma^2)$

In this exercise, the numerical results can be checked by calculating  $\langle E_{\text{pot}} \rangle$  and  $\langle E_{\text{kin}} \rangle$  also analytically:

$$\langle E_{\text{pot}} \rangle = \frac{\langle \psi | \frac{1}{2}x^2 | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\int \frac{1}{2}x^2 |\psi(x)|^2 dx}{\int |\psi(x)|^2 dx} = \frac{1}{2}\sigma^2$$

$$\langle E_{\text{kin}} \rangle = \frac{\langle \psi | -\frac{1}{2}\nabla^2 | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\int \left( \frac{1}{4\sigma^2} - \frac{x^2}{8\sigma^4} \right) |\psi(x)|^2 dx}{\int |\psi(x)|^2 dx} = \frac{1}{8\sigma^2}$$





# Correlations

Consider a random sequence of configurations.

How many Monte Carlo steps are required between two configurations to be considered **uncorrelated**?

=> study the **autocorrelation** function:

$$C(j) = \frac{\langle x_{i+j} x_i \rangle - \langle x_i \rangle^2}{\langle x_i^2 \rangle - \langle x_i \rangle^2}$$

where:

$\langle \dots \rangle$ : average over the random sequence (index i)

$$C(j = 0) = 1$$

$C(j \neq 0) = 0$  expected for **totally uncorrelated** points,  
since in that case  $\langle x_i x_j \rangle = \langle x_i \rangle \langle x_j \rangle = \langle x_i \rangle^2$

# Correlations

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How many Monte Carlo steps are required between two configurations to be considered **uncorrelated**?

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$$C(j) = \frac{\langle x_{i+j} x_i \rangle - \langle x_i \rangle^2}{\langle x_i^2 \rangle - \langle x_i \rangle^2}$$

It is not always the case, but at least for *ergodic* simulations we should expect that the autocorrelation function approaches 0 as  $j \rightarrow \infty$ .

$C(j \neq 0) = 0$  expected for **totally uncorrelated** points, since in that case  $\langle x_i x_j \rangle = \langle x_i \rangle \langle x_j \rangle = \langle x_i \rangle^2$

# Origin of correlations

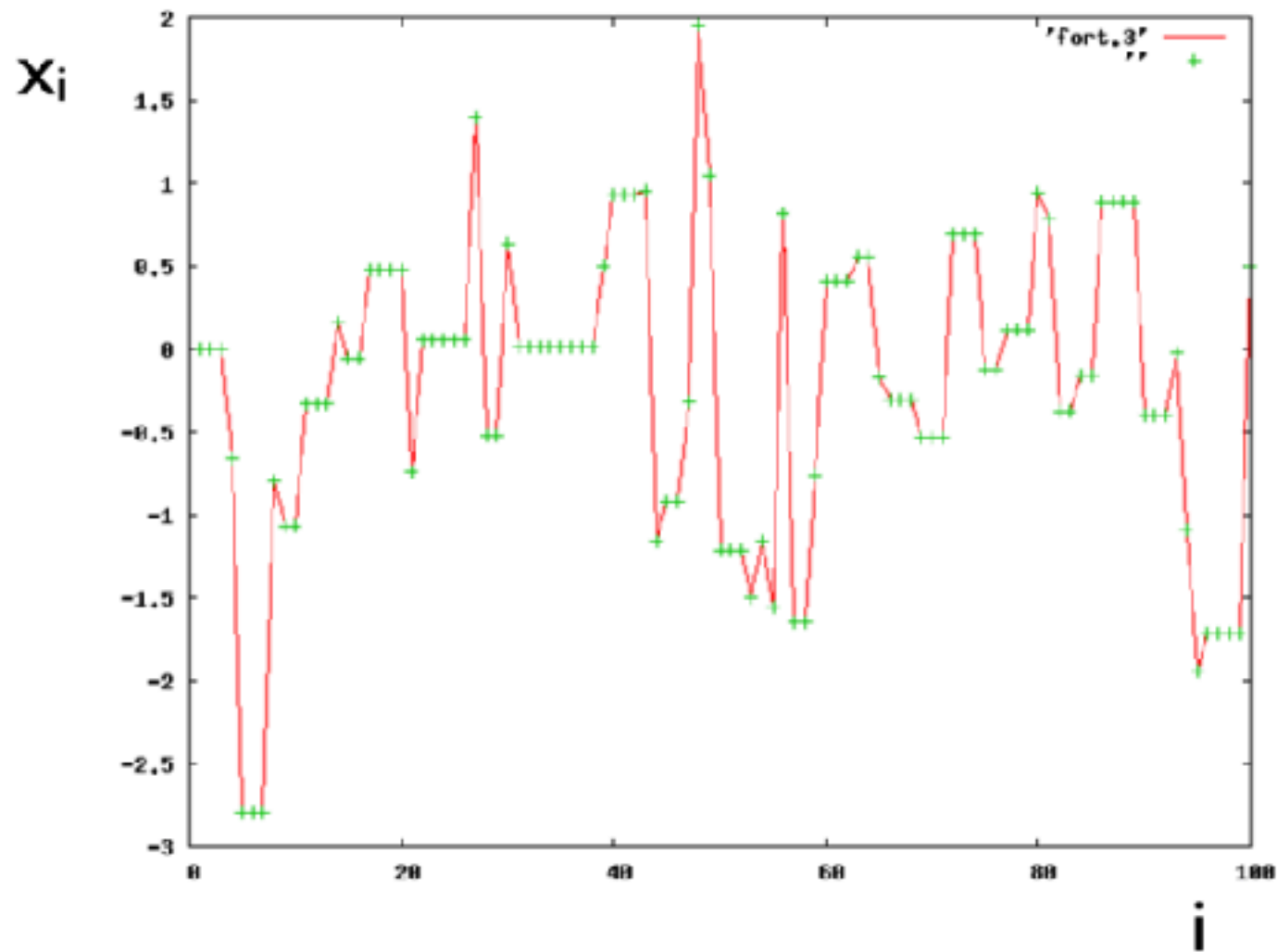
Metropolis algorithm:

