Amortised Analysis

Chapters 17.1-17.2 of Cormen's book

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Algorithmic Design
a.y. 2022/2023

Amortised analysis

In an amortised analysis, we average the time required to perform a sequence of operations over all the operations performed.

With amortised analysis, we can show that the average cost of an operation over a sequence of operations is small, even though a single operation within the sequence might be expensive.

Probability is not involved; an amortised analysis guarantees the average performance of each operation in the worst case.

Graphs

Chapter 22 of Cormen's book

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BFS: Pseudocode

BFS(G,s) - G is represented by the adjacency lists $\mathrm{Adj}[\cdot]$ of its vertices

for each $u \in V \setminus \{s\}$
    $u.color \leftarrow \mathrm{white};$
    $u.distance \leftarrow \infty;$
$s.color \leftarrow \mathrm{gray};$
$s.distance \leftarrow 0;$
$Q \leftarrow \emptyset;$
enqueue(Q,s);
while $Q \neq \emptyset$
    $u \leftarrow$ dequeue(Q);
    for each $v \in \mathrm{Adj}[u]$
        if $v.color = \mathrm{white}$
            $v.color \leftarrow \mathrm{gray};$
            $v.distance \leftarrow u.distance + 1;$
            enqueue(Q,v);
    $u.color \leftarrow \mathrm{black};$

Initialisation

Visit of the source

Visit of the other vertices

BFS: Complexity

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    $u.color \leftarrow$ black;

$O(1)$

$O(|\mathrm{Adj}[u]|)$

Initialisation: $O(|V|)$

Visit of the source: $O(1)$

Visit of the other vertices: each iteration of the **for** loop enqueues $v \in \mathrm{Adj}[u]$ only if it is white, and it immediately turns its color to gray $\Longrightarrow$ each vertex is inserted in Q at most once. Cost of the **while** loop:

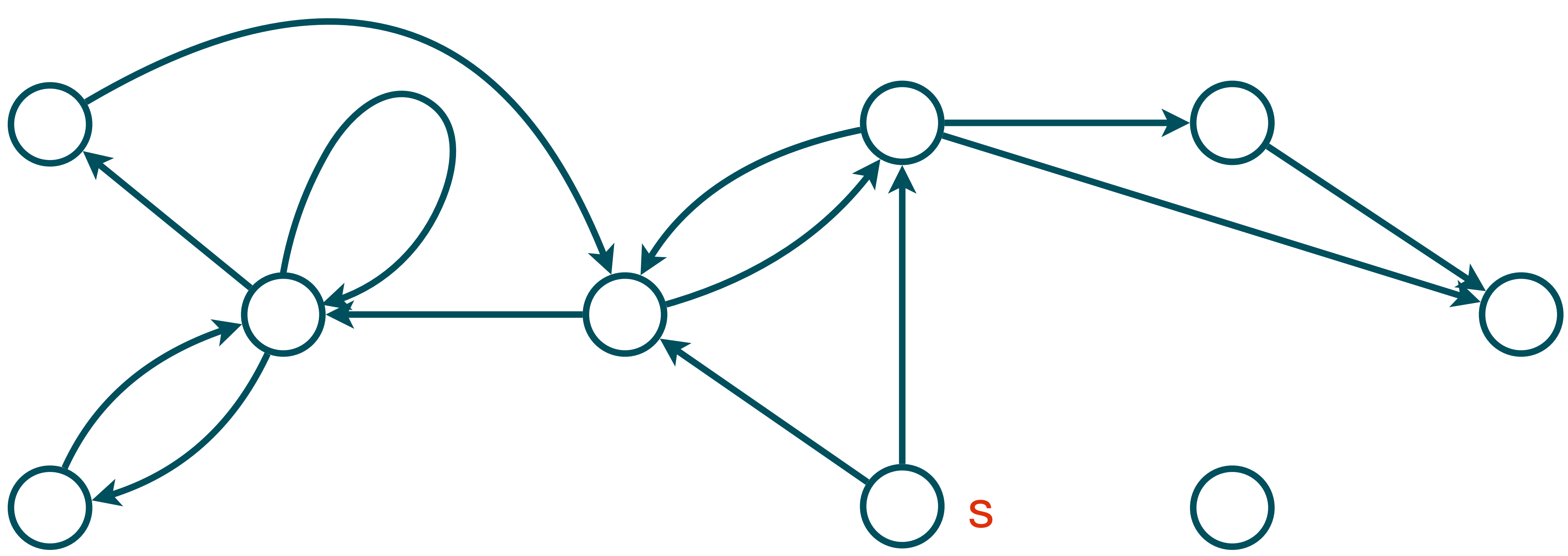
$$O\left(\sum_{u \in V} |\mathrm{Adj}[u]|\right) = O(|E|)$$

Breadth-First Search

The visiting order is related to the distance from a source node: the closer a node to the source, the sooner it will be visited

BFS produces a breadth-first tree: the tree consisting of the shortest paths from the source to any reachable node

White nodes have not been discovered yet;



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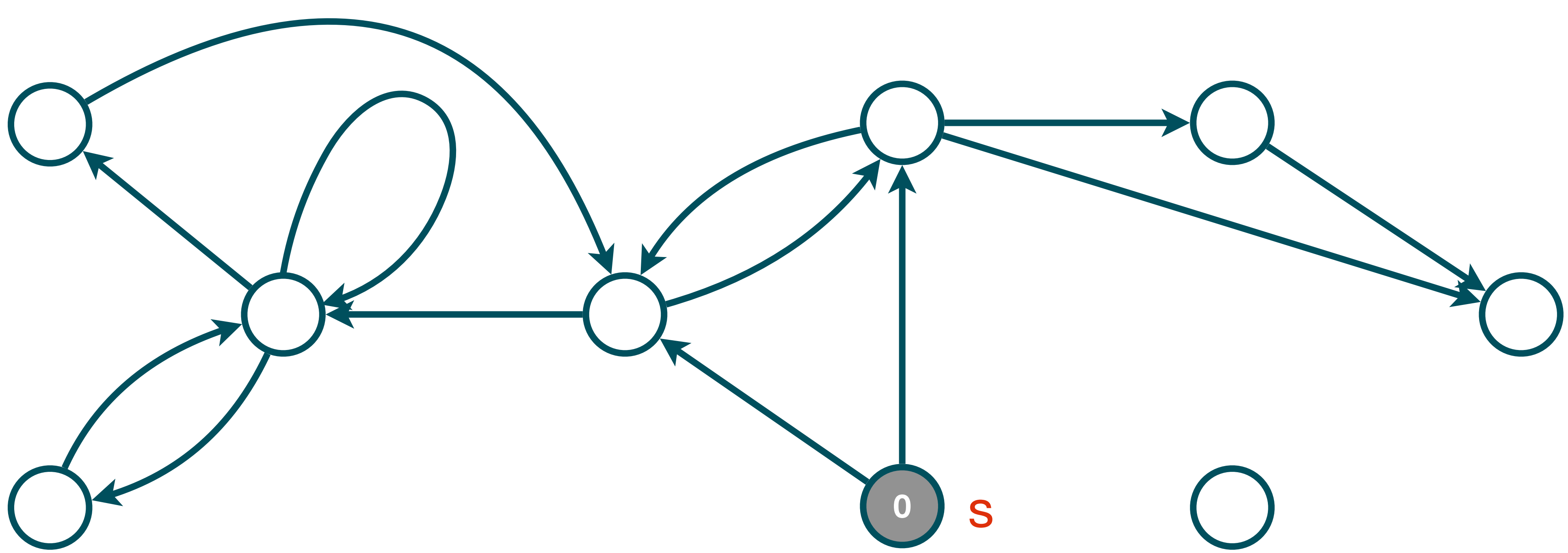

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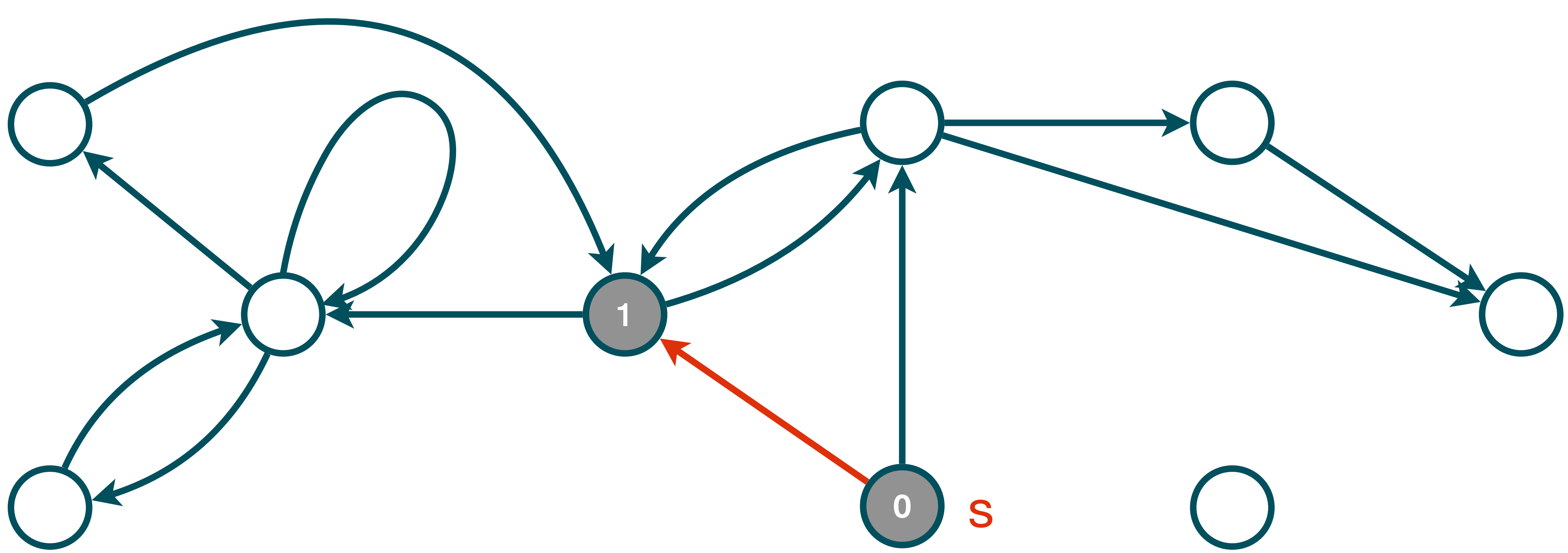

