- Day 3

  Modern Control Theory:

  Optimal Control, MPC
  Elements of Robust and Nonlinear Control

Summary

- **<u>Optimal Control / LQR</u>**
- MPC
- Robust Control via SM Generation

(Nonlinear) Optimal Control

$$\dot{x} = f(x, u, t)$$
$$x \in \mathbb{R}^n, u \in \mathbb{R}^m$$
$$x(t_0) = x_0$$

- Minimization of cost function $J[u(t)]$ over time interval $[t_0, t_1]$

$$J[u(t)] = \underbrace{S(x(t_1), t_1)}_{\text{Final State Rating}} + \underbrace{\int_{t_0}^{t_1} L(x, u, t) dt}_{\text{Integral Cost}}$$

- Find solution $\underline{x} := \begin{bmatrix} x \\ u \end{bmatrix}$

LQR Control (finite time, discrete)

$$x(k+1) = Ax(k) + Bu(k), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

$$J = x(T)'Qx(T) + \sum_{k=0}^{T-1} [x(k)'Qx(k) + u(k)'Ru(k)], \quad Q, R > 0$$

- Solution

LQR Control (finite time, discrete)

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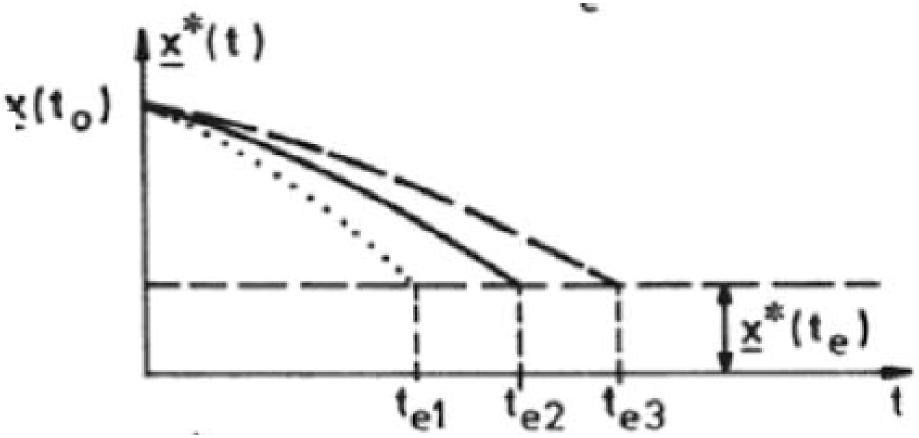
- Solution

$$u(k) = -K(k)x(k)$$

  - Depends on final time T



LQR Control (finite time, discrete)

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$$J = x(T)'Qx(T) + \sum_{k=0}^{T-1} [x(k)'Qx(k) + u(k)'Ru(k)], \quad Q, R > 0$$

$$\Rightarrow \quad J = \phi(A, B, Q, R, x_0, u_0, \dots, u_{T-1})$$

- Solution

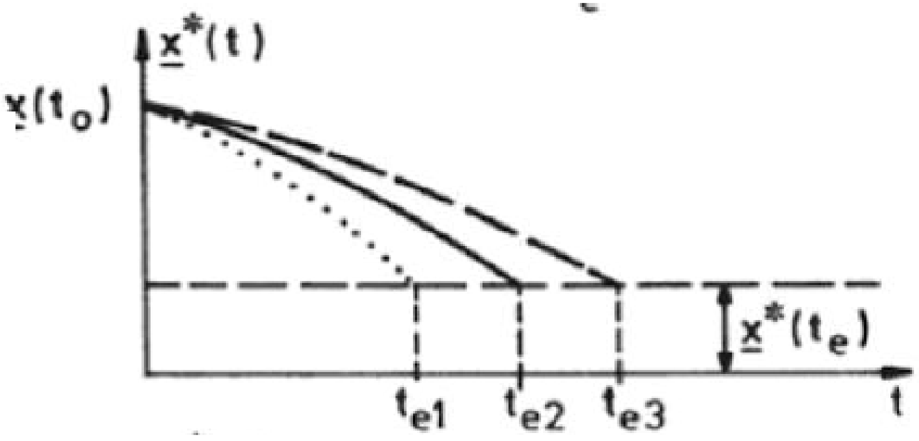
$$u(k) = -K(k)x(k)$$

  - Depends on final time T

$$P(T) = Q$$

$$K(k) = (R + B'P(k+1)B)^{-1}(B'P(k+1)A)$$

$$P(k-1) = A'P(k)A - (A'P(k)B)(R + B'P(k)B)^{-1}(B'P(k)A) + Q$$



LQR Control (finite time, discrete)

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$$J = x(T)'Qx(T) + \sum_{k=0}^{T-1} [x(k)'Qx(k) + u(k)'Ru(k)], \quad Q, R > 0$$

$$\Longrightarrow \quad J = \phi(A, B, Q, R, \cancel{x_0}, u_0, \ldots, u_{T-1})$$

- Solution

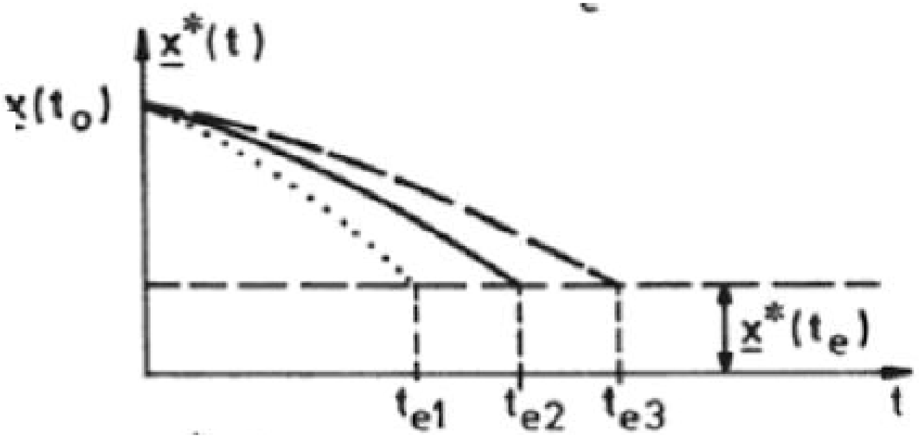
$$u(k) = -K(k)x(k)$$

  - Does not depend on initial condition!