necessarily the points of the walker are correlated  
each other **over a short “time” scale** (measured in  
terms of Monte Carlo steps; at least 1 time step!)

Correlation exponentially decaying with a certain  
characteristic “time”  $\tau$

Only points separated by  $2\tau$  or  $3\tau$  can be  
considered statistically independent

example of a sequence of points randomly (Gaussian) distributed  
generated according Metropolis



by its nature, Metropolis method introduces (at least short-range) correlations



## ex.VI.3 - Correlations

[a very simple calculation if the array points(n) is stored]

```
do j = 0, jmax
  si=0._dp ; si2=0._dp ; sij=0._dp
  do i = 1,n-j
    si = si + points(i)
    si2 = si2 + points(i)**2
    sij = sij + points(i)*points(i+j)
  end do
  si = si/(n-j)
  si2 = si2/(n-j)
  sij = sij/(n-j)

  write(2,*), j, (sij-si**2)/(si2-si**2)

end do
```

$$C(j) = \frac{\langle x_{i+j}x_i \rangle - \langle x_i \rangle^2}{\langle x_i^2 \rangle - \langle x_i \rangle^2}$$

# Calculation of correlations

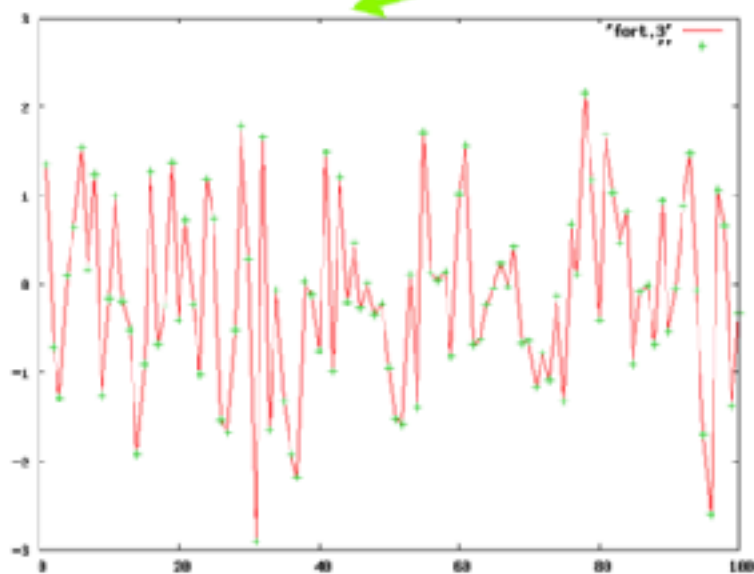
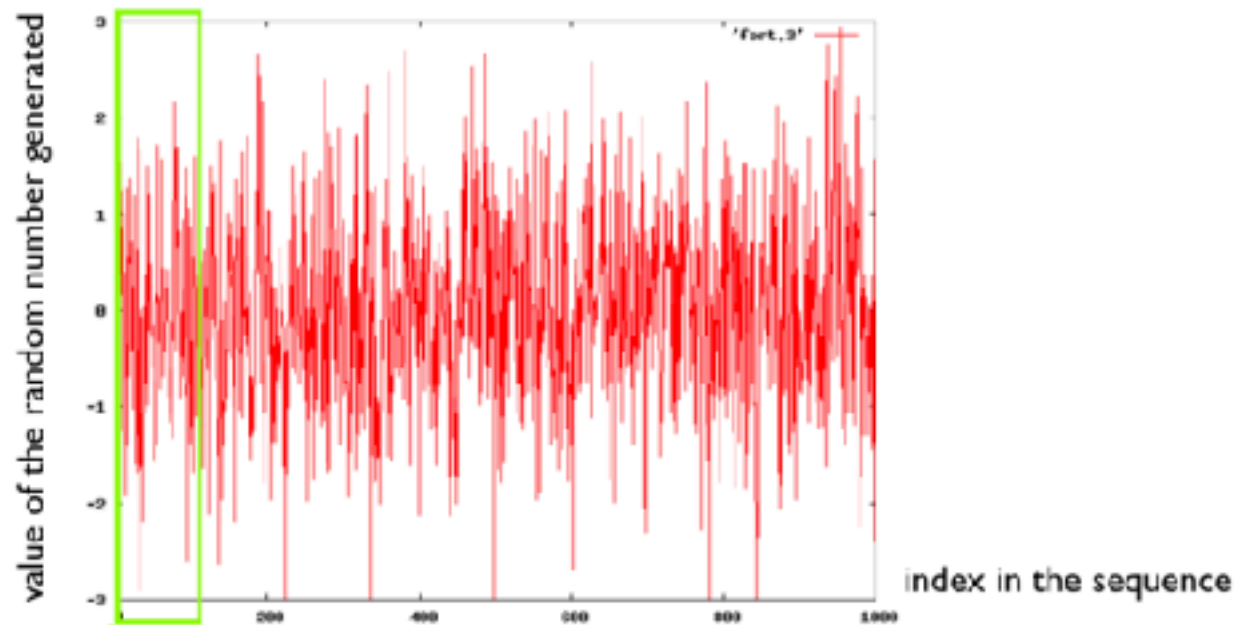
In Metropolis simulations, the autocorrelation time is often measured when the simulation is running:

- Create an array, say “corr” with  $j$  elements and initialize it to zero.
- Maintain a list of the  $j$  most recently computed values of the observable  $C$ . This can be an array of length  $j$  in which the value of  $C_n$  is stored at index “ $n \bmod j$ ”.
- At each step  $n \geq j$  accumulate the values of  $C_{n-j}C_{n-j+i}$  for  $i = 0, 1, \dots, j-1$  in the array elements  $\text{corr}[i]$ .
- At the end of the run, divide each  $\text{corr}[i]$  by  $n - j$ , subtract  $\langle C \rangle^2$ , and divide by  $\text{corr}[0]$  to normalize.

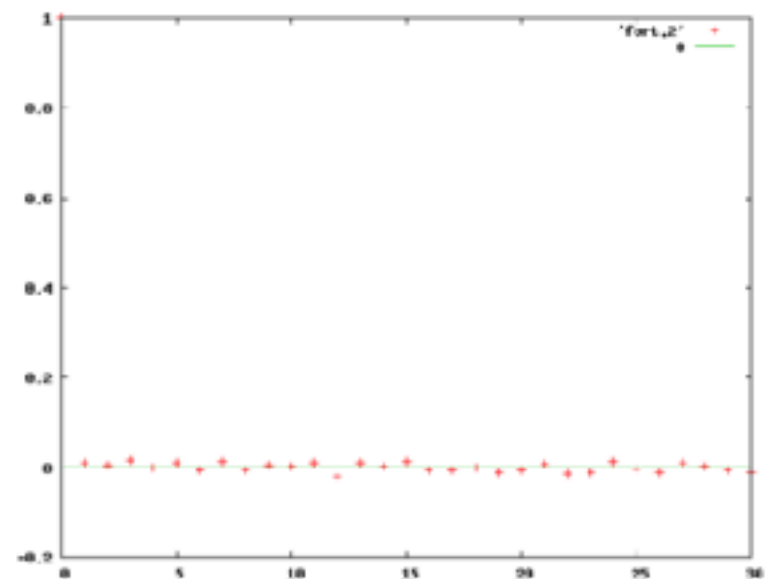
(do the code yourself!)