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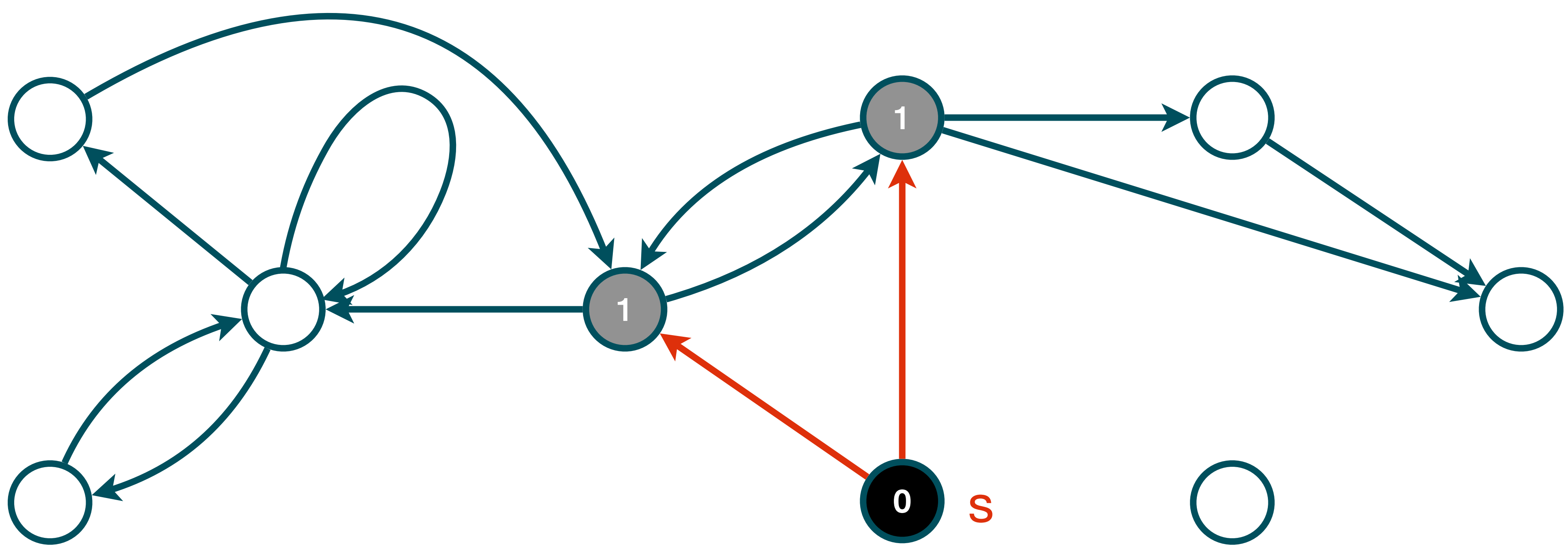

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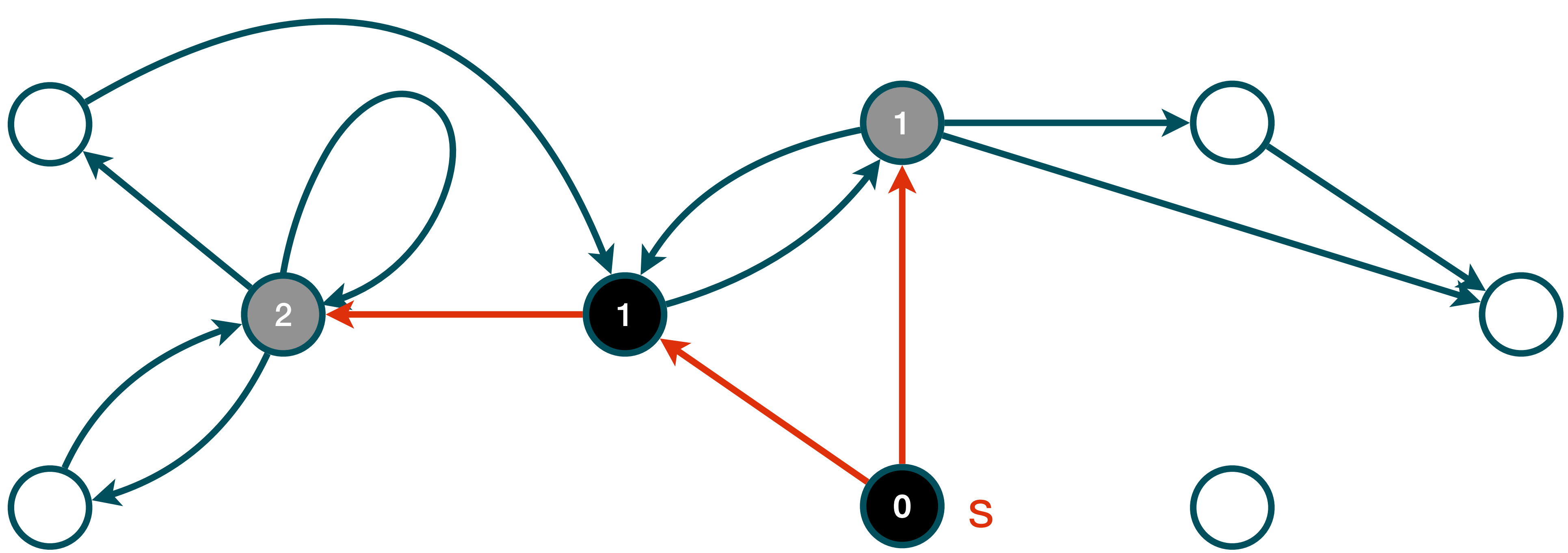

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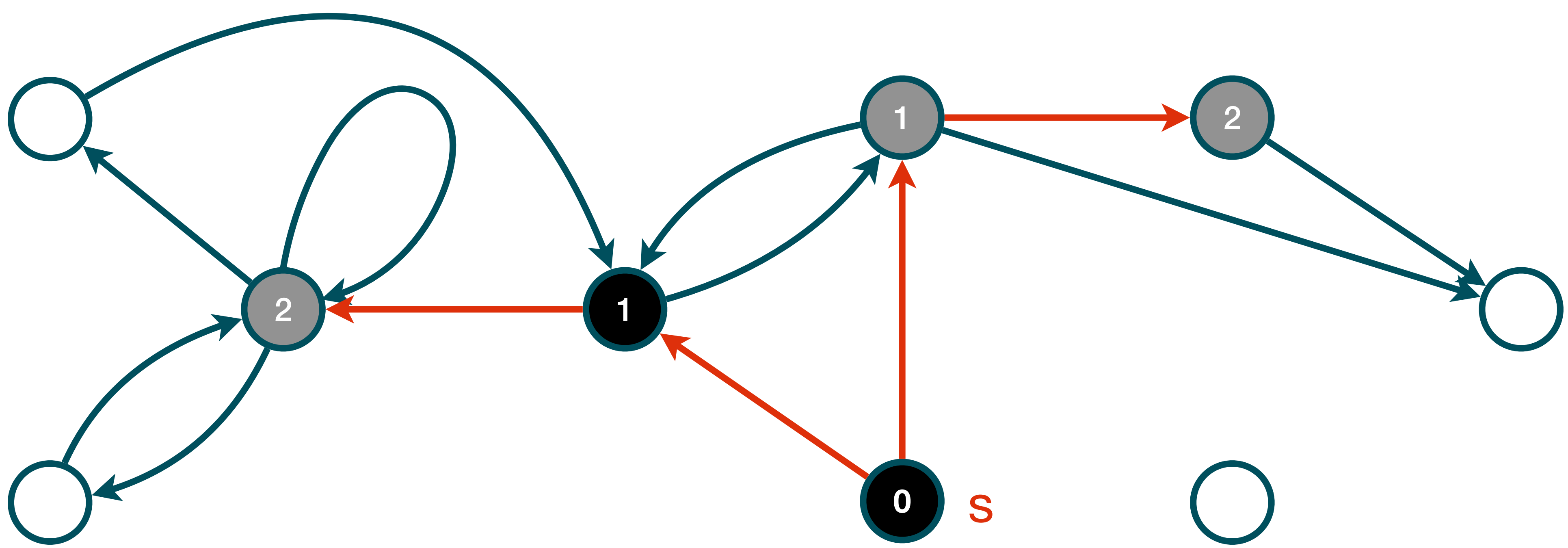

