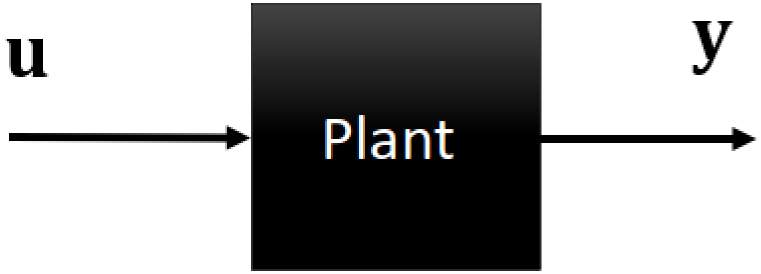
Day 4

Kalman Filtering

Control Laboratory

What is state estimation?



- Given a “black box” component, we can try to use a linear or nonlinear system to model it (maybe based on physics, or data-driven)
- Model may posit that the plant has internal states, but we typically have access only to the outputs of the model (whatever we can measure using a sensor)
- May need internal states to implement controller: how do we estimate them?
- State estimation: Problem of determining internal states of the plant

Deterministic vs. Noisy case

- Typically sensor measurements are noisy (manufacturing imperfections, environment uncertainty, errors introduced in signal processing, etc.)
- In the absence of noise, the model is deterministic: for the same input you always get the same output
- Can use a simpler form of state estimator called an observer (e.g. a Luenberger observer)
  - $\frac{d\hat{\mathbf{x}}}{dt} = A\hat{\mathbf{x}} + B\mathbf{u} + L(\mathbf{y} - \hat{\mathbf{y}})$
  - $\hat{\mathbf{y}} = C\hat{\mathbf{x}} + D\mathbf{u}$
  - $\mathbf{u}(t) = -K_{\mathrm{lqr}}\hat{\mathbf{x}}(t),$

$$\Longleftrightarrow \quad \dot{e} = (A - LC)e$$

- In the presence of noise, we use a state estimator, such as a Kalman Filter
- Kalman Filter is one of the most fundamental algorithm that you will see in autonomous systems, robotics, computer graphics, …

Random variables and statistics refresher

- For random variable w, $\mathbb{E}[w]$: expected value of w, also known as mean
- Suppose $\mathbb{E}[x] = \mu$: then var(w) : variance of w, is $\mathbb{E}[(w - \mu)^2]$
- For random variables x and y, $\text{cov}(x, y)$: covariance of x and y
  - $\text{cov}(x, y) = \mathbb{E}[(x - \mathbb{E}(x)(y - \mathbb{E}(y)]$
- For random ***vector*** $\mathbf{x}$, $\mathbb{E}[\mathbf{x}]$is a vector
- For random vectors, $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$, cross-covariance matrix is $m \times n$ matrix: $\text{cov}(\mathbf{x}, \mathbf{y}) = \mathbb{E}[(\mathbf{x} - \mathbb{E}[\mathbf{x}])(\mathbf{y} - \mathbb{E}[\mathbf{y}])^{\text{T}}]$
- $w \sim N(\mu, \sigma^2)$: w is a normally distributed variable with mean μ and variance σ

Multi-variate sensor fusion

- Instead of estimating one quantity, we want to estimate n quantities, then:
- Actual value is some vector $\mathbf{x}$
- Measurement noise for i^{th} sensor is $v_i \sim N(\boldsymbol{\mu}_i, \Sigma_i)$, where $\boldsymbol{\mu}_i$ is the mean vector, and Σ_i is the covariance matrix
- $\Lambda = \Sigma^{-1}$ is the information matrix
- For the two-sensor case:
  - $\hat{\mathbf{x}} = (\Lambda_1 + \Lambda_2)^{-1}(\Lambda_1 \mathbf{z}_1 + \Lambda_2 \mathbf{z}_2)$, and $\hat{\Sigma} = (\Lambda_1 + \Lambda_2)^{-1}$

Motion makes things interesting

- What if we have one sensor and making repeated measurements of a moving object?

- Measurement differences are not all because of sensor noise, some of it is because of object motion

- Kalman filter is a tool that can include a motion model (or in general a dynamical model) to account for changes in internal state of the system

- Combines idea of prediction using the system dynamics with correction using weighted average (Bayesian inference)