Calculate  $C(j)$  for different  $j$  and check when it approaches 0

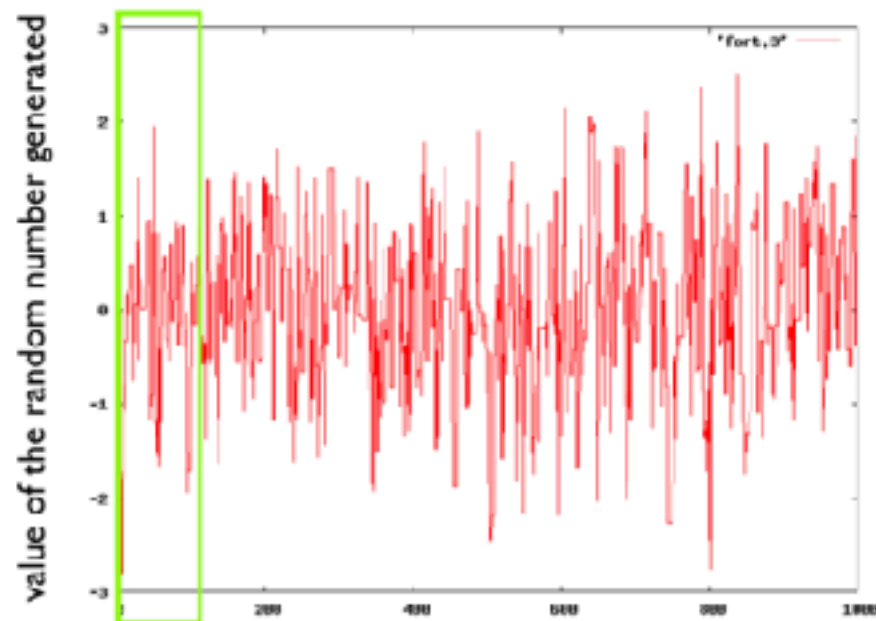
# Correlations - Box-Muller algorithm



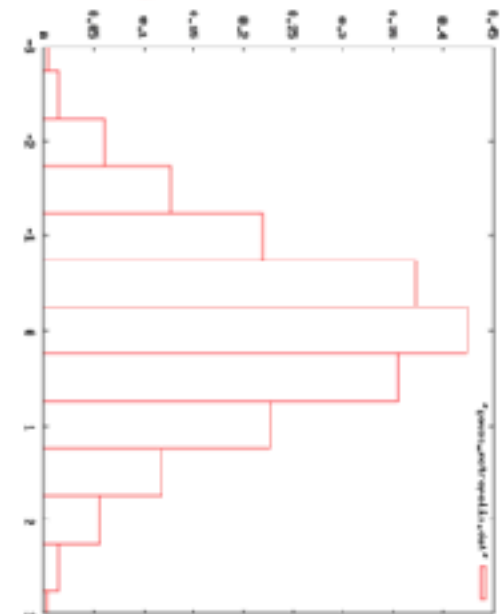
no correlations



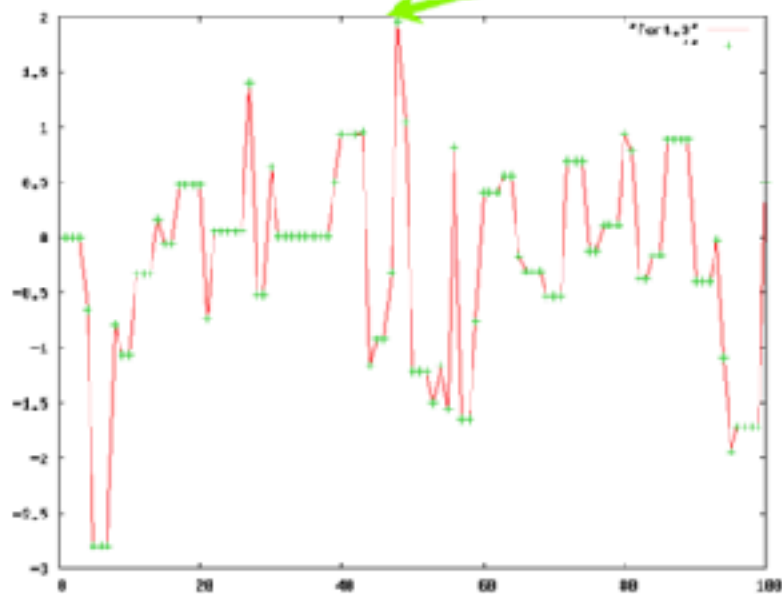
# Correlations - Metropolis algorithm



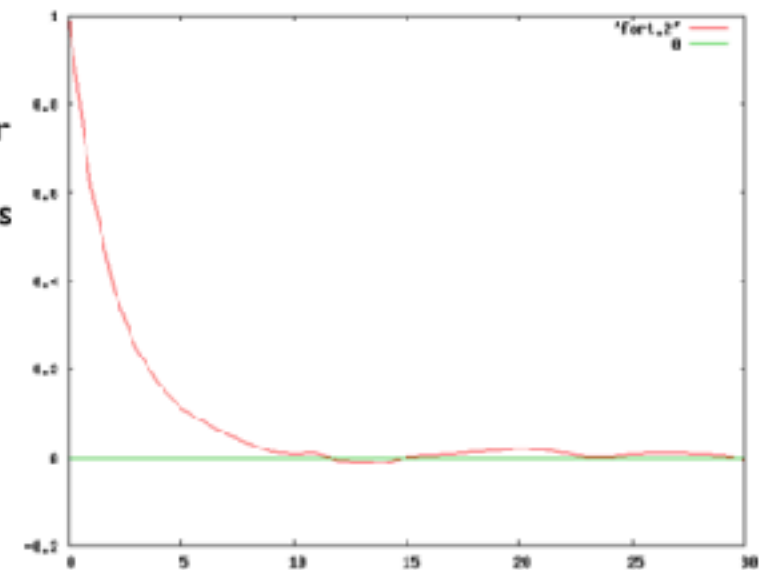
the first 1000 points



histogram over 10000 points



the short-term behavior  
of the Markov chain  
explains the correlations



# Notes - I

- \* Correlation may cause fictitiously a variance of the averages much smaller than the actual error!