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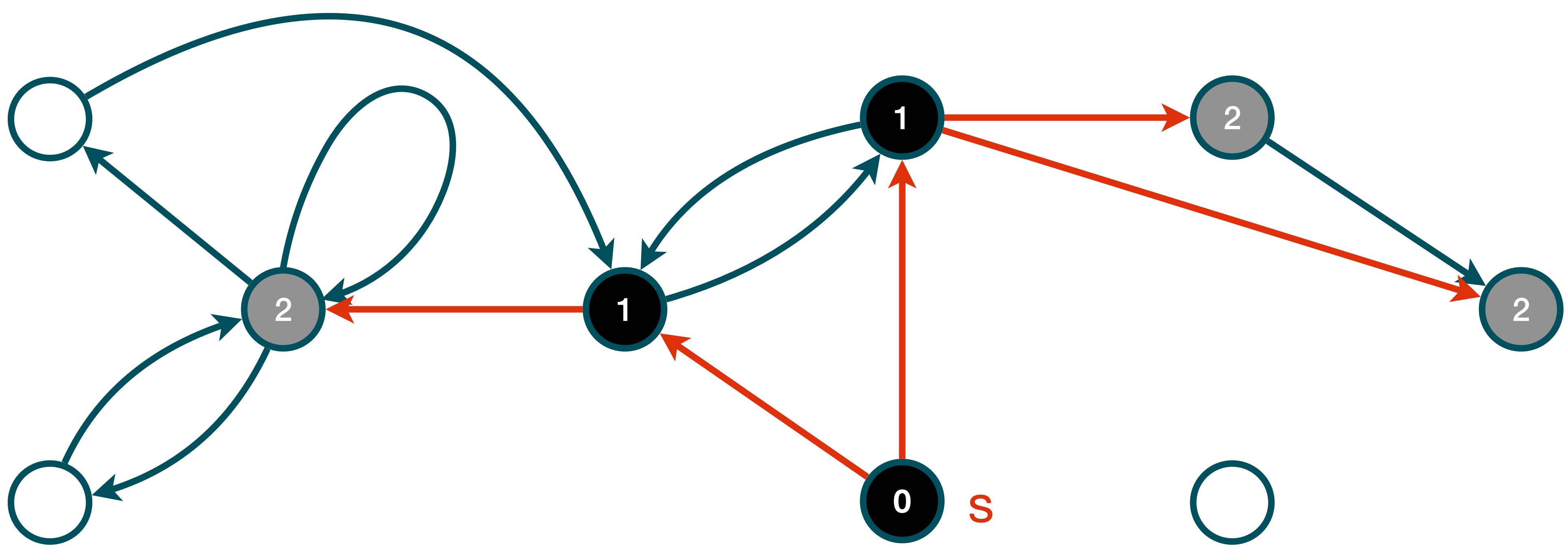

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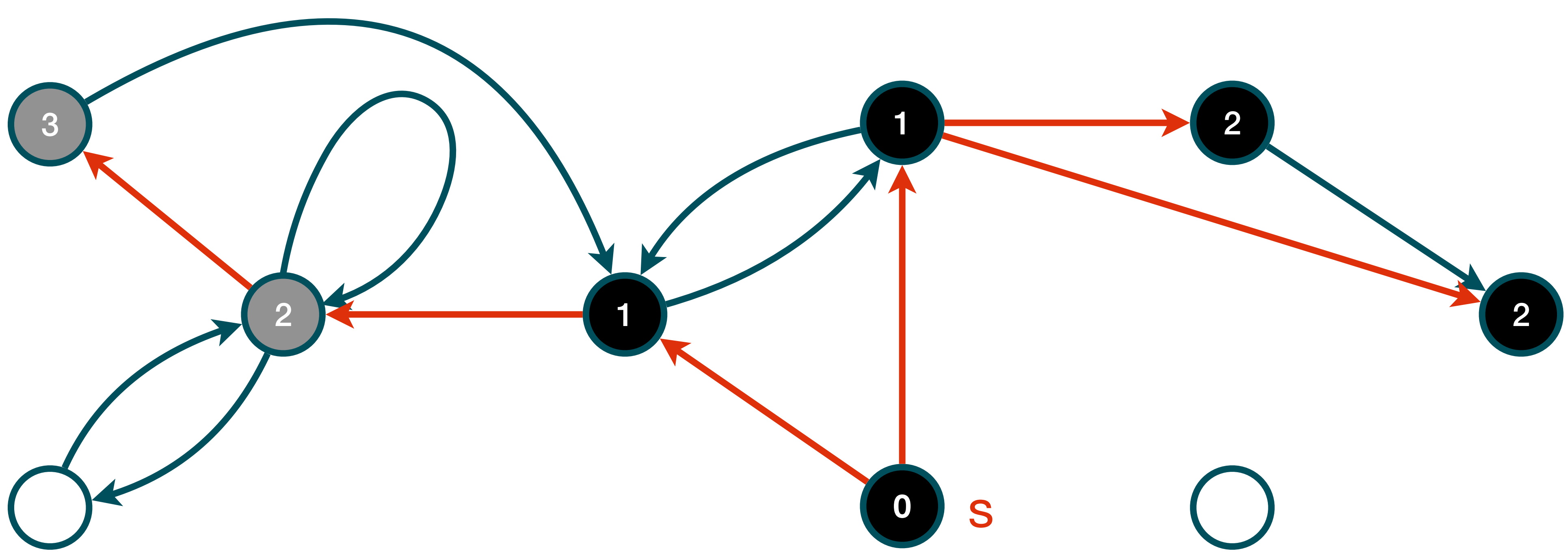

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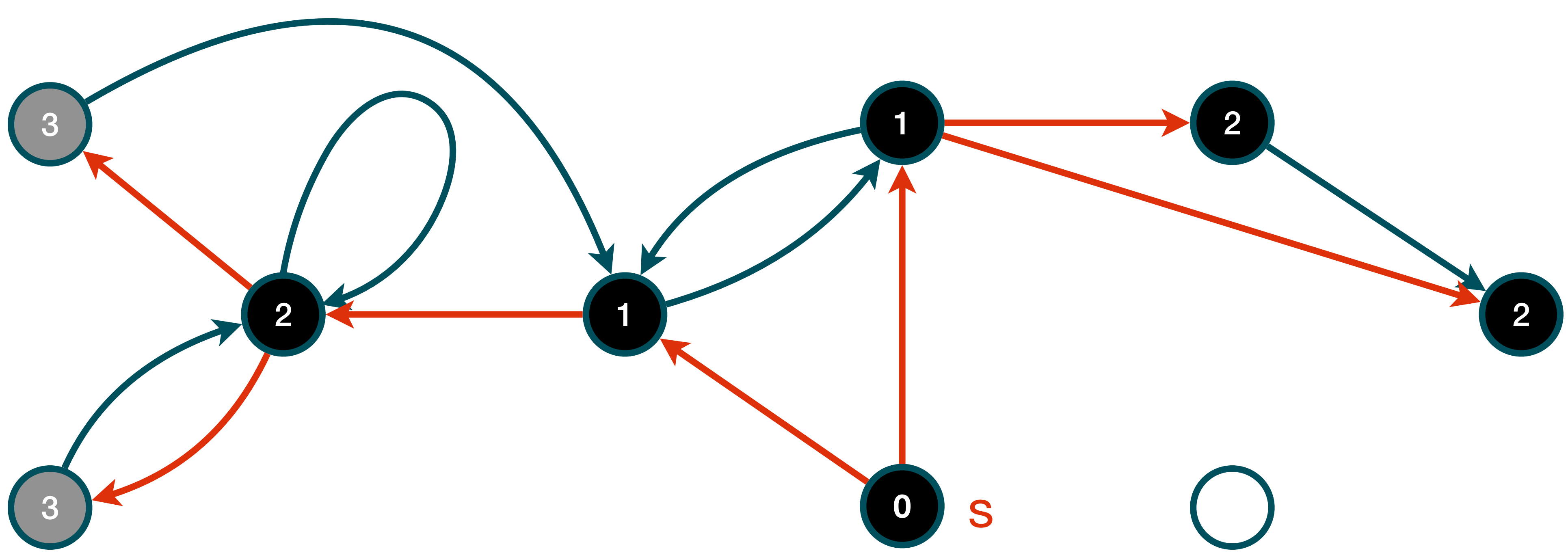

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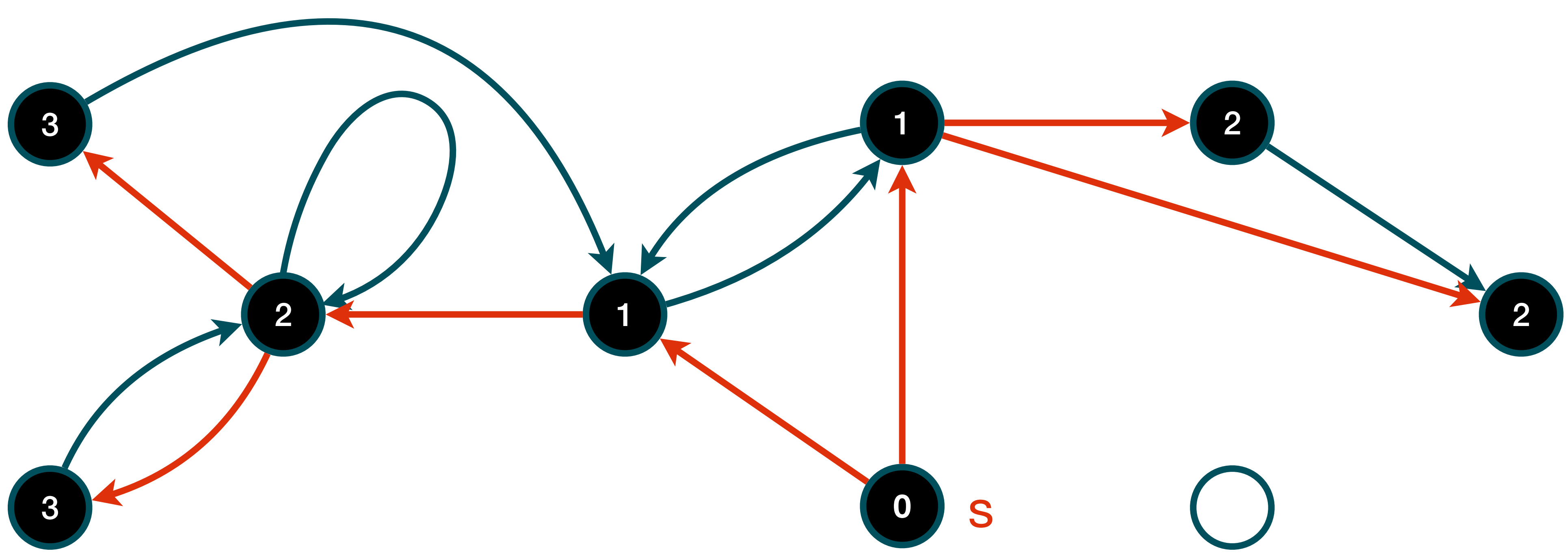

