DECADIMENTO DEL MUONE

Riprendiamo la Lagrangiana di int. debole di Fermi:

$$\mathcal{L}_{CC}^{eff} = -4\frac{G_F}{\sqrt{2}} J_c^\mu J_{c\mu}^\dagger \qquad G_F \simeq 1{,}66 \times 10^{-5}\ GeV^{-2}$$

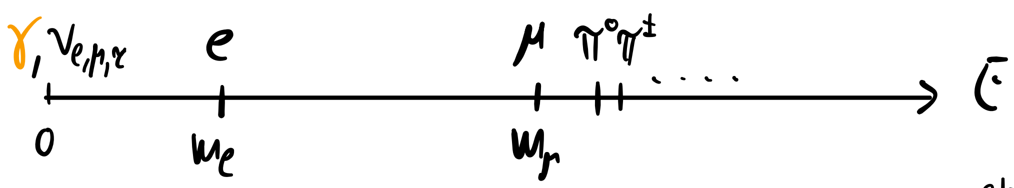
$$J_c^\alpha = V_{ij}\,\bar{u}_{i_L}\gamma^\alpha d_{jL} + \bar{\nu}_{iL}\gamma^\alpha e_{iL}$$

Questa contiene il termine:

$$\mathcal{L}_{CC}^{eff} \supset -4\frac{G_F}{\sqrt{2}}\left(\bar{\nu}_{\mu L}\gamma^\alpha \mu_L\right)\left(\bar{e}_L\gamma_\alpha \nu_{e_L}\right) + h.c.$$

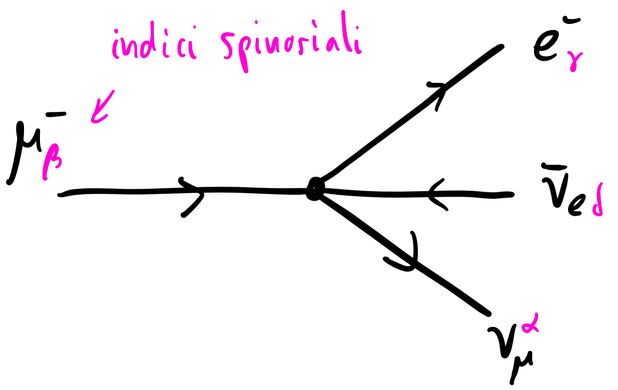
$$= -4\frac{G_F}{\sqrt{2}}\left(\bar{\nu}_\mu \gamma^\alpha P_L \mu\right)\left(\bar{e}\gamma_\alpha P_L \nu_e\right) + h.c. \,, \qquad P_{R,L} \equiv \frac{1 \pm \gamma_5}{2}$$

Andando a studiare lo spettro delle particelle leggere,



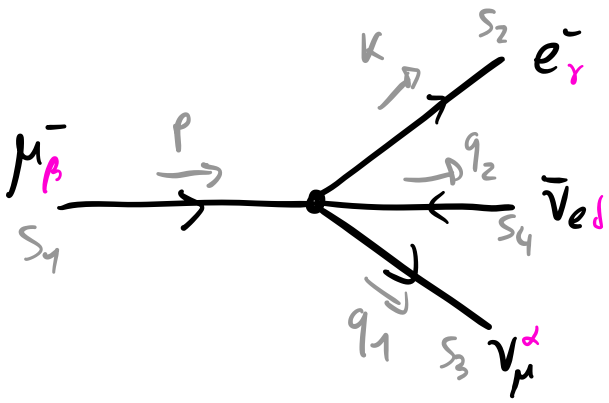
vediamo che il muone, tramite $\mathcal{L}_{CC}^{eff}$, può decadere solo nel canale: $\boxed{\mu^- \to e^- \bar{\nu}_e \nu_\mu}$

REGOLA DI FEYNMAN DA $\mathcal{L}_{CC}^{eff}$:



$$= -4i\frac{G_F}{\sqrt{2}}\left(\gamma^\alpha P_L\right)_{\alpha\beta}\left(\gamma_\alpha P_L\right)_{\gamma\delta}$$

AMPIEZZA DI DECADIMENTO



$$i\mathcal{M} = -i\,4\frac{G_F}{\sqrt{2}}\left[\bar{u}^{s_3}_{\nu_\mu}(q_1)\gamma^\alpha P_L u^{s_1}_\mu(p)\right]\left[\bar{u}^{s_2}_e(k)\gamma_\alpha P_L v^{s_4}_{\nu_e}(q_2)\right]$$