- \* The relationship

$$\sigma_n / \sqrt{n} \approx \sigma_m \approx \sigma_s / \sqrt{s}$$

is based on the assumption of **uncorrelated** data  
(at least, for the block average, uncorrelated among  
different blocks)

# Notes - II

Not good to use correlated points...

How to estimate correlations? How to estimate  $\tau$ ?

How to control the reliability of the statistical sampling?

Use **block averages** with different block size and compare the numerical error estimates  $\sigma_s/\sqrt{s}$  and  $\sigma_{s'}/\sqrt{s'}$ . If they coincide, the correlation time is smaller than the smallest block size used.

Suppose  $n=1000$  data.

- 1) do blocking of  $s=20$  sets with 50 points and calculate averages and errors
  - 2) do blocking of  $s'=10$  sets of 100 points and calculate averages and errors
- 50 is therefore the smallest block size used.

If  $\sigma_{20}/\sqrt{20} \approx \sigma_{10}/\sqrt{10}$ , this means that  $\tau \ll 50$

# Metropolis Sampling

Using a method to generate a distribution  $p(x)$ , we can efficiently sample integrals of the form

$$\langle f \rangle = \frac{\int p(x) f(x) dx}{\int p(x) dx}$$

Particularly useful integrals (or averages) are those related to **ensemble averages**

# Metropolis algorithm

- 1) to generate random points with a given distribution
- 2) to calculate averages with importance sampling
- 3) in particular: in the canonical ensemble



# Review of some concepts of statistical mechanics

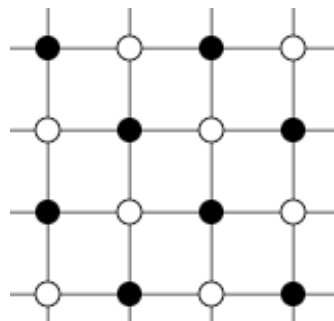
*(microstate / macrostate / trajectory / statistical ensemble;  
statistical and temporal averages)*

# Microstates

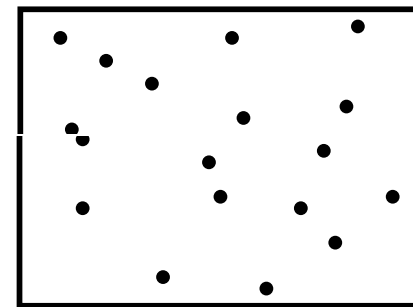
I) Examples of a **microstate**: characterized by:

- $|\Psi_\nu\rangle$  ( $\nu$  index related to  $N$  particles) in the Hilbert space in Quantum Mechanics
- a point in the phase space (large number of variables) in Classical Mechanics

examples of microstates:



distribution of **spins on a lattice**  
(open circles: spin up;  
closed circles: spin down)



distribution of **particles in a box**  
(list of positions)

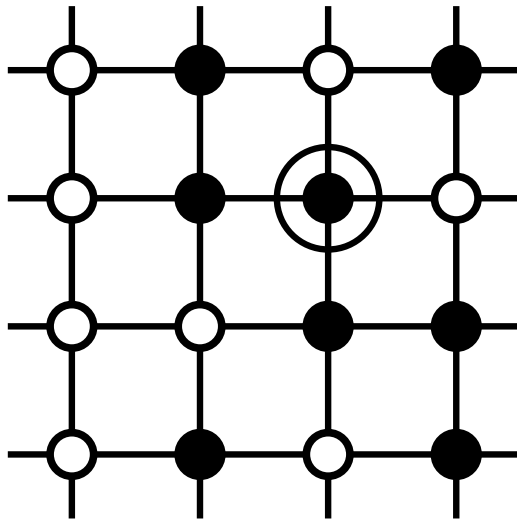
# Macrostate and Trajectory

2) For a classical system, the temporal evolution (with possible changes of microstates) is a **trajectory** (a line) in the phase space:

- along the trajectory, some parameters or variables such as  $N$ , or  $V$ , or  $T$  .... are fixed (constraints - **macrostate**)
- others do change