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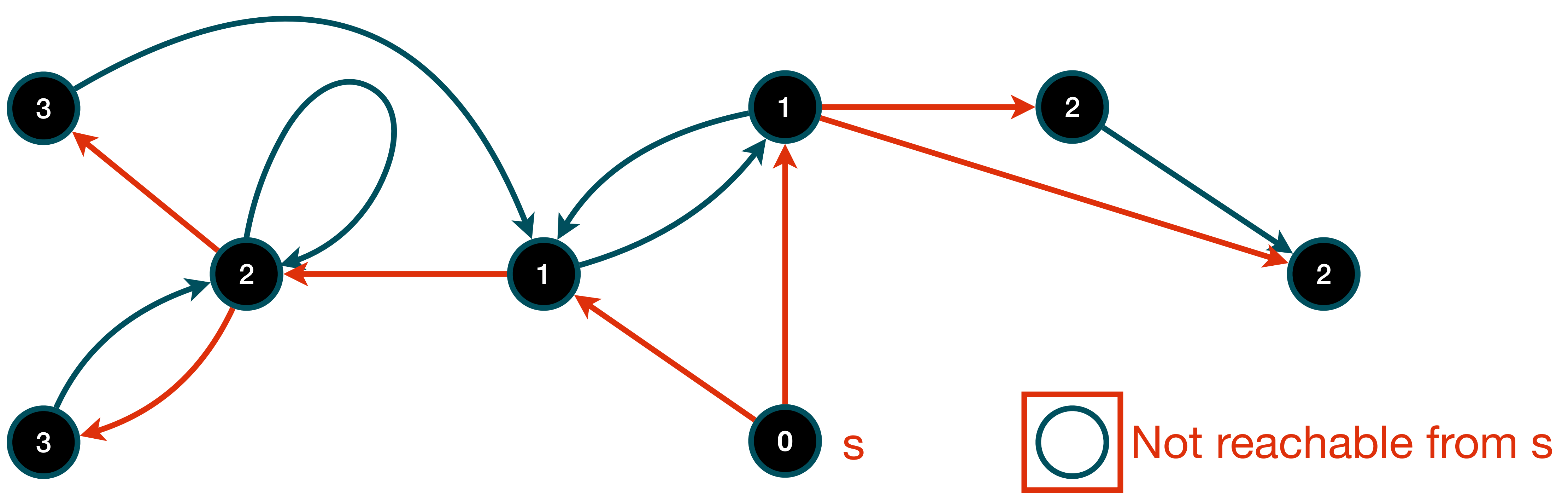


BFS: Properties

Lemma 1. The time complexity of BFS is $O(|V|+|E|)$ (linear in the size of the adjacency-list representation of G)

Lemma 2. Let $Q=[v_1,\ldots,v_n]$ be the queue at any iteration of BFS. Then $v_i.distance \leq v_{i+1}.distance$ and $v_n.distance \leq v_1.distance+1$, for all $i=1,\ldots,n-1$

Lemma 2 tells us that, at any iteration, if the head node of Q is at distance *d* from *s*, Q only contains nodes at distance *d* or *d*+1 from *s*; possible nodes at distance *d*+2 will be only enqueued after all nodes at distance *d* have been dequeued.

Lemma 3. Let $d(v,s)$ be the distance between *v* and *s,* for any $v \in V$. Then:

(i) $v.distance \neq \infty \iff v$ is reachable from *s*

(ii) if $v.distance \neq \infty \implies v.distance = d(v,s)$

DFS: Pseudocode

DFS(G) - G is represented by the adjacency lists Adj[·] of its vertices

for each $u \in V$
  $u.color \leftarrow$ white;
$t \leftarrow 0$;

Initialisation

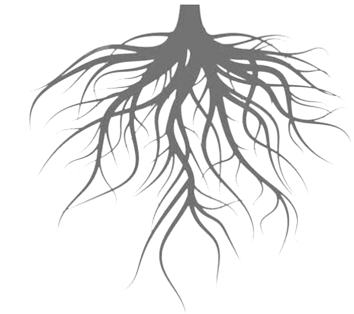
for each $u \in V$
  if $u.color =$ white
    DFS_visit(G,u)

Start the search from a new source

DFS_visit(G,u)
  $t \leftarrow t+1$;
  $u.d \leftarrow t$;
  $u.color \leftarrow$ gray;
  for each $v \in$ Adj[u]
    if $v.color =$ white
      DFS_visit(G,v);
  $v.color \leftarrow$ black;
  $t \leftarrow t+1$;
  $u.f \leftarrow t$;

Visit the graph recursively

DFS: Complexity



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    if $u.color =$ white
        DFS_visit(G,u)

Initialisation: $O(|V|)$

Start the search from a new source: this only happens when a vertex is white $\Longrightarrow$ $O(|V|)$ calls

DFS_visit(G,u)
  $t \leftarrow t+1$;
  $u.d \leftarrow t$;
  $u.color \leftarrow$ gray;
  for each $v \in$ Adj[u]
    if $v.color =$ white
      DFS_visit(G,v);
  $v.color \leftarrow$ black;
  $t \leftarrow t+1$;
  $u.f \leftarrow t$;

Visit the graph recursively: this procedure is only called on white vertices, which are immediately painted gray

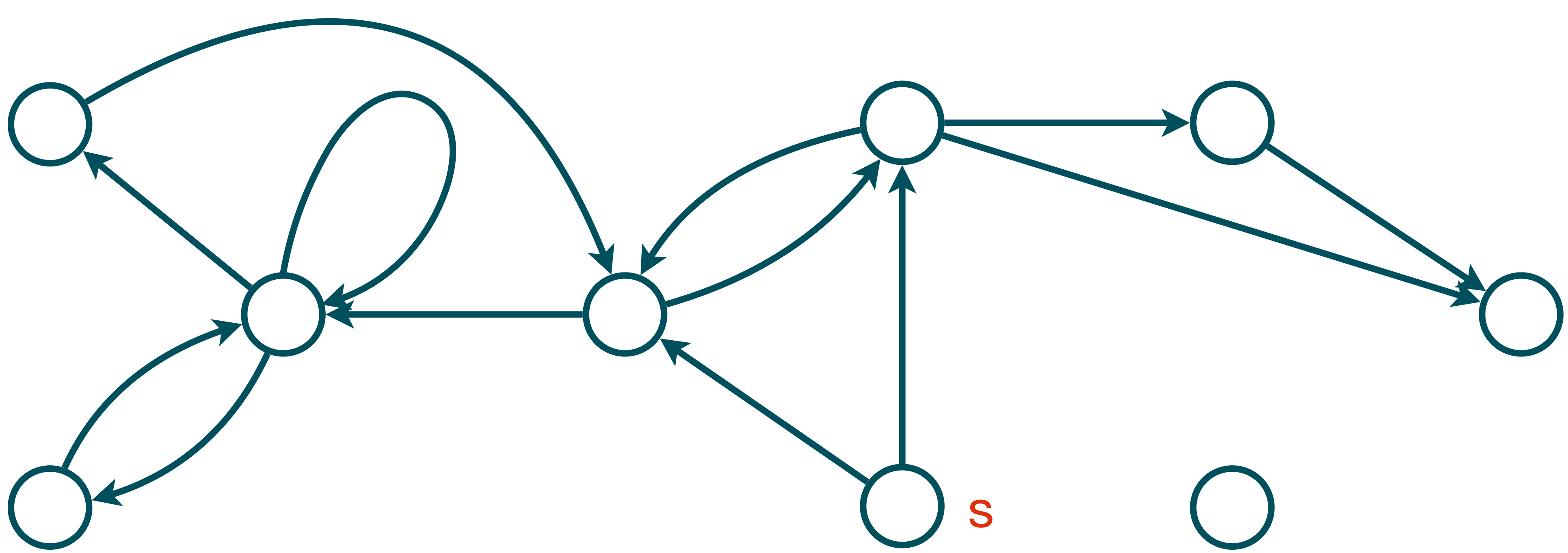
$$\Longrightarrow O\left(\sum_{u \in V} |\mathrm{Adj}[u]|\right) = O(|E|)$$

Depth-First Search

Much like BFS, DFS colors the nodes of G during the visit.

Again, white nodes have not been visited yet; gray nodes have been discovered but have undiscovered neighbours; black nodes have been discovered and their neighbours too.

DFS assigns two timestamps to each node *v*: *v.d* records when *v* becomes gray, *v.f* records when it becomes black.