$$P(T) = Q$$

$$K(k) = (R + B'P(k+1)B)^{-1}(B'P(k+1)A)$$

$$\boxed{P(k-1) = A'P(k)A - (A'P(k)B)(R + B'P(k)B)^{-1}(B'P(k)A) + Q}$$



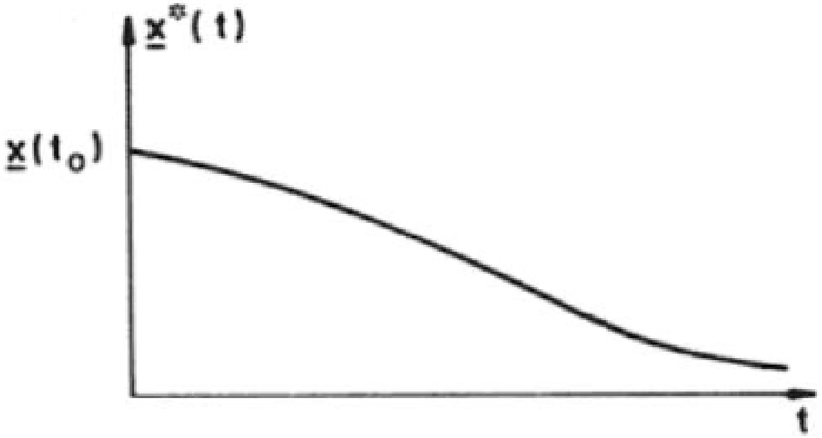
LQR Control (infinite time, discrete)

$$x(k+1) = Ax(k) + Bu(k), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

$$J = \sum_{k=0}^{\infty} [x(k)'Qx(k) + u(k)'Ru(k)], \quad Q, R > 0$$

- Solution

$$u(k) = -Kx(k)$$



$$K = (R + B'PB)^{-1}(B'PA)$$

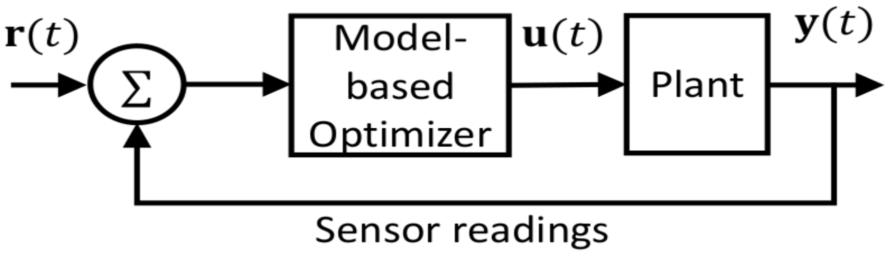
$$\boxed{P = A'PA - (A'PB)(R + B'PB)^{-1}(B'PA) + Q \qquad \text{ARE}}$$

Summary

- Optimal Control / LQR
- **MPC**
- Robust Control via SM Generation

Model Predictive Control

- Main idea: Use a dynamical model of the plant (inside the controller) to predict the plant’s future evolution, and optimize the control signal over possible futures



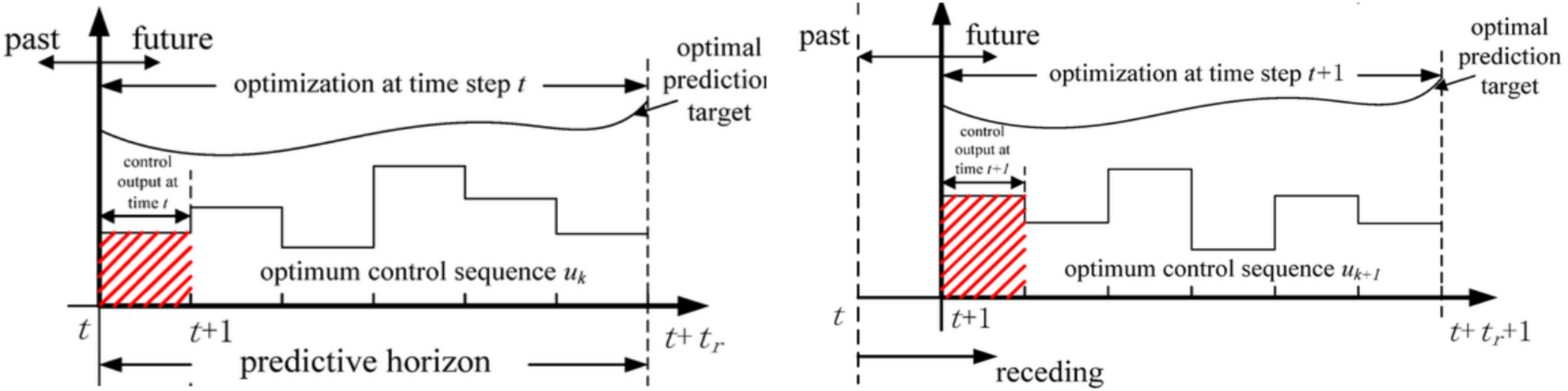


Image from: https://tinyurl.com/yaej43x5

Why MPC?

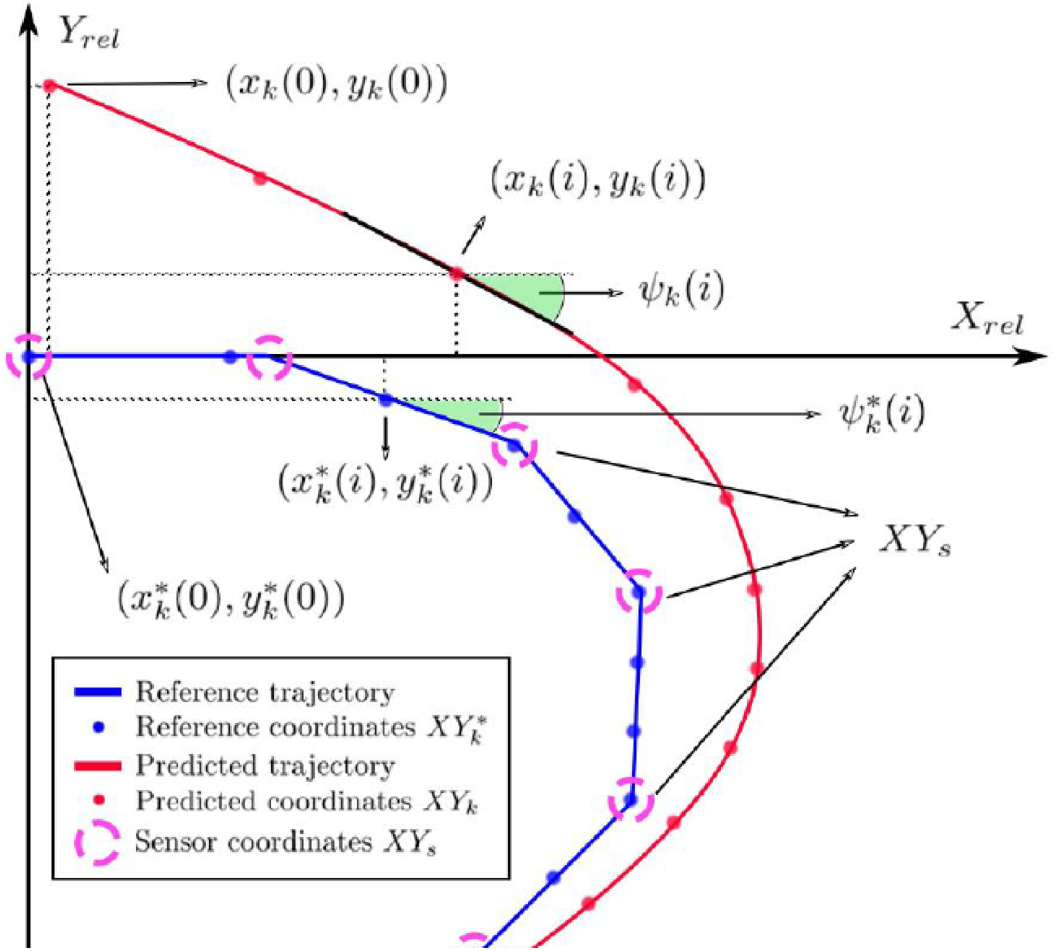
- Optimal control with constraints (input, output and states)
- ideal for MIMO (Multi Input Multi Output) systems
- linear and nonlinear models