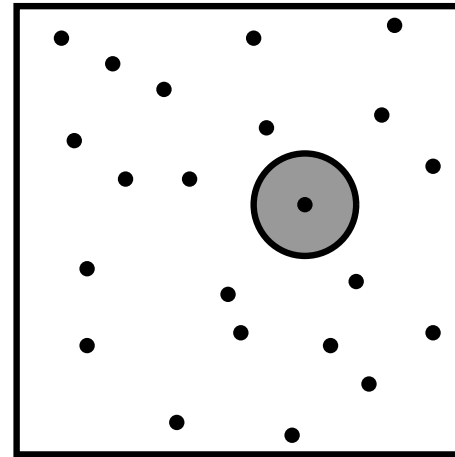
The trajectory is on a certain **surface** in the phase space (typically still high-dimensional), determined by the constraints

## examples of changes of microstates:



**flip of one spin** in a lattice  
(open circles: spin up;  
closed circles: spin down)

(the macrostate here is  
characterized by the  
temperature  $T$  and  
by the total number of lattice sites)

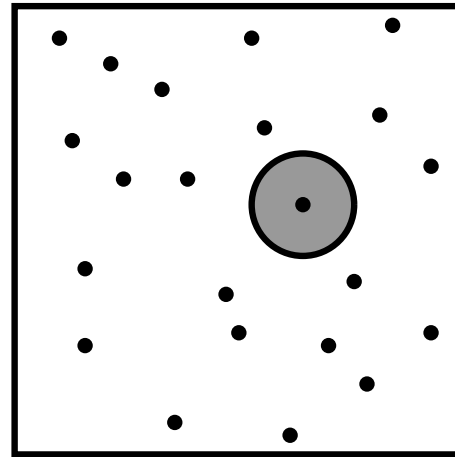
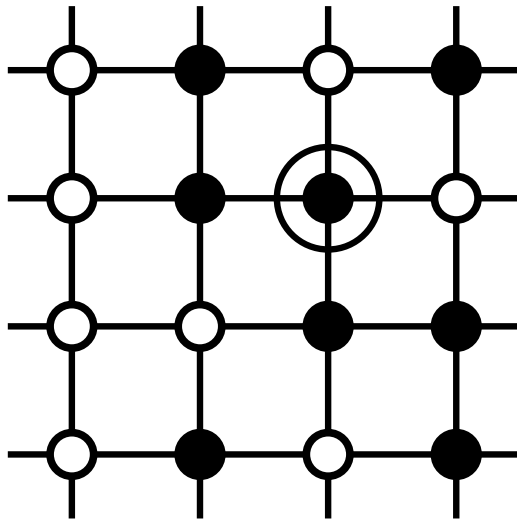


**moving one particle** in a box

(the macrostate here is  
characterized by the  
number of particles  $N$ )

(random choice of spin flip/particle move in case of Markov chain)

# Stochastic processes and Markov chains



## **Stochastic process:**

evolves through a series (a Markov chain) of well defined configurations (microstates in a given ensemble)

$C = \{C_1, C_2, \dots, C_N\}$  stochastically generated, i.e.  $C_{k+1}$  is obtained from the previous one,  $C_k$ , by making some random changes on the former.

# Statistical averages

3) which info from the **trajectory** in the phase space?

After a sufficiently long time, the system will assume **all possible microstates** compatible with the constraints i.e. with the **macrostate** : the ensemble of such microstates is a **statistical ensemble**

Suppose to make N independent measurements of an observable G:

$$G_{obs} = \frac{1}{N} \sum_{a=1}^N G_a \leftarrow \text{the measure is on a microstate}$$

$$= \sum_s \left[ \frac{1}{N} \cdot \left( \begin{array}{c} \text{\# of times in which the} \\ \text{microstate } s \text{ is observed} \end{array} \right) \right] G_s$$

(M is the number of the possible different microstates s)

$$= \sum_{s=1}^M P_s G_s = \underbrace{\langle G \rangle}_{\substack{\text{statistical average} \\ \text{or} \\ \text{ensemble average}}}$$

( $P_s$  depends of the microstate s but also on the macrostate)

**statistical average = temporal average**

is a fundamental assumption of the statistical mechanics;  
OK if:

- the system is **ergodic** (after a sufficient long time, the trajectory visits **all** the possible microstates)
- observation time is **long** ( $T \gg \tau_{relax \text{ or } equil}$ )
- observations are **independent** ( $d \gg d_{correl}$ )

# The canonical ensemble

**(N,V,T) fixed.** The probability that the system is in the microstate  $s$  with energy  $E_s$  is given by:

(here the energy identifies different microstates, it is not a characteristic of the macrostate)

$$P_s = \frac{1}{Z} e^{-\beta E_s}, \text{ (Boltzmann distribution)}$$

where  $\beta = 1/kT$ , and  $Z$  is a normalization constant. Because  $\sum P_s = 1$ ,

$$Z = \sum_{s=1}^M e^{-E_s/kT}$$

*partition function*  
( $M$ : all accessible microstates of the system)  
characterized by different  $E_s$ )

Ensemble averages (e.g. for the energy):

$$\langle E \rangle = \sum_{s=1}^M E_s P_s = \frac{1}{Z} \sum_{s=1}^M E_s e^{-\beta E_s}$$



# Averages in the canonical ensemble

We can generate only a finite number  $m$  of the total number  $M$  of accessible microstates; we hope that:

$$\langle A \rangle \approx A_m = \frac{\sum_{s=1}^m A_s e^{-\beta E_s}}{\sum_{s=1}^m e^{-\beta E_s}}$$

(\*)

(Note:  $m$ , not  $M$  !)