Rate di decadimento <u>non polarizzata</u>:

- Sommare sulle pol. finali
- Mediare " " iniziali

$$\frac{1}{2}\sum_{s_1,s_2,s_3,s_4}\mathcal{M}\mathcal{M}^\dagger = 4G_F^2\sum_{\text{spin}}\left[\bar{u}^{s_3}_{\nu_\mu}(q_1)\gamma^\alpha P_L u^{s_1}_\mu(p)\,\bar{u}^{s_1}_\mu(p)\gamma^\beta P_L u^{s_3}_{\nu_\mu}(q_1)\right]$$
$$\left[\bar{u}^{s_2}_e(k)\gamma_\alpha P_L v^{s_4}_{\nu_e}(q_2)\,\bar{v}^{s_4}_{\nu_e}(q_2)\gamma_\beta P_L u^{s_2}_e(k)\right]$$

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p}+m\,,\qquad \sum_s v^s(p)\bar{v}^s(p) = \not{p}-m$$

$P_L\gamma^\mu = \gamma^\mu P_R$

$\{\gamma^5,\gamma^\mu\}=0$

$$= 4G_F^2\,\mathrm{Tr}\left[\not{q}_1\gamma^\alpha P_L(\not{p}+\cancel{m_\mu})\gamma^\beta P_L\right]\mathrm{Tr}\left[(\not{k}+\cancel{m_e})\gamma_\alpha P_L\not{q}_2\gamma_\beta P_L\right]$$

$$= 4G_F^2\,\mathrm{Tr}\left[\not{q}_1\gamma^\alpha\not{p}\gamma^\beta P_L\right]\mathrm{Tr}\left[\not{k}\gamma_\alpha\not{q}_2\gamma_\beta P_L\right]$$

Tensore antisimm. di Levi-Civita

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta P_L\right] = 2\left(g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}-g^{\mu\nu}g^{\alpha\beta}+i\varepsilon^{\mu\alpha\nu\beta}\right)$$

$$\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\rho\sigma} = -2\left(\delta^\mu_\rho\delta^\nu_\sigma-\delta^\mu_\sigma\delta^\nu_\rho\right)$$

$$= 16\, G_F^2 \left(q_1^{\alpha} p^{\beta} + q_1^{\beta} p^{\alpha} - (q_1 p) g^{\alpha\beta} + i\, \varepsilon^{\mu\alpha\nu\beta} q_{1\mu} p_{\nu} \right)$$

$$\left(k_{\alpha} q_{2\beta} + k_{\beta} q_{2\alpha} - (k \cdot q_2) g_{\alpha\beta} + i\, \varepsilon_{\rho\alpha\sigma\beta} k^{\rho} q_2^{\sigma} \right) =$$

$$= 16\, G_F^2 \Big[2(q_1 k)(p q_2) + 2(q_1 q_2)(p k) + \cancel{(-2-2+4)(q_1 p)}(k \cdot q_2)$$

$$+ i^2 (-1)^2 \varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\alpha\beta\rho\sigma} q_{1\mu} p_{\nu} k^{\rho} q_2^{\sigma} \Big] =$$

$$= 16\, G_F^2 \Big[2(q_1 k)(p q_2) \cancel{+ 2(q_1 q_2)(p k)} + 2(q_1 k)(p q_2) \cancel{- 2(q_1 q_2)(p k)} \Big] =$$

$$\boxed{= 64\, G_F^2 (q_1 k)(p q_2) = \frac{1}{2} \sum_{spin} \mathcal{M} \mathcal{M}^{\dagger}}$$

RATE DI DECADIMENTO

$$d\Gamma = \frac{1}{2m_\mu}\left(\frac{1}{2}\sum_{spin} \mathcal{M}\mathcal{M}^+\right)(2\pi)^4 \delta^4(p-k-q_1-q_2)\frac{d^3q_1}{(2\pi)^3 2E_1}\frac{d^3q_2}{(2\pi)^3 2E_2}\frac{d^3k}{(2\pi)^3 2E_e}$$

- Vogliamo calcolare inizialmente $\frac{d\Gamma}{dE_e}$: spettro di energia dell'elettrone
- Dopodiché integreremo in dE_e per ottenere $\Gamma(\mu^- \to e^- \nu_\mu \bar{\nu}_e) = \tau^{-1}$

$$d\Gamma = \frac{4G_F^2}{m_\mu (2\pi)^5} k_\mu p_\nu q_1^\mu q_2^\nu \, \delta^4(\overbrace{p-k}^{q}-q_1-q_2)\frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2}\frac{d^3k}{E_e}$$