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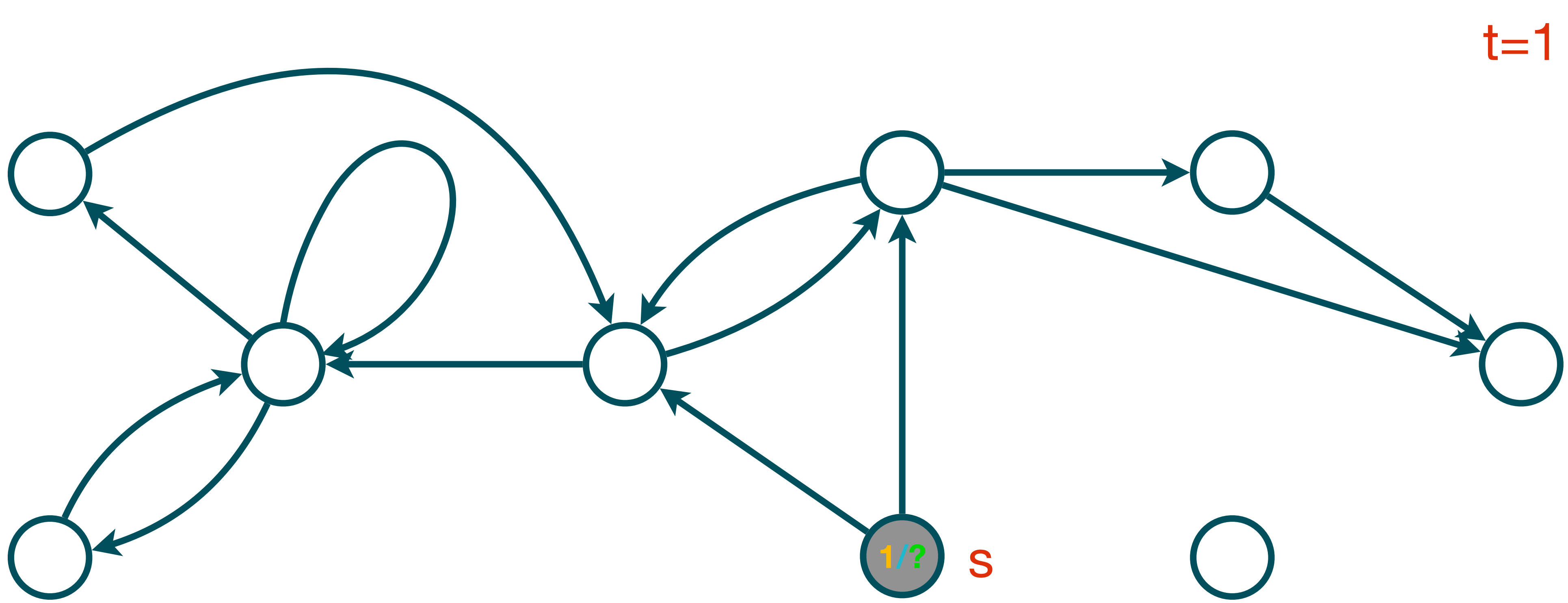

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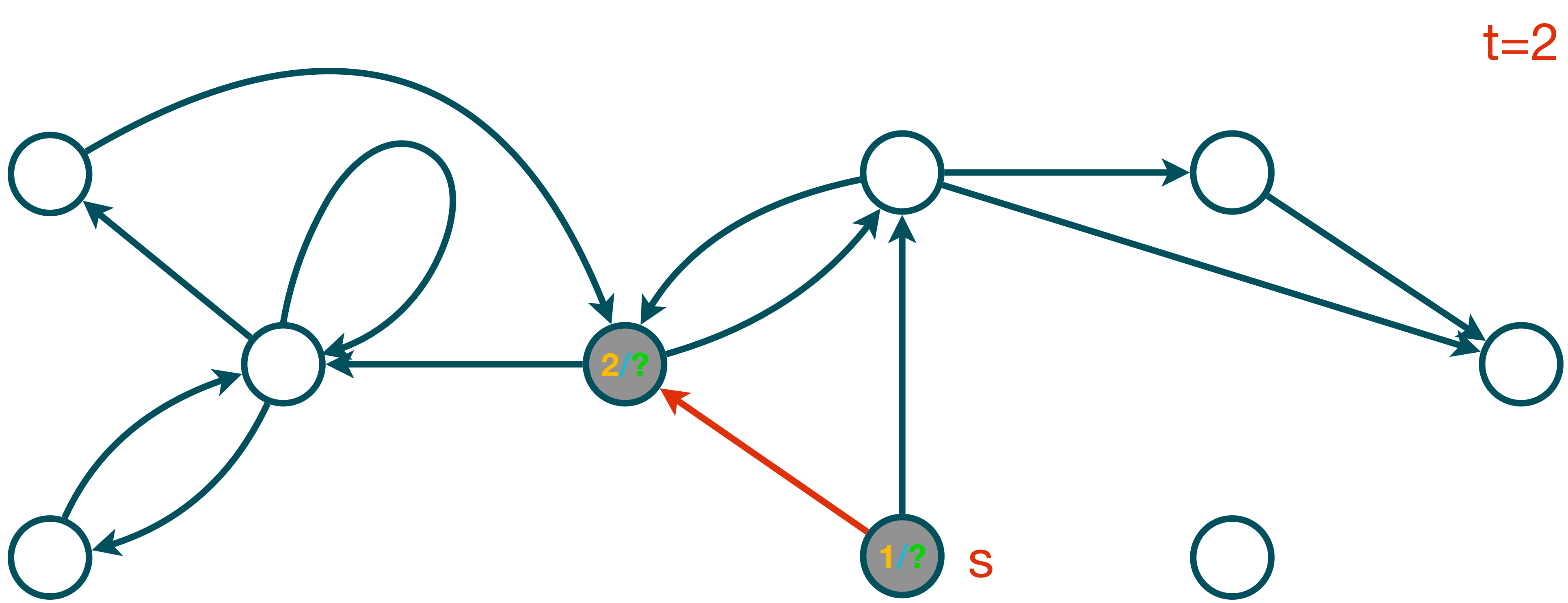

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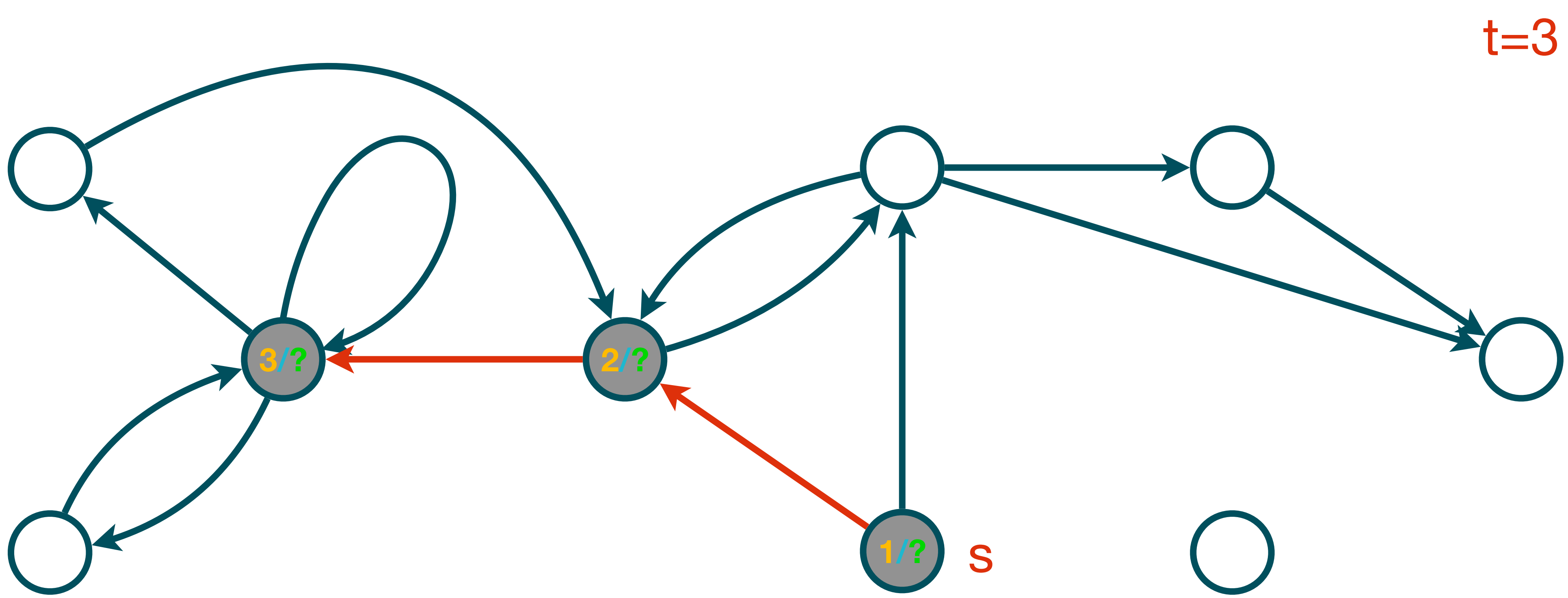

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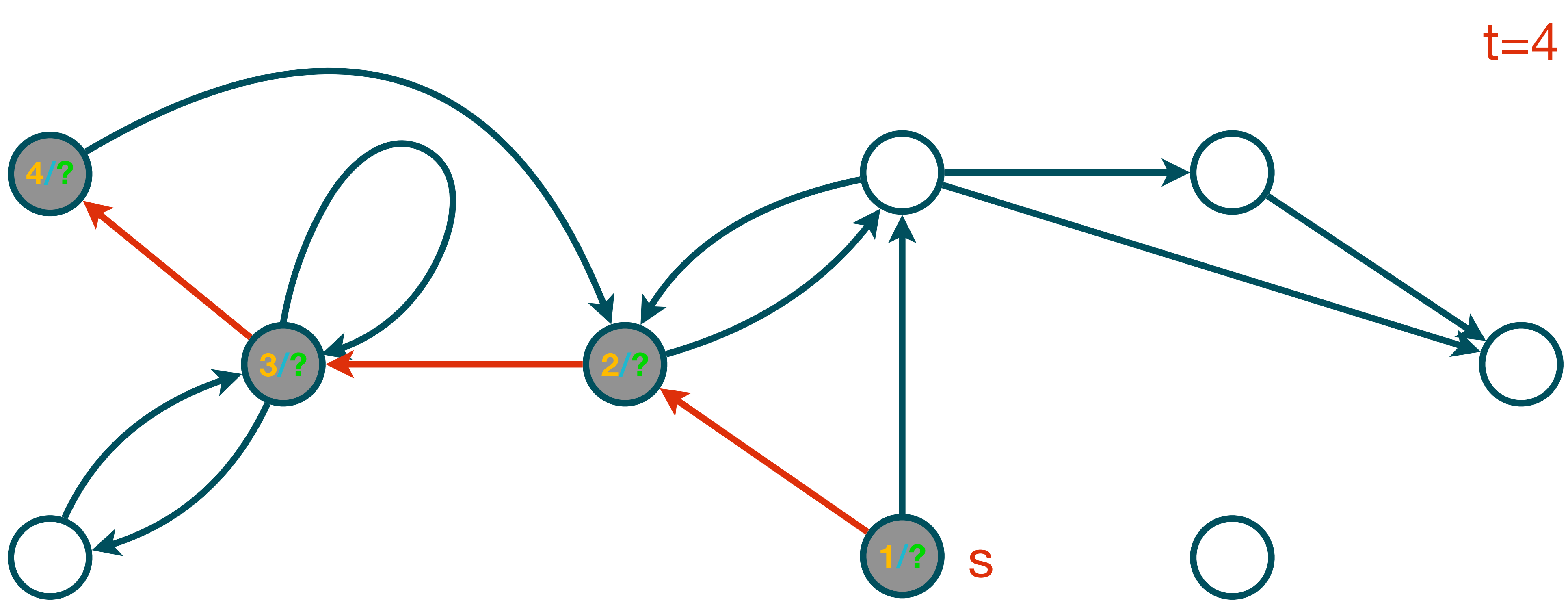