## A crude MonteCarlo procedure:

generate a microstate  $s$  at random, calculate  $E_s$ ,  $A_s$ , and  $e^{-\beta E_s}$ , and evaluate (\*)

Poorly efficient!  $P_s$  for the generated microstates could be rather small. An importance sampling method is better!

# Importance sampling in the canonical ensemble

$$\langle A \rangle \approx A_m = \frac{\sum_{s=1}^m A_s e^{-\beta E_s}}{\sum_{s=1}^m e^{-\beta E_s}} = \sum_{s=1}^m A_s \pi_s \quad \text{with} \quad \pi_s = \frac{e^{-\beta E_s}}{\sum_{s=1}^m e^{-\beta E_s}}$$

(\*) **NO** importance sampling:  
generate m configurations  
with uniform random distribution,  
then make a **weighted** average of A) (Note :  $\pi_s \neq P_s$  !)

If we **generate** microstates according to  **$\pi_s$** , the calculation of  $A_m$  reduces to:

$$A_m = \frac{1}{m} \sum_{s=1}^m A_s$$

**with importance sampling**

(generate m configurations  
with random distribution  $\pi_s$ ,  
then make a **simple** average of A)

(much more efficient than (\*))

# Importance sampling in the canonical ensemble

Therefore, summarising: we aim at calculating

$$\langle A \rangle \approx \frac{1}{m} \sum_{s=1}^m A_s \text{ with microstates } s \text{ generated according to } \pi_s$$

The transition matrix that generates microstates  $s$  according to  $\pi_s$  is :

$$T_{old,new} = \min \left[ 1, \frac{\pi_{new}}{\pi_{old}} \right] = \min \left[ 1, \frac{p_{new}}{p_{old}} \right] = \min \left[ 1, \frac{e^{-\beta E_{new}}}{e^{-\beta E_{old}}} \right]$$

hence, using **Metropolis**, the procedure is as following:

# Metropolis algorithm in the canonical ensemble

1. Establish an initial microstate.
2. Make a random trial change in the microstate. For example, choose a spin at random and flip it. Or choose a particle at random and displace it a random distance.
3. Compute  $\Delta E \equiv E_{\text{trial}} - E_{\text{old}}$ , the change in the energy of the system due to the trial change.
4. If  $\Delta E$  is less than or equal to zero, accept the new microstate and go to step 8.
5. If  $\Delta E$  is positive, compute the quantity  $w = e^{-\beta \Delta E}$ .
6. Generate a random number  $r$  in the unit interval.
7. If  $r \leq w$ , accept the new microstate; otherwise retain the previous microstate.
8. Determine the value of the desired physical quantities.
9. Repeat steps (2) through (8) to obtain a sufficient number of microstates.
10. Periodically compute averages over microstates.

# Metropolis algorithm in the canonical ensemble

Steps 2 to 7 give really the desired distribution using:

$$T(i \rightarrow j) = \min(1, e^{-\beta \Delta E}) \quad (\text{Metropolis algorithm}),$$

where  $\Delta E = E_j - E_i$ .

A few remarks:

1) ERGODICITY implicitly assumed!

2) TEMPERATURE:

If  $E_B > E_A$ , accept the new (higher energy) configuration with probability  $p = e^{-(E_B - E_A)/k_B T}$ . This means that when the temperature is high, we don't mind taking steps in the “wrong” direction, but as the temperature is lowered, we are forced to settle into the lowest configuration we can find in our neighborhood.

# Metropolis algorithm in the canonical ensemble: other remarks

I) Because it is necessary to evaluate only the ratio  $P_j/P_i = e^{-\beta\Delta E}$ ,  
it is not necessary to normalize the probability.

$$(P_j/P_i = \pi_j/\pi_i)$$

2) Other choices of  $\pi_s$  are possible. Instead of writing:

$$\langle A \rangle \approx A_m = \frac{\sum_{s=1}^m A_s e^{-\beta E_s}}{\sum_{s=1}^m e^{-\beta E_s}} = \sum_{s=1}^m \underbrace{A_s}_{\text{(no importance sampling)}} \pi_s = \frac{1}{m} \sum_{s=1}^m A_s \quad \text{(with importance sampling)}$$

rewrite:

$$A_m = \frac{\sum_{s=1}^m \underbrace{(A_s / \pi_s)}_{\text{(no importance sampling)}} e^{-\beta E_s} \pi_s}{\sum_{s=1}^m \underbrace{(1 / \pi_s)}_{\text{(no importance sampling)}} e^{-\beta E_s} \pi_s} \quad \text{(no importance sampling)}$$