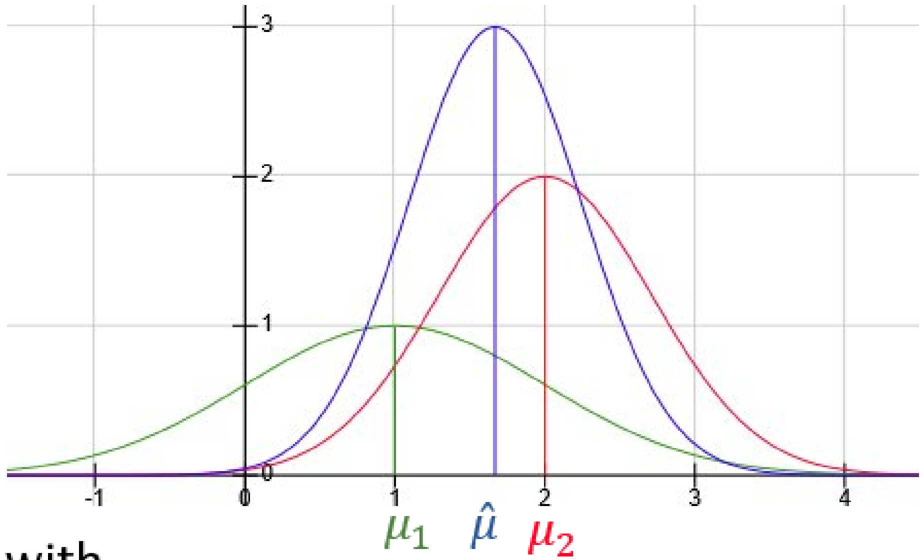
Data fusion example

- Using radar and a camera to estimate the distance to the lead car:
  - Measurement is never free of noise
  - Actual distance: x
  - Measurement with radar: $z_1 = x + v_1$ ($v_1 \sim N(\mu_1, \sigma_1^2)$ is radar noise)
  - With camera: $z_2 = x + v_2$ ($v_2 \sim N(\mu_2, \sigma_2^2)$ is camera noise)
  - How do you combine the two estimates?
- Use a weighted average of the two estimates, prioritize more likely measurement
  - $\hat{x} = \frac{(z_1/\sigma_1^2) + (z_2/\sigma_2^2)}{(1/\sigma_1^2) + (1/\sigma_2^2)} = kz_1 + (1-k)z_2$, where $k = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$
  - $\hat{\sigma}^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$ $\quad \hat{\mu} = k\mu_1 + (1-k)\mu_2$
- Observe: uncertainty reduced, and mean is closer to measurement with lower uncertainty

$\mu_1 = 1, \sigma_1^2 = 1$

$\mu_2 = 2, \sigma_2^2 = 0.5$

$\hat{\mu} = 1.67, \sigma^2 = 0.33$



Stochastic Difference Equation Models

- We assume that the plant (whose state we are trying to estimate) is a stochastic discrete dynamical process:

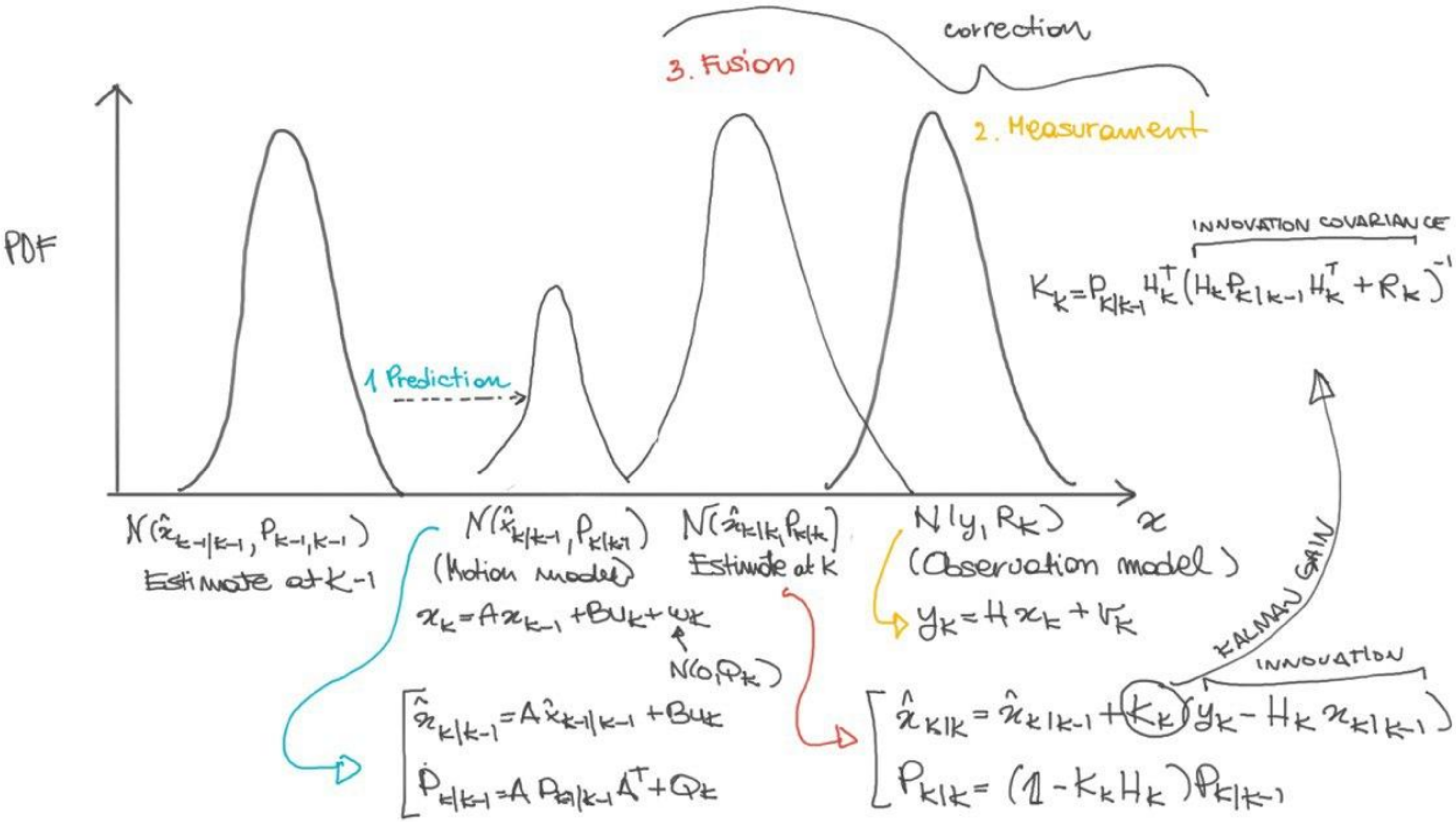
$$\mathbf{x}_k = A\mathbf{x}_{k-1} + B\mathbf{u}_k + \mathbf{w}_k \text{ (Process Model)}$$