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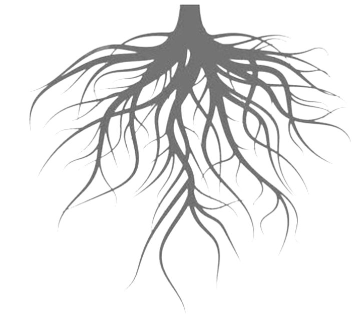

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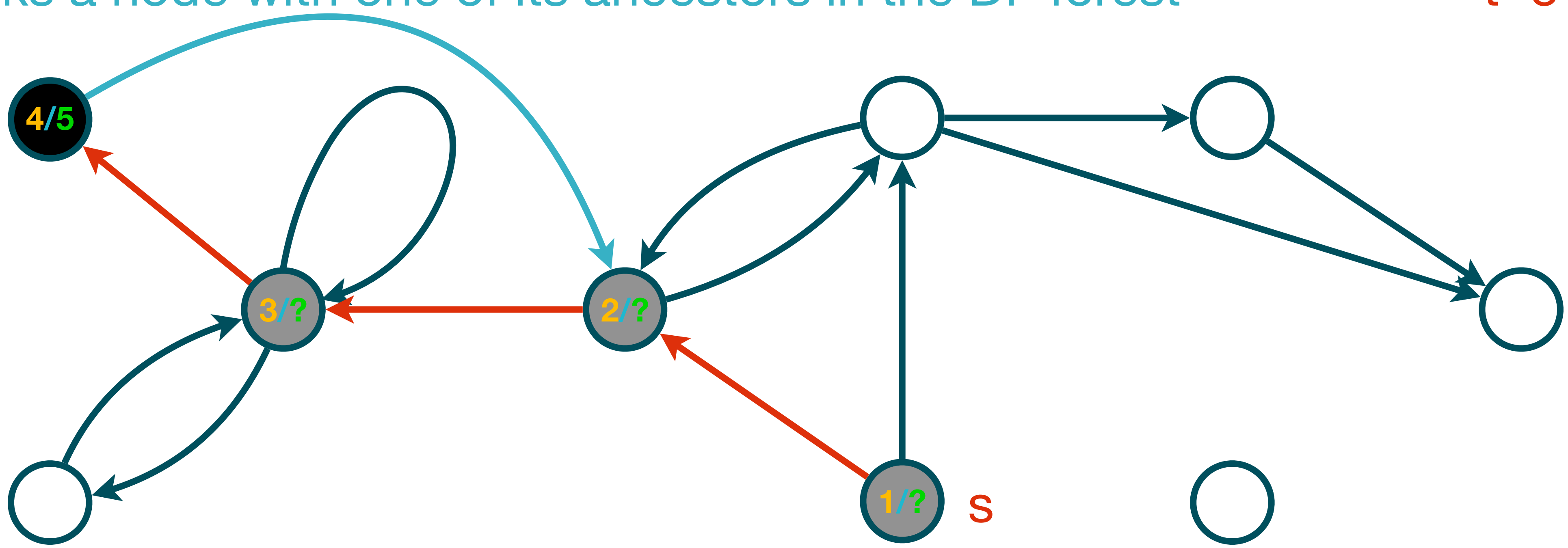
Back edge: links a node with one of its ancestors in the DF forest

t=5



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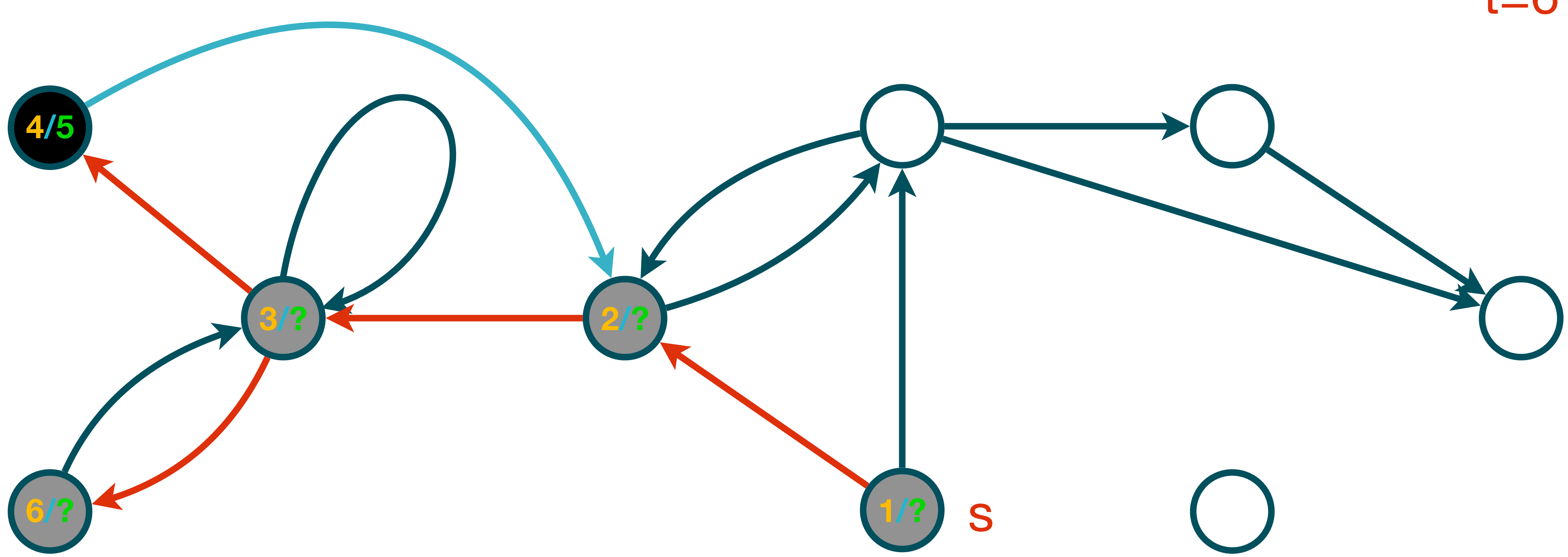

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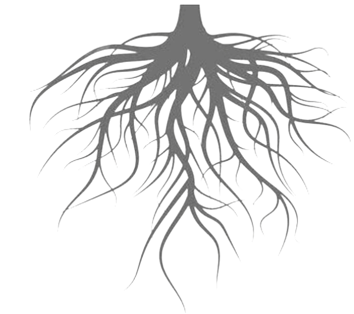

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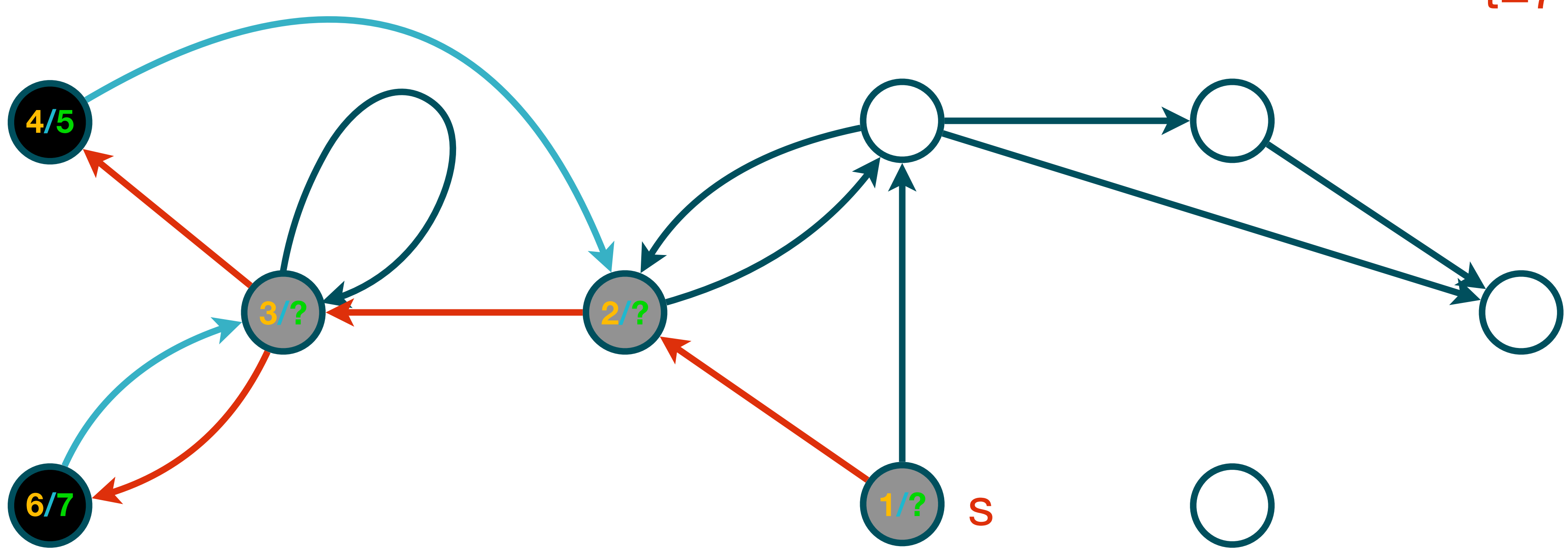