$q \equiv p-k = q_1+q_2$

Integriamo sullo spazio delle fasi dei neutrini:

$$\int \frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2} q_1^\mu q_2^\nu \, \delta^4(q-q_1-q_2) = I^{\mu\nu}(q)$$

$I^{\mu\nu}(q)$ dipende solo da q ed è simmetrico in $\mu \leftrightarrow \nu$

↳ Ansatz $I^{\mu\nu}(q) = g^{\mu\nu} A(q^2) + q^\mu q^\nu B(q^2)$ funz. scalare

$$\begin{cases} g_{\mu\nu} I^{\mu\nu} = 4A + q^2 B \\ q_\mu q_\nu I^{\mu\nu} = q^2 A + (q^2)^2 B \end{cases}$$

$m_\nu = 0 \to q_1^2 = q_2^2 = 0 \to q^2 = (p-k)^2 = (q_1+q_2)^2 = +2(q_1 q_2)$

$q\, q_1 = (q_1+q_2) q_1 = q_1 q_2 = \frac{q^2}{2} = q\, q_2$

$$\bullet\ g_{\mu\nu} I^{\mu\nu} = \int \frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2}(q_1 q_2)\,\delta^4(q-q_1-q_2) = \frac{q^2}{2}\int \frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2}\,\delta^4(q-q_1-q_2)$$

Nel sist. centro di massa di $\bar{\nu}_e \nu_\mu$: $\vec{q}_1 = -\vec{q}_2$

$\Rightarrow E_1 = E_2 = \omega = q^0/2$

$$= \frac{q^2}{2}\int \frac{d^3q_1}{\omega^2}\,\delta(q^0-2\omega) = \frac{q^2}{2}\int \frac{\cancel{\omega^2}\,d\omega\, d\Omega}{\cancel{\omega^2}}\,\frac{1}{2}\,\delta\!\left(\omega-\frac{q^0}{2}\right) =$$

$$= \frac{q^2}{4}\int d\Omega = \boxed{\pi q^2 = 4A + q^2 B}$$

$$\bullet\ q_\mu q_\nu I^{\mu\nu} = \int \frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2}(qq_1)(qq_2)\,\delta^4(q-q_1-q_2) = \left(\frac{q^2}{2}\right)^2 \int \frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2}\,\delta^4(q-q_1-q_2)$$

$$= \frac{1}{2}\pi (q^2)^{\cancel{2}} = \cancel{q^2} A + (q^2)^{\cancel{2}} B \rightarrow \boxed{\frac{1}{2}\pi q^2 = A + q^2 B}$$

Risolvendo per A e B

$$I^{\mu\nu}(q) = \frac{\pi}{6}\left(g^{\mu\nu}q^2 + 2q^\mu q^\nu\right) \rightarrow \text{Valido in qualsiasi sistema di riferimento}$$

Inserendo in $d\Gamma$:

$$d\Gamma = \frac{4G_F^2}{m_\mu (2\pi)^5} K_\mu P_\nu \frac{\pi}{6}\left(g^{\mu\nu}q^2 + 2q^\mu q^\nu\right)\frac{d^3k}{E_e} =$$

$$= \frac{2\pi}{3}\frac{G_F^2}{m_\mu(2\pi)^5}\left((kp)q^2 + 2(kq)(pq)\right)\frac{d^3k}{E_e}$$

Nel sistema di riposo del muone:

$P = (m_\mu, \vec{0})$, $\vec{k} = -\vec{q}_1 - \vec{q}_2 = -\vec{q}$, $q^0 = m_\mu - E_e$

$(kp) = m_\mu E_e$, $(kq) = E_e(m_\mu - E_e) + |\vec{k}|^2$, $(pq) = m_\mu^2 - m_\mu E_e$

$q^2 = (m_\mu - E_e)^2 - |\vec{k}|^2 = (m_\mu - E_e)^2 - (E_e^2 - m_e^2) = m_\mu^2 + m_e^2 - 2E_e m_\mu$

$$d^3k = |\vec{k}|^2 d|\vec{k}| d\Omega = |\vec{k}|^2 \frac{d|\vec{k}|}{dE_e} dE_e\, d\Omega = |\vec{k}|\, E_e\, dE_e\, d\Omega$$