- RECEDING HORIZON PRINCIPLE

"At any time instant k, based on the available process information, solve the optimization problem with respect to the future control sequence $[u(k), ..., u(k+N-1)]$ and apply only its first element $u^o(k)$. Then, at next time instant $k+1$, a new optimization problem is solved, based on the process information available at time $k+1$, along the prediction horizon $[k+1, k+N]$." (Camacho)

Receding Horizon Principle

- Closed Loop solution (no constraints, LQR)
- Open Loop solution (constraints)



$$J(x(k),u(\cdot),k) = \sum_{i=0}^{N-1} \left(||x(k+i)||_Q^2 + ||u(k+i)||_R^2 \right) + ||x(k+N)||_S^2$$

Linear MPC (1)

$$x(k+1) = Ax(k) + Bu(k), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

$$x(k+i) = A^i x(k) + \sum_{j=0}^{i-1} A^{i-j-1} Bu(k+j), \quad i > 0$$

$$X(k) = \mathcal{A}x(k) + \mathcal{B}U(k) \quad \Rightarrow \quad A\underline{x} = b$$

$$X(k) = \begin{bmatrix} x(k+1) \\ x(k+2) \\ \vdots \\ x(k+N-1) \\ x(k+N) \end{bmatrix}, \quad U(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N-2) \\ u(k+N-1) \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^{N-1} \\ A^N \end{bmatrix},$$

Linear MPC (2)

$$x(k+1) = Ax(k) + Bu(k), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

$$x(k+i) = A^i x(k) + \sum_{j=0}^{i-1} A^{i-j-1} Bu(k+j), \quad i > 0$$

$$X(k) = \mathcal{A}x(k) + \mathcal{B}U(k) \qquad \Rightarrow \qquad A\underline{x} = b$$

$$\mathcal{B} = \begin{bmatrix} B & 0 & 0 & \cdots & 0 & 0 \\ AB & B & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A^{N-2}B & A^{N-3}B & A^{N-4}B & \cdots & B & 0 \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \cdots & AB & B \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} I^{(nN)} \;, & -\mathcal{B} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} X(k) \\ U(k) \end{bmatrix}}_{\underline{x}} = \underbrace{\mathcal{A}x(k)}_{b}$$

(Non-)Linear MPC

$$s = [x, u, \Delta u]^T$$

$$J_{MPC} = \sum_{i=1}^{N} (||x(i) - x^*(i)||_Q^2 + ||u(i) - u^*(i)||_R^2 + ||\Delta u(i) - \Delta u^*(i)||_{\Delta R}^2)$$

- Linear formulation:

$$\begin{aligned} \underset{s}{\text{minimize}} \quad & J_{MPC}(s) \\ \text{subject to} \quad & A_{eq}s = b_{eq}, \\ & A_{ineq}s \leq b_{ineq} \end{aligned}$$

- Nonlinear formulation:

$$\begin{aligned} \underset{x,u}{\text{minimize}} \quad & J_{MPC}(x, u) \\ \text{subject to} \quad & \\ & x(k+1) = f(x(k), u(k)), \\ & h(x(k), u(k)) \leq 0 \end{aligned}$$

Issues with MPC

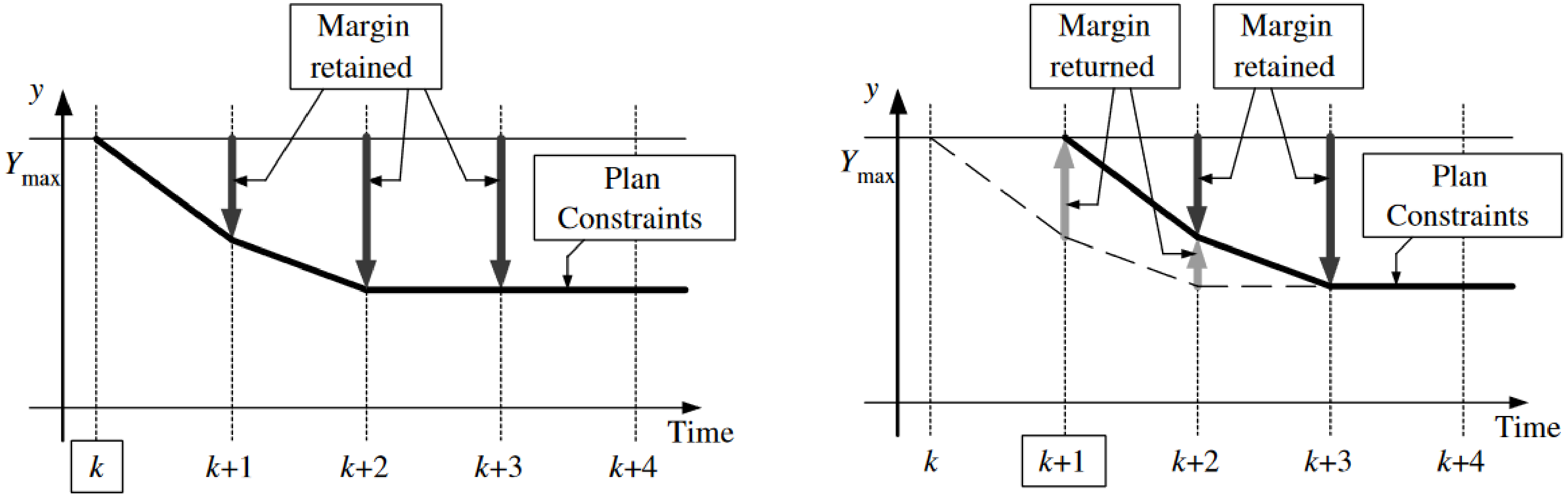
- Feasibility
- Stability
- Computation

Conflicting Requirements

(several solutions depending on needs)