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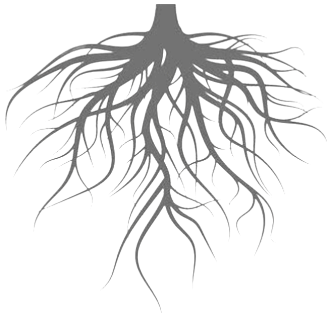

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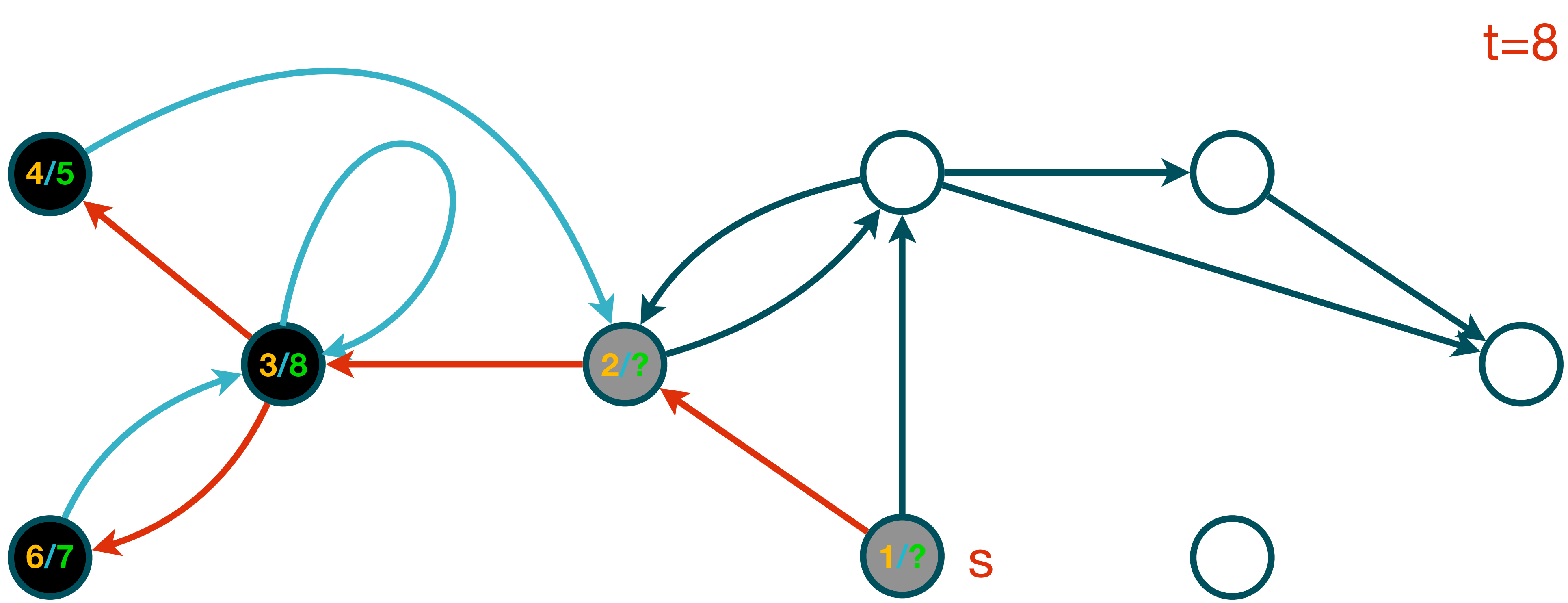

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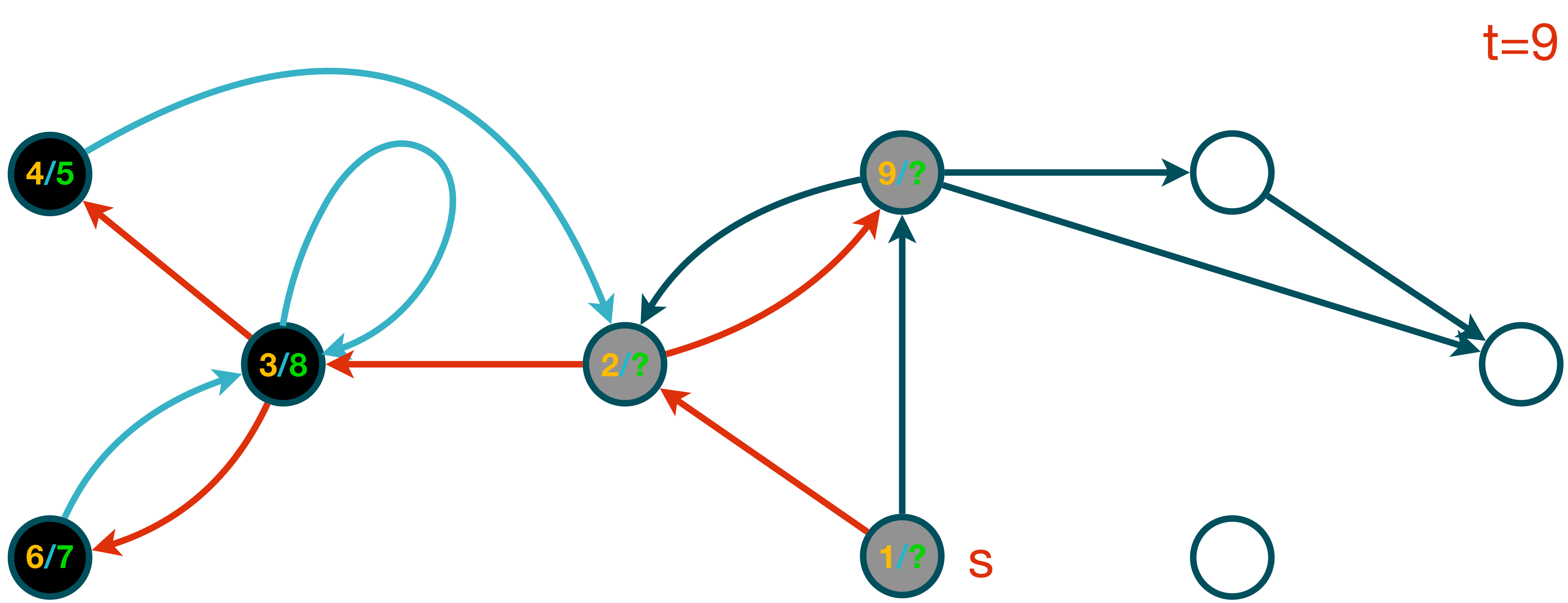

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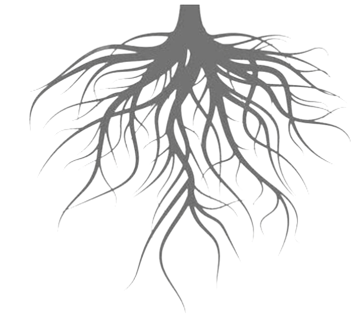

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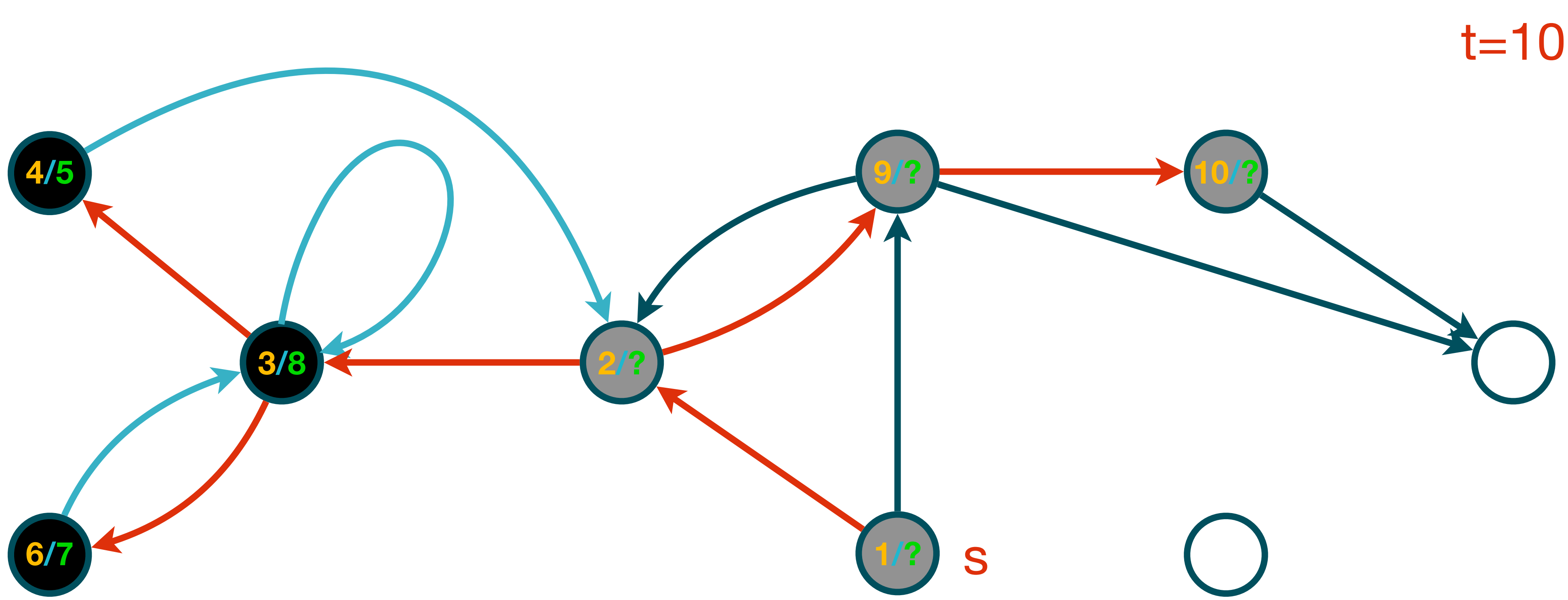