$$\boldsymbol{y}_k = H\mathbf{x}_k + \mathbf{v}_k \text{ (Measurement Model)}$$

$\mathbf{x}_k, \mathbf{x}_{k-1}$	State at time k,$k-1$
$\mathbf{u}_k$	Input at time k
$\mathbf{w}_k$	Random vector representing noise in the plant, $\mathbf{w} \sim N(\mathbf{0}, Q_k)$
$\mathbf{v}_k$	Random vector representing sensor noise, $\mathbf{v} \sim N(\mathbf{0}, R_k)$
$\mathbf{z}_k$	Output at time k

n	Number of states
m	Number of inputs
p	Number of outputs
A	$n \times n$ matrix
B	$n \times m$ matrix
H	$p \times n$ matrix

Kalman Filter



Step I: Prediction

- We assume an estimate of $\mathbf{x}$ at time $k-1$, fusing information obtained by measurements till time $k-1$: this is denoted $\hat{\mathbf{x}}_{k-1|k-1}$
- We also assume that the error between the estimate $\hat{\mathbf{x}}_{k-1|k-1}$ and the actual $\mathbf{x}_{k-1}$ has 0 mean, and covariance $P_{k-1|k-1}$
- Now, we use these values and the state dynamics to predict the value of $\mathbf{x}_k$
- Because we are using measurements only up to time $k-1$, we can denote this predicted value as $\hat{\mathbf{x}}_{k|k-1}$, and compute it as follows:

$$\hat{\mathbf{x}}_{k|k-1} := A\hat{\mathbf{x}}_{k-1|k-1} + B\mathbf{u}_k$$

Step I: Prediction

$$\begin{aligned} P_{k|k-1} &= \mathrm{cov}\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}\right) = \mathrm{cov}\left(A\mathbf{x}_{k-1} + B\mathbf{u}_k + w_k - A\hat{\mathbf{x}}_{k-1|k-1} - B\mathbf{u}_k\right) \\ &= A\mathrm{cov}\left(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1}\right)A^T + cov(w_k) \\ &= AP_{k-1|k-1}A^T + Q_k \end{aligned}$$

- Thus, the state and error covariance prediction are:

$$\begin{aligned} \hat{\mathbf{x}}_{k|k-1} &:= A\hat{\mathbf{x}}_{k-1|k-1} + B\mathbf{u}_k \\ P_{k|k-1} &:= AP_{k-1|k-1}A^T + Q_k \end{aligned}$$

Step II: Correction

- This is where we basically do data fusion between new measurement and old prediction to obtain new estimate
- Note that data fusion is not straightforward like before because we don't really observe/measure $\mathbf{x}_k$ directly, but we get measurement $\boldsymbol{y}_k$, for an observable output!
- Idea remains similar: Do a weighted average of the prediction $\hat{\mathbf{x}}_{k|k-1}$ and new information
- We integrate new information by using the difference between the predicted output and the observation