- Robustness formulation: system affected by process and measurement noise

Constraints tightening



Summary

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- MPC
- **Robust Control via SM Generation**

Sliding Mode Control (dummy case)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x) + g(x)u \end{cases}$$

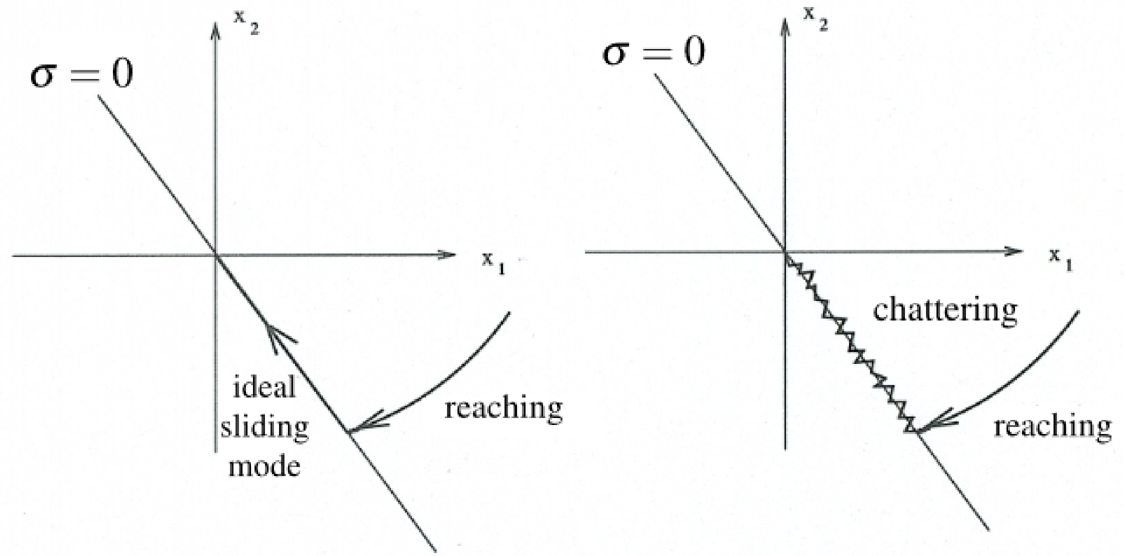
$$0 < G \leq g(x), \quad |f(x)| \leq F < \infty$$

- Surface $\sigma(x) = \alpha x_1 + x_2, \quad \alpha > 0$
- Manifold $\sigma(x) = 0, \quad \text{dimension} \quad m - n$
- Lyapunov stability condition

$$V = \frac{1}{2}\sigma^2 \Rightarrow \dot{V} = \sigma\dot{\sigma}$$

- Control Law satisfying Lyapunov:

$$u = -Ksign(\sigma(x)), \quad K > \frac{F + \alpha|x_2|}{G}$$



Integral Sliding Mode – Robustifying (Matched Disturbances)

- ISM:

$$\dot{x} = f(x) + g(x)u$$

$$u(t) = u_0(t) + u_{1,eq}(t)$$

$$u_{1,eq}(t) = LP(s) \cdot u_1(t)$$

$$u_1(t) = -sign(\sigma_0(x) + z)$$

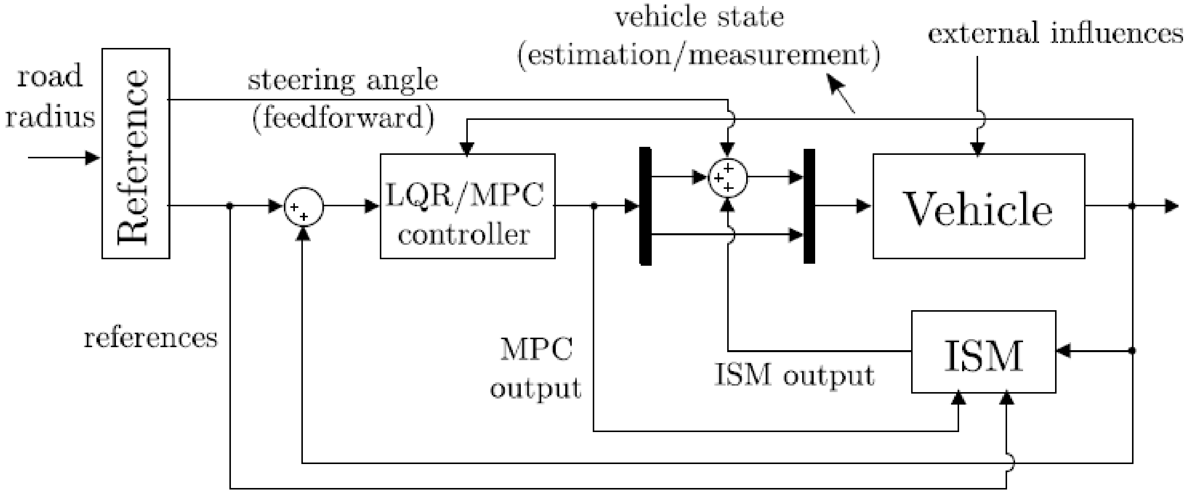
row

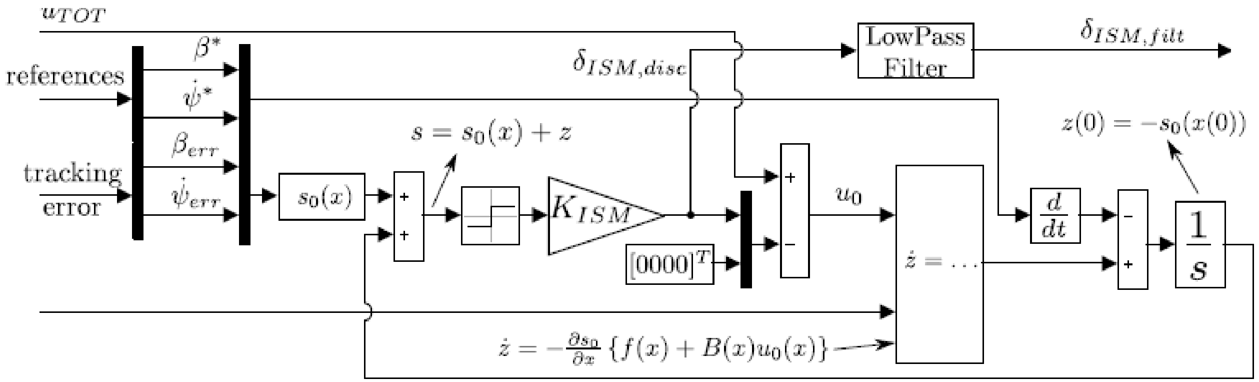
scalar

$$\dot{z} = -\frac{\partial \sigma_0}{\partial x}\{f(x) + g(x)(u - u_1)\}$$

column

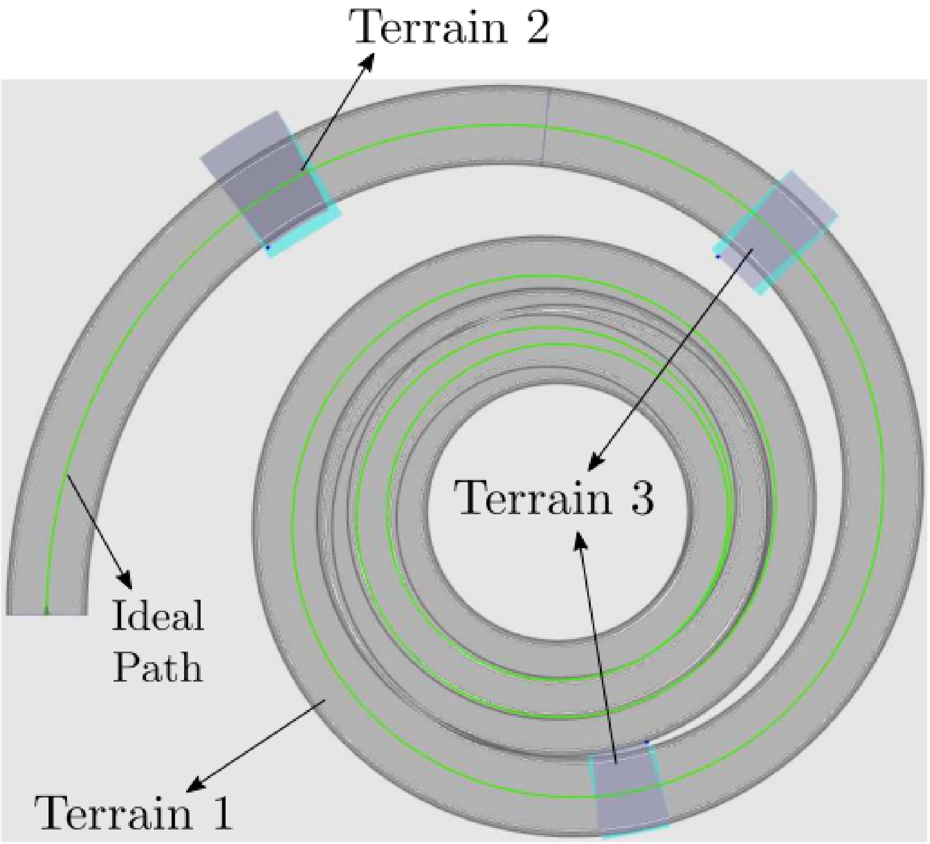
$$z(0) = -\sigma_0(x(0))$$

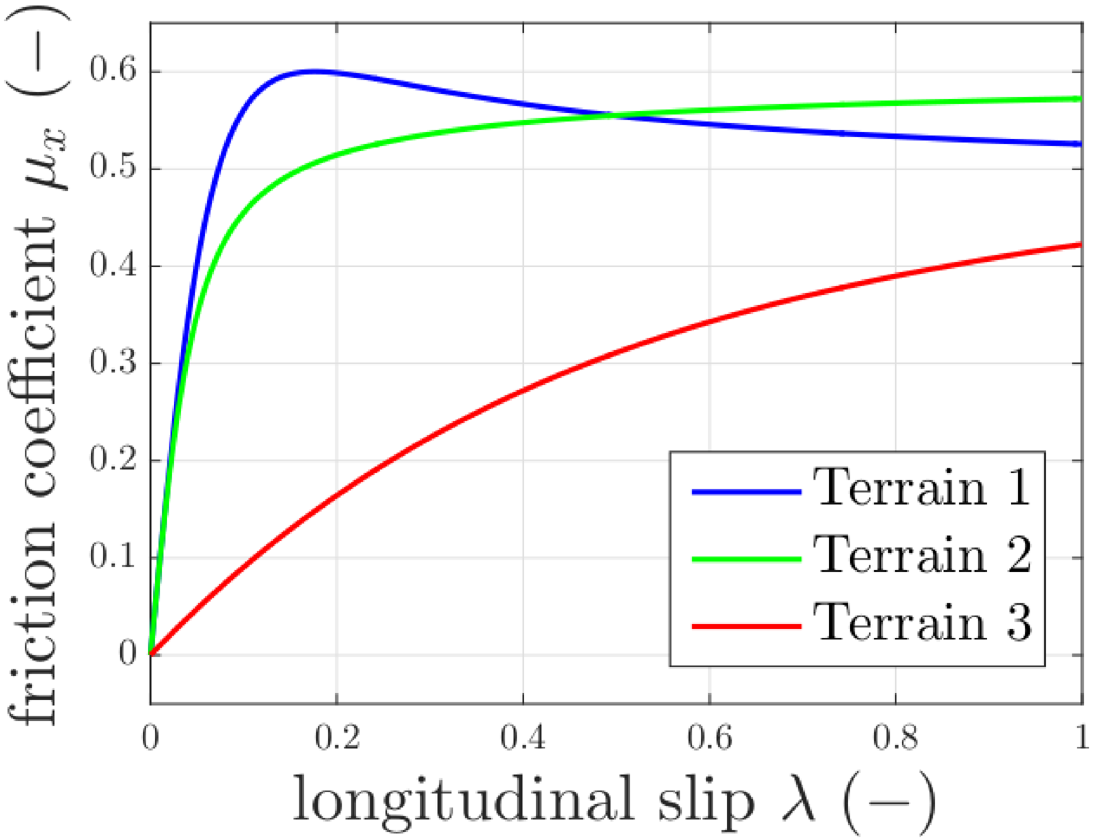




Control schemes from:
E. Regolin, M. Zambelli, A. Ferrara, "A multi-rate ISM approach for robust vehicle stability control during cornering", Proceedings of the 15th IFAC Symposium on Control in Transportation Systems (CTS 2018), 6-8 June, Savona, Italy.