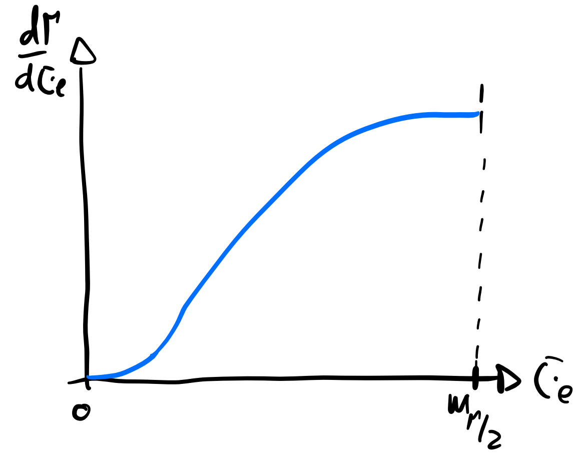
$$\Rightarrow \frac{d\Gamma}{dE_e} = \frac{2\pi}{3} \frac{G_F^2}{m_\mu (2\pi)^5} \left((kp) q^2 + 2(kq)(pq) \right) |\vec{k}|\, 4\pi$$

Infine:

$$\boxed{\frac{d\Gamma}{dE_e} = \frac{G_F^2}{12\pi^3} \sqrt{E_e^2 - m_e^2} \left(E_e (m_\mu^2 + m_e^2 - 2 m_\mu E_e) + 2 (E_e m_\mu - m_e^2)(m_\mu - E_e) \right)}$$

Prendiamo per semplicità il limite $m_e \simeq 0$:

$$\frac{d\Gamma}{dE_e} \overset{m_e = 0}{\simeq} \frac{G_F^2}{4\pi^3} m_\mu E_e^2 \left(m_\mu - \frac{4}{3} E_e \right)$$



Vediamo i limiti di E_e ($m_e \simeq 0$)

$$E_1 = |\vec{q}_1| = |\vec{k} + \vec{q}_2| = \sqrt{E_e^2 + E_2^2 + 2 E_2 E_e \cos\theta} \in \left[|E_e - E_2|,\ E_e + E_2 \right]$$

La conservazione dell'energia mi da:

$$|E_e - E_2| \leq E_1 = m_\mu - E_e - E_2 \leq E_e + E_2$$

$$|E_e - E_2| + E_e + E_2 \leq m_\mu \leq 2(E_e + E_2)$$

Il lato dx da il limite inf: $E_e \geq \frac{m_\mu}{2} - E_2$

Il lato sx " " sup per $E_2 = 0$ $\rightarrow$ $E_e \leq \frac{m_\mu}{2}$

Ripetendo lo stesso per E_2: $E_2 \leq \frac{m_\mu}{2}$

Sfruttandolo nel limite inf per E_2: $E_e \geq 0$

$$\Rightarrow E_e \in \left[0, \frac{m_\mu}{2}\right]$$

Integrando in ϵ_e:

$$\Gamma(\mu\to e\nu\nu) = \int_0^{\frac{m_\mu}{2}} \frac{d\Gamma}{d\epsilon_e}\, d\epsilon_e = \int_0^{\frac{m_\mu}{2}} \frac{G_F^2}{4\pi^3}\, m_\mu\, \epsilon_e^2 \left(m_\mu - \frac{4}{3}\epsilon_e\right) d\epsilon$$

$$\boxed{\Gamma(\mu^- \to e^- \bar{\nu}_e \nu_\mu) = \frac{G_F^2\, m_\mu^5}{192\,\pi^3} = \tau_\mu^{-1}}$$

Sfruttando l'ANALISI DIMENSIONALE:

$\Gamma \propto |M|^2 \propto G_F^2$

$[\Gamma] = 1 \quad , \quad [G_F^2] = -4 \quad \Rightarrow \quad \Gamma \propto \dfrac{G_F^2\, m_\mu^5}{16\,\pi^3}$

per compensare il -4 da G_F^2. È l'unica scala di E se $m_e = 0$.

$16\pi^3 \to$ spazio delle fasi generico in 3 part.

Dalla misura $\tau_\mu \simeq 2{,}2 \times 10^{-6}$ s e $m_\mu \simeq 105{,}7$ MeV $= 0{,}1057$ GeV

Fattore di conversione $1\ \text{GeV} \simeq 6{,}58 \times 10^{-25}\ s^{-1}$

$$G_F = \left(\frac{6{,}58 \times 10^{-25}}{\tau_\mu[s]} \; \frac{192\,\pi^3}{m_\mu^5[\text{GeV}]} \right)^{1/2} = 1{,}16 \times 10^{-5}\ \text{GeV}^{-2}$$