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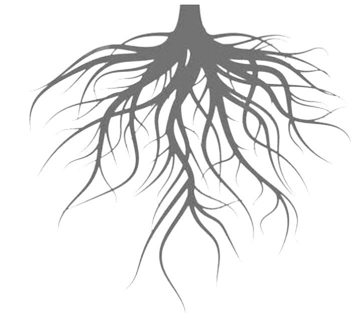

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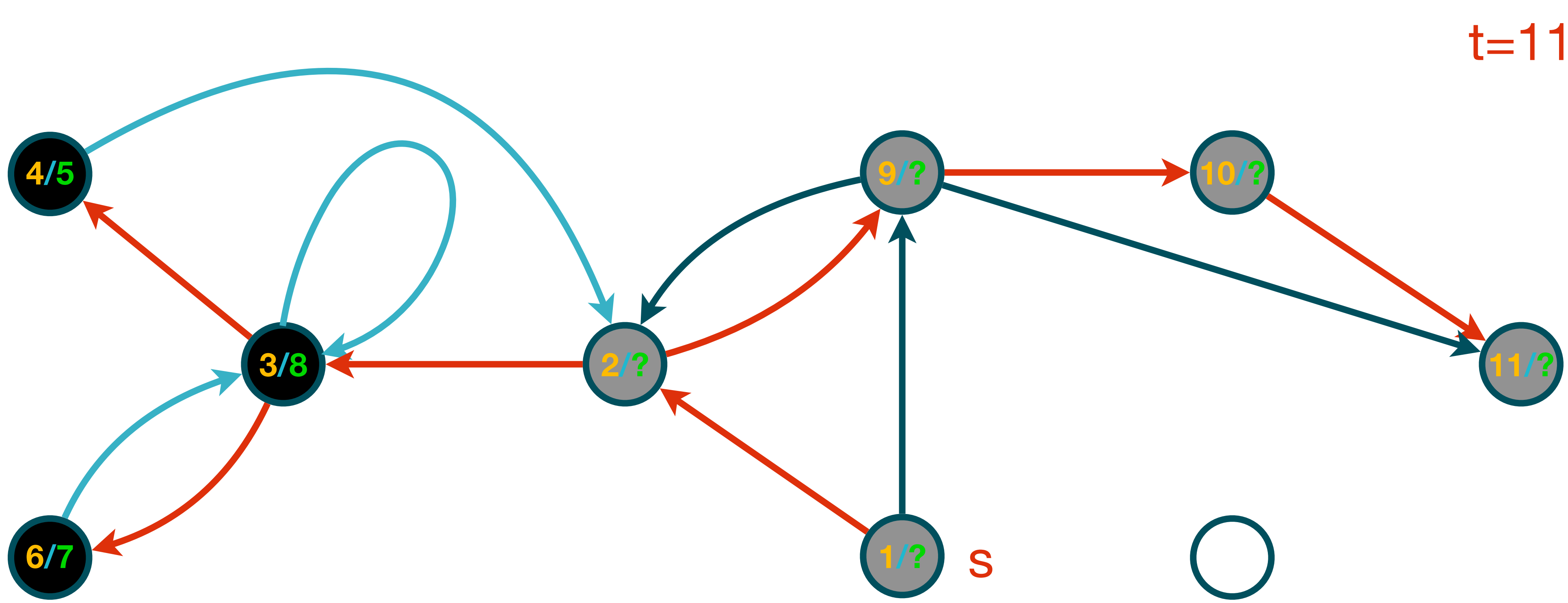

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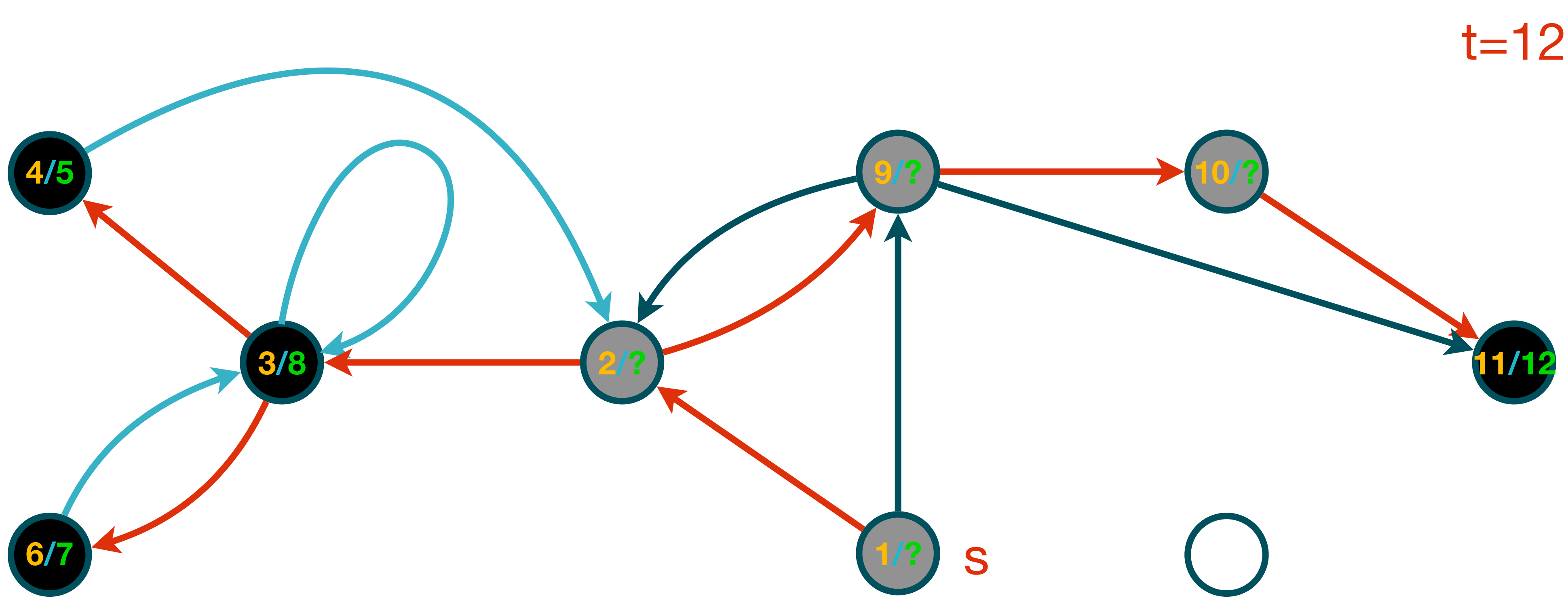

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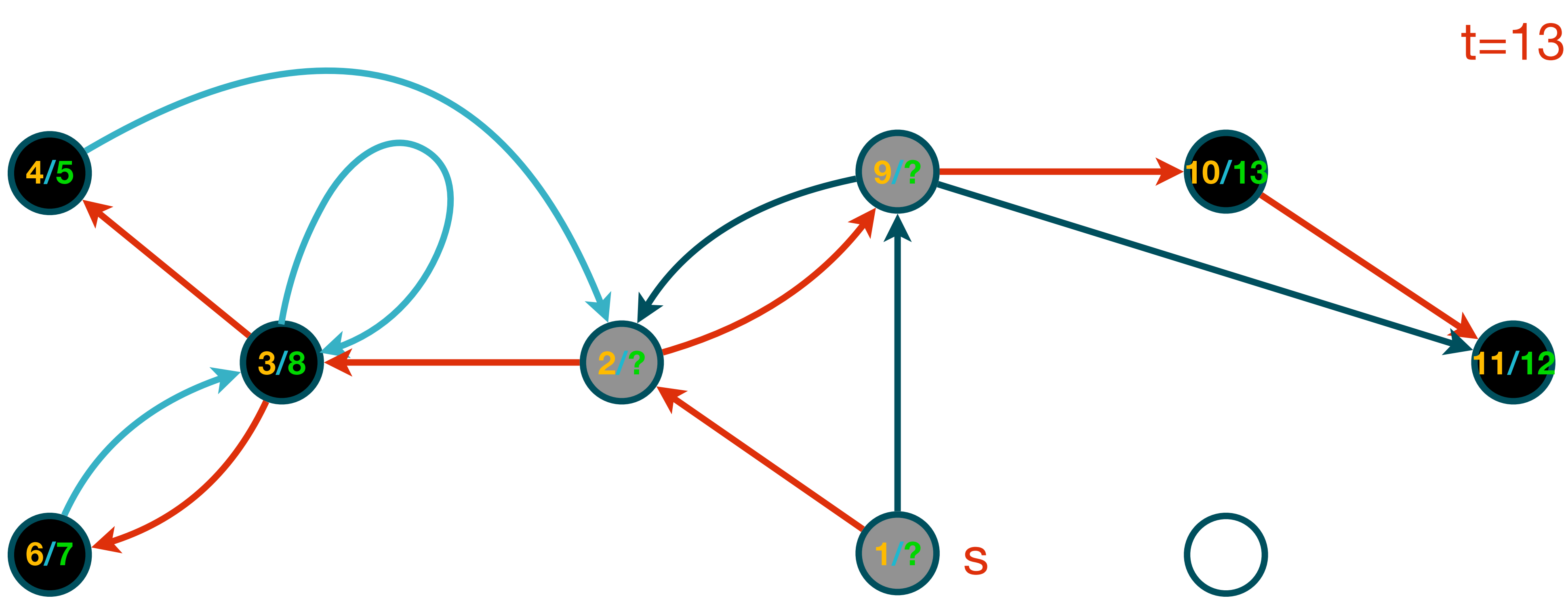

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