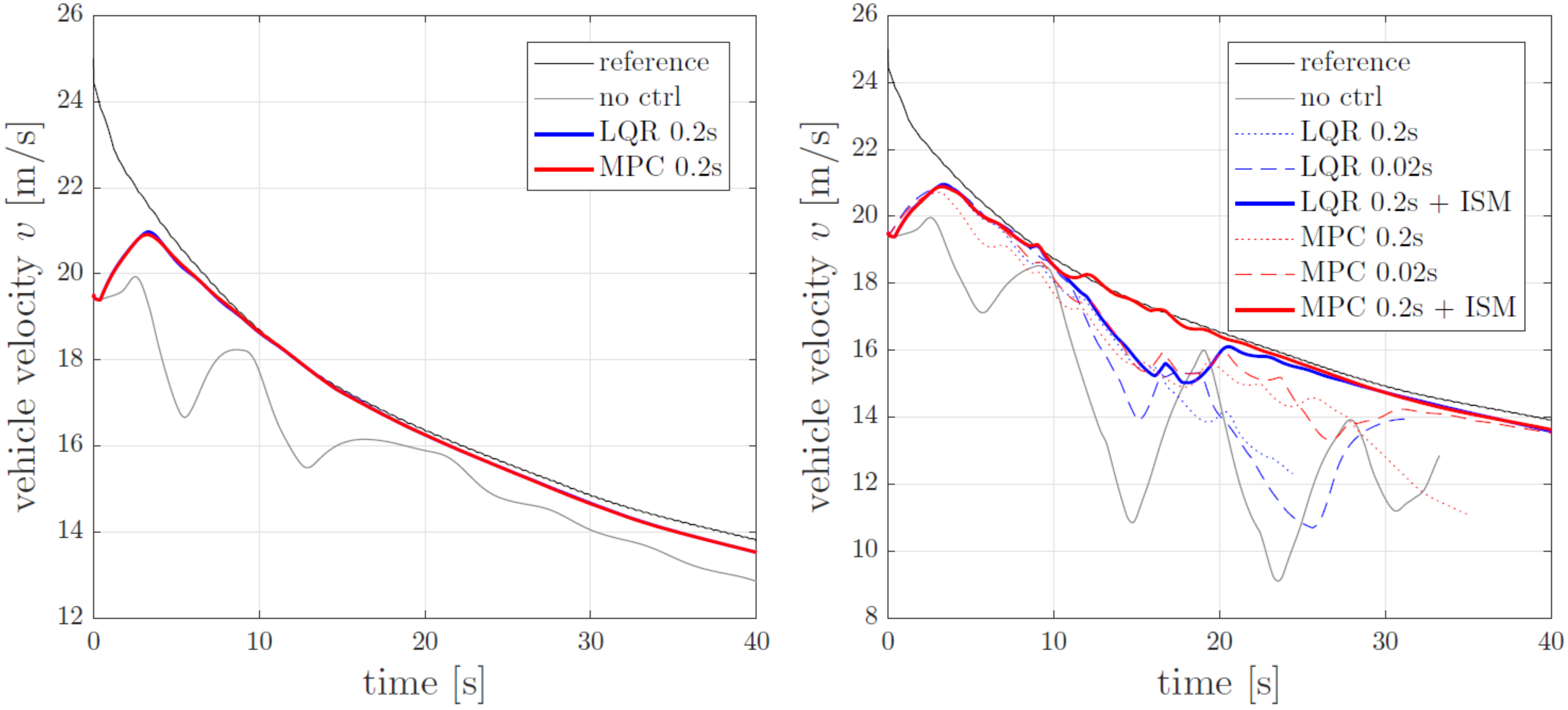
ISM Effect: Constant Steering Example



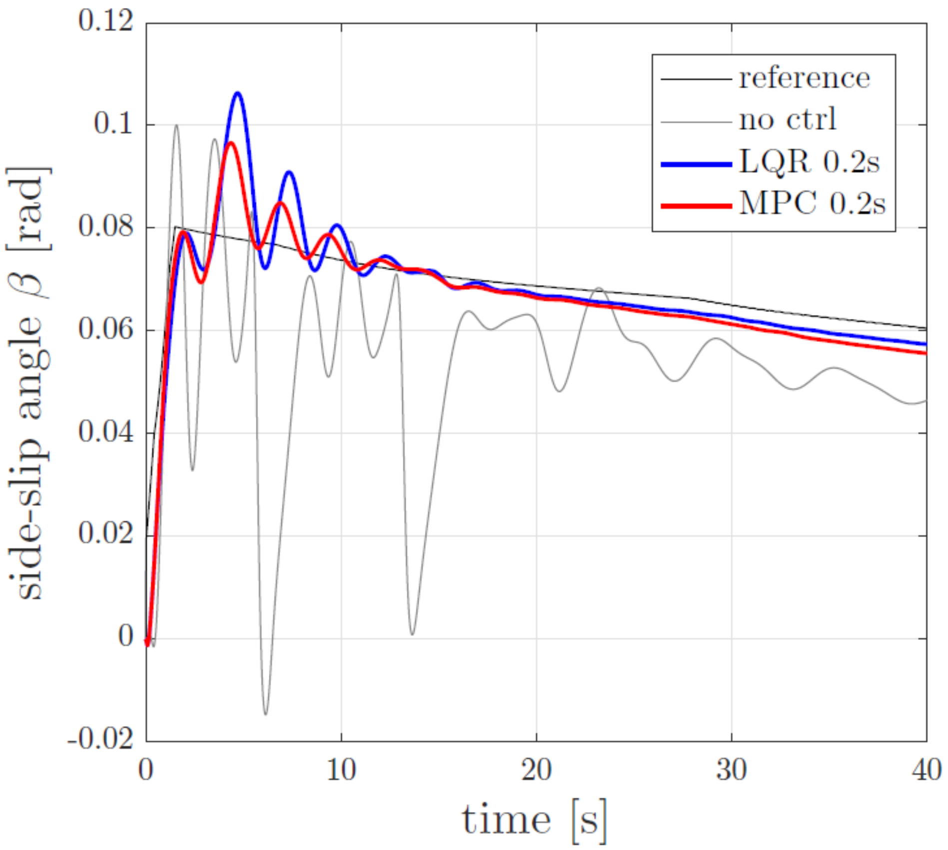


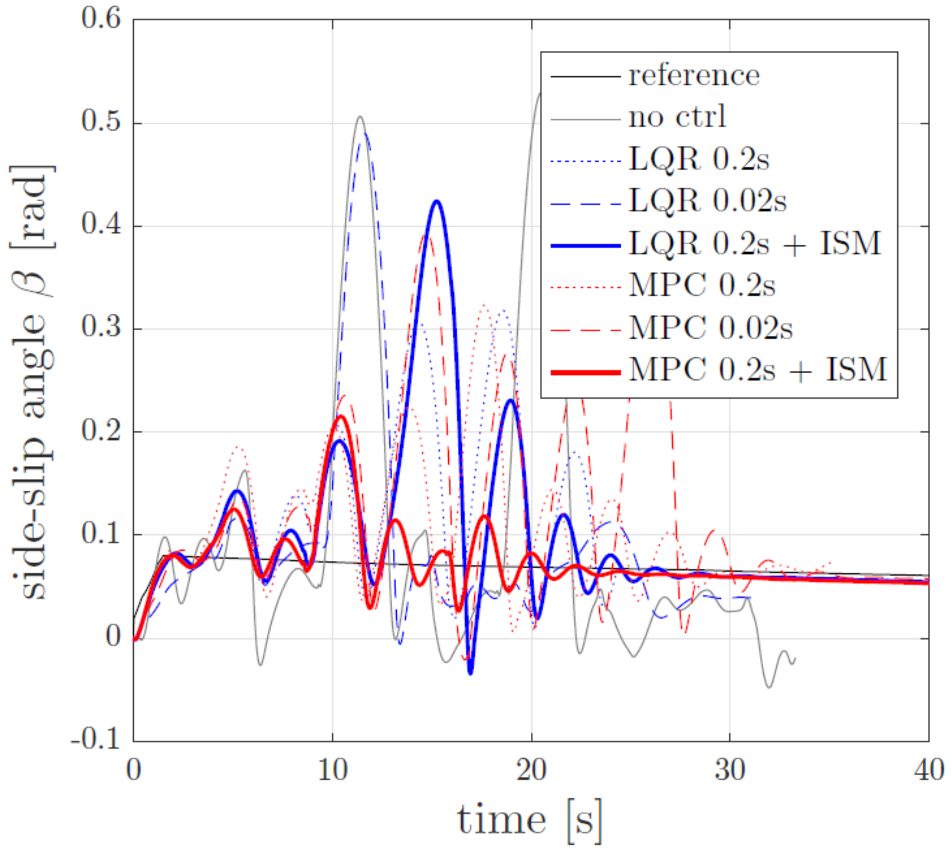
Zambelli, Massimo, and Antonella Ferrara. "Robustified distributed model predictive control for coherence and energy efficiency-aware platooning." *2019 American Control Conference (ACC)*. IEEE, 2019.

ISM Effect : Results (1)

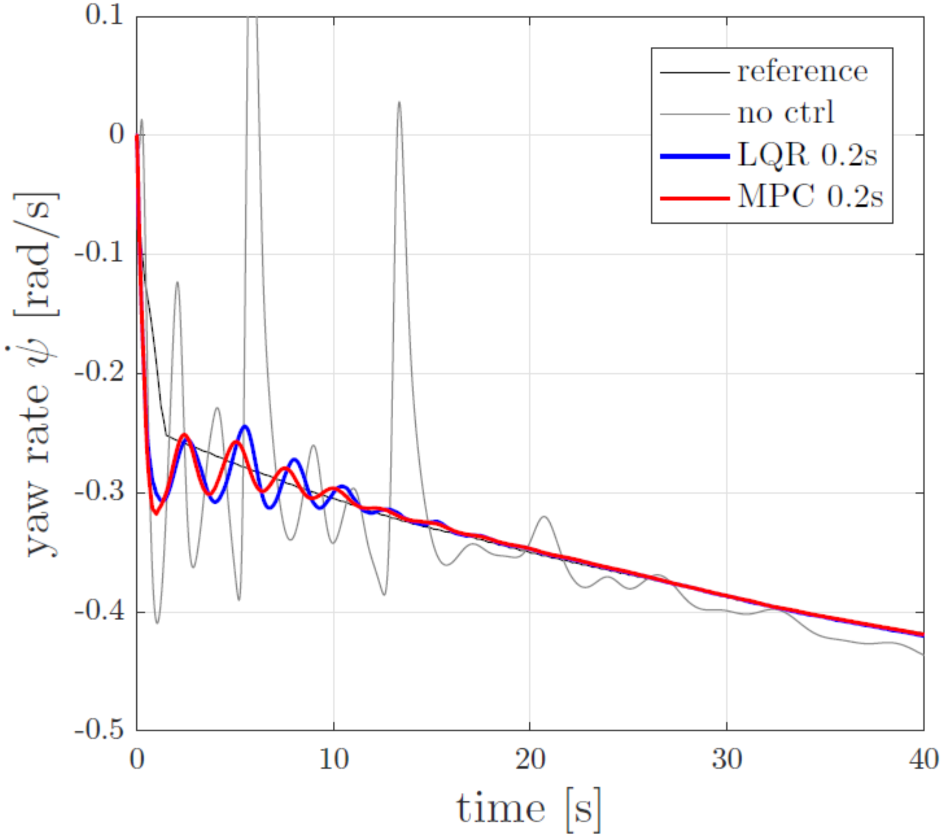


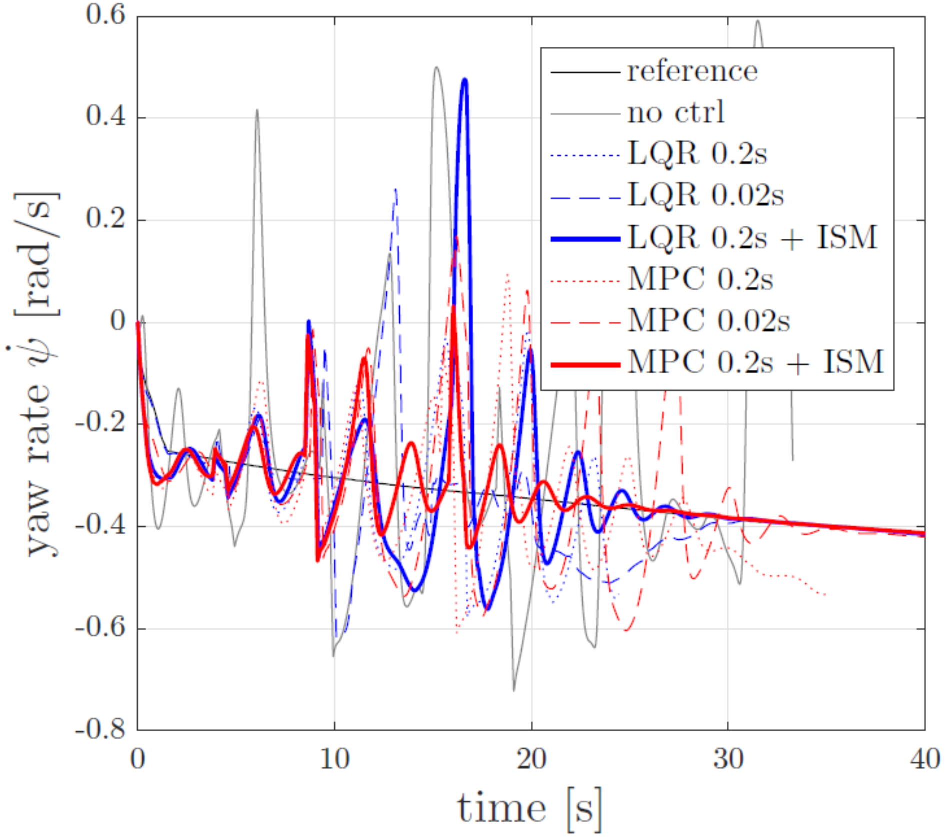
ISM Effect : Results (2)