Step II: Correction

- Predicted output: $\hat{\boldsymbol{y}}_k = H_k \hat{\mathbf{x}}_{k|k-1}$
- We denote the error in predicted output as the *innovation*
$$\mathbf{z}_k := \mathbf{y}_k - H_k \hat{\mathbf{x}}_{k|k-1}$$
- Covariance of innovation
$$S_k = \mathrm{cov}\,(\mathbf{z}_k) = cov(H_k \mathbf{x}_k + \mathbf{v}_k - H_k \hat{\mathbf{x}}_{k|k-1}) = R_k + H_k P_{k|k-1} H_k^T$$
- Then to do data fusion is given by:
$$\hat{\boldsymbol{x}}_{k|k} := \hat{\boldsymbol{x}}_{k|k-1} + K_k z_k$$
- Where, K_k = $P_{k|k-1} H_k^T S_k^{-1}$ is the (optimal) Kalman gain. It minimizes the least square error
- Finally, the updated error covariance estimate is given by:
$$P_{k|k} = (I - K_k H_k) P_{k|k-1}$$

Step II: Correction

Innovation	$\mathbf{z}_k := \mathbf{y}_k - H_k \hat{\mathbf{x}}_{k\|k-1}$
Innovation Covariance	$S_k := R_k + H_k P_{k\|k-1} H_k^T$
Optimal Kalman Gain	$K_k := P_{k\|k-1} H_k^T S_k^{-1}$
State estimate at time k	$\hat{\boldsymbol{x}}_{k\|k} := \hat{\boldsymbol{x}}_{k\|k-1} + K_k \mathbf{z}_k$
Covariance estimate at time k	$P_{k\|k} = (I - K_k H_k) P_{k\|k-1}$

one-dimensional example

- Let's take a simple one-dimensional example
- Kalman filter prediction equations become:
  - $\hat{x}_{k|k-1} := a\hat{x}_{k-1|k-1} + bu$; $\sigma^2_{k|k-1} := \underbrace{a^2\sigma^2_{k-1|k-1}}_{\text{prior uncertainty in estimate}} + \underbrace{\sigma^2_q}_{\text{uncertainty in process}}$
- Also, the correction equations become:
  - Innovation: $z_k := y_k - \hat{x}_{k|k-1}$, $\mathrm{S_k} = \sigma^2_r + \sigma^2_{k|k-1}$
  - Optimal gain: $k = \sigma^2_{k|k-1} / (\sigma^2_r + \sigma^2_{k|k-1})$,
  - Updated state estimate: $\hat{x}_{k|k} := \hat{x}_{k|k-1} + k(y_k - \hat{x}_{k|k-1})$
  - I.e. updated state estimate: $\hat{x}_{k|k} := (1-k)\ \hat{x}_{k|k-1} + ky_k$ (Weighted average!)

Extended Kalman Filter

- We skipped derivations of equations of the Kalman filter, but a fundamental property assumed is that the process model and measurement model are both linear.
- Under linear models and Gaussian process/measurement noise, a Kalman filter is an *optimal* state estimator (minimizes mean square error between estimate and actual state)
- In an EKF, state transitions and observations need not be linear functions of the state, but can be any differentiable functions
- I.e., the process and measurement models are as follows:

$$\mathbf{x}_k = f(x_{k-1}, u_k) + w_k$$
$$y_k = h(x_k) + v_k$$

EKF updates

- Functions f and h can be used directly to compute state-prediction, and predicted measurement, but cannot be directly used to update covariances
- So, we instead use the Jacobian of the dynamics at the predicted state
- This linearizes the non-linear dynamics around the current estimate
- Prediction updates:

$$\hat{\mathbf{x}}_{k|k-1} := f(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_k)$$
$$P_{k|k-1} := F_k P_{k-1|k-1} F_k^T + Q_k$$

$$F_k := \left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}_{k|k-1}, \mathbf{u}=\mathbf{u}_k}$$

EKF updates

- Correction updates:

$$H_k := \left.\frac{\partial h}{\partial \mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}_{k|k-1}}$$

Innovation	$\mathbf{z}_k := \mathbf{y}_k - h(\hat{\mathbf{x}}_{k\|k-1})$
Innovation Covariance	$S_k := R_k + H_k P_{k\|k-1} H_k^T$
Near-Optimal Kalman Gain	$K_k := P_{k\|k-1} H_k^T S_k^{-1}$
A posteriori state estimate	$\hat{\boldsymbol{x}}_{k\|k} := \hat{\boldsymbol{x}}_{k\|k-1} + K_k \mathbf{y}_k$
A posteriori error covariance estimate	$P_{k\|k} = (I - K_k H_k) P_{k\|k-1}$