If we generate microstates with probability  $\pi_s$ , eq. becomes:

$$A_m = \frac{\sum_{s=1}^m (A_s / \pi_s) e^{-\beta E_s}}{\sum_{s=1}^m (1 / \pi_s) e^{-\beta E_s}} \quad \text{(importance sampling)}$$

**3)** If  $T(i \rightarrow j)e^{-\beta E_i} = T(j \rightarrow i)e^{-\beta E_j}$  (detailed balance),  
Metropolis algorithm generates states with Boltzmann distribution

(we will prove it empirically : see exercise)



# Some programs:

in [moodle2.units.it](http://moodle2.units.it):

`gauss_metropolis.f90`

`metropolis_sampling.f90`

`direct_sampling.f90`

**`boltzmann_metropolis.f90`**

# Boltzmann distribution in the canonical ensemble

The Metropolis algorithm really produces microstates with the Boltzmann distribution:

application to ideal classical 1D gas (exercise n. 4)

1 free particle: Energy:  $E = \frac{1}{2}mv^2$

in this case, velocity or energy **labels a microstate**

(the energy with a factor of 2, due to +/- sign of  $v$ );

we generate different microstates by random variations of the velocity and we accept/reject with Metropolis

Important quantities are the probabilities:

$P(v)dv$  that the system has a velocity between  $v$  and  $v+dv$

or  $P(E)dE$  that the system has an energy between  $E$  and  $E+dE$

# ideal classical 1D gas

A particle moving randomly has in each direction a distribution of the component of the velocity:

$$f(v_x) = \left( \frac{m}{2\pi k_B T} \right)^{1/2} e^{-mv_x^2/2k_B T} \quad (1)$$

$$\langle v_x^2 \rangle = \int_{-\infty}^{+\infty} v_x^2 f(v_x) dv_x = \frac{k_B T}{m} \quad (2)$$

**In 1D:**

$$f(v)2dv = P(E)dE$$

that gives:

$$P(E) = \frac{1}{(\pi k_B T)^{1/2}} \frac{1}{\sqrt{E}} e^{-E/k_B T}$$

**In 3D:**

$$P(E) = \frac{2}{\sqrt{\pi}} \frac{1}{(k_B T)^{3/2}} \sqrt{E} \exp\left(-\frac{E}{k_B T}\right) \quad (3D)$$

# Boltzmann distribution in the canonical ensemble

```
# T      : 1.00000
# <E0>   : .000000
# <v0>   : .000000
# dvmax  : 2.00000
# deltaE : 5.000000E-02
# nbin   : 79
```

$$T = 1 \rightarrow \langle E \rangle (expected) = 0.5 \quad (m = 1)$$

```
=> boltzmann.1K <==
# nMCsteps: 1000
# <E>      : .501263
# <v>      : 7.456664E-02
# accept.  : .692000
# sigma    : .713780
```

$$\sigma / \sqrt{n} = 0.022 \quad (\sigma \text{ is the variance of the energy})$$

```
=> boltzmann.10K <==
# nMCsteps: 10000
# <E>      : .507580
# <v>      : 3.366172E-02
# accept.  : .707700
# sigma    : .726145
```

$$\sigma / \sqrt{n} = 0.007$$

```
=> boltzmann.1M <==
# nMCsteps: 1000000
# <E>      : .500138
# <v>      : 1.833840E-04
# accept.  : .693837
# sigma    : .707472
```

$$\sigma / \sqrt{n} = 0.0007$$

NOTE:

- Accuracy of ~ 1% on <E> and 10% on <v> : NMCS=1000 is enough
- NOT ENOUGH to well reproduce the BOLTZMANN DISTRIBUTION! (1M needed!)
- ACCEPTANCE RATIO: constant, depends only on dvmax
- SIGMA also

# Boltzmann distribution in the canonical ensemble

**many particles:** Energy:  $E = \sum_{i=1}^N \frac{1}{2} m_i v_i^2$   
in this case, the energy is NOT a label of a microstate  
(there are several **microstates with the same total energy**)

*Note: the energy histogram is NOT the distribution of microstates!*

$$P(E) = \sum_{\substack{\text{states } s \\ \text{with } E_s = E}} P_s \quad \text{with} \quad P_s = \frac{1}{Z} e^{-\beta E_s}$$

$$P(E) \propto e^{\frac{-(E - \langle E \rangle)^2}{2\sigma^2}} \quad \text{with } \langle E \rangle \text{ average over all the microstates}$$

**What is P(E)? (exercise)**

$P(E)$ , distribution of the kinetic energy of the system for different  $T$