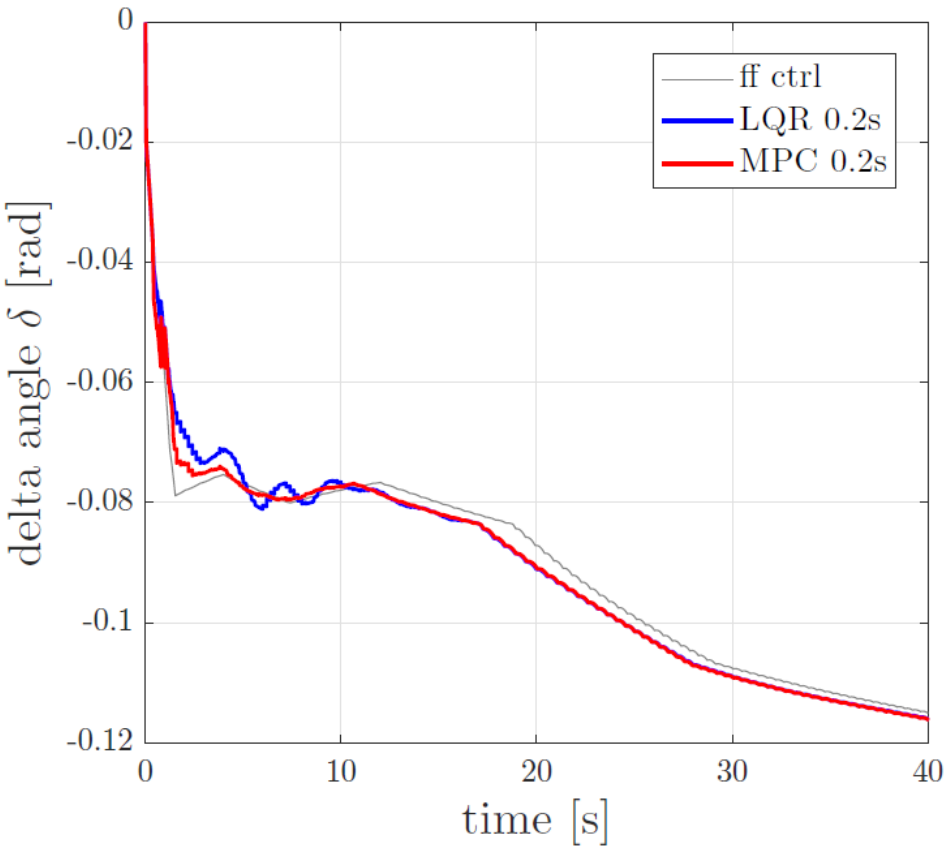


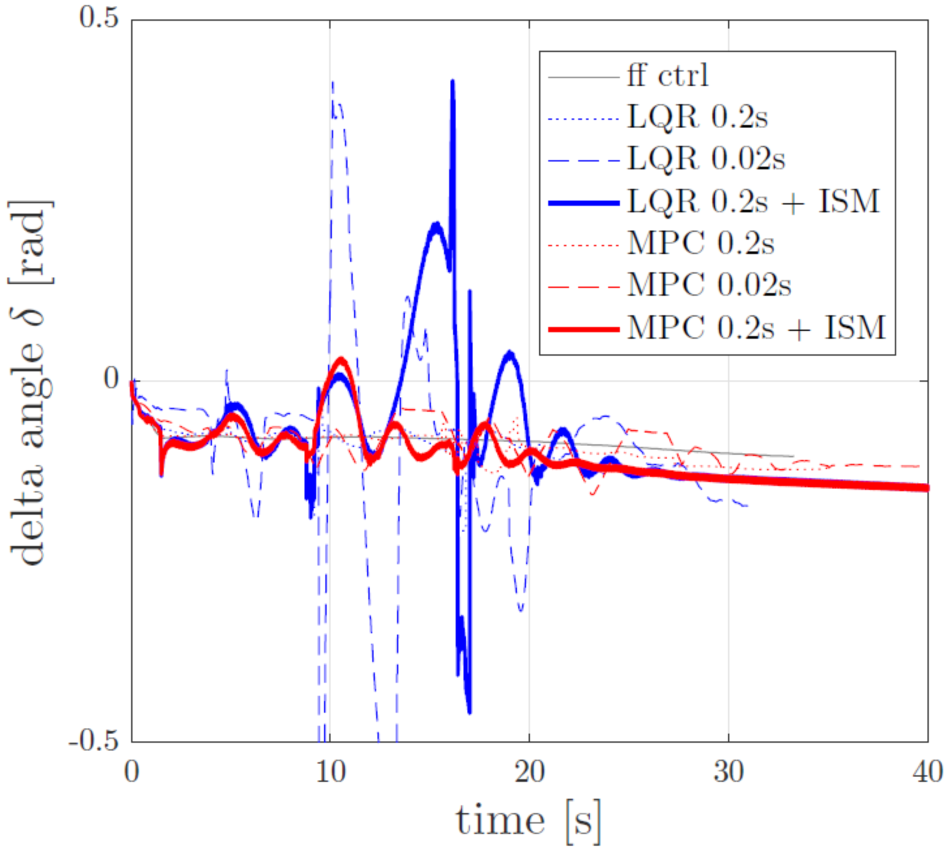
ISM Effect : Results (3)





ISM Effect : Results (4)





ISM Effect : Results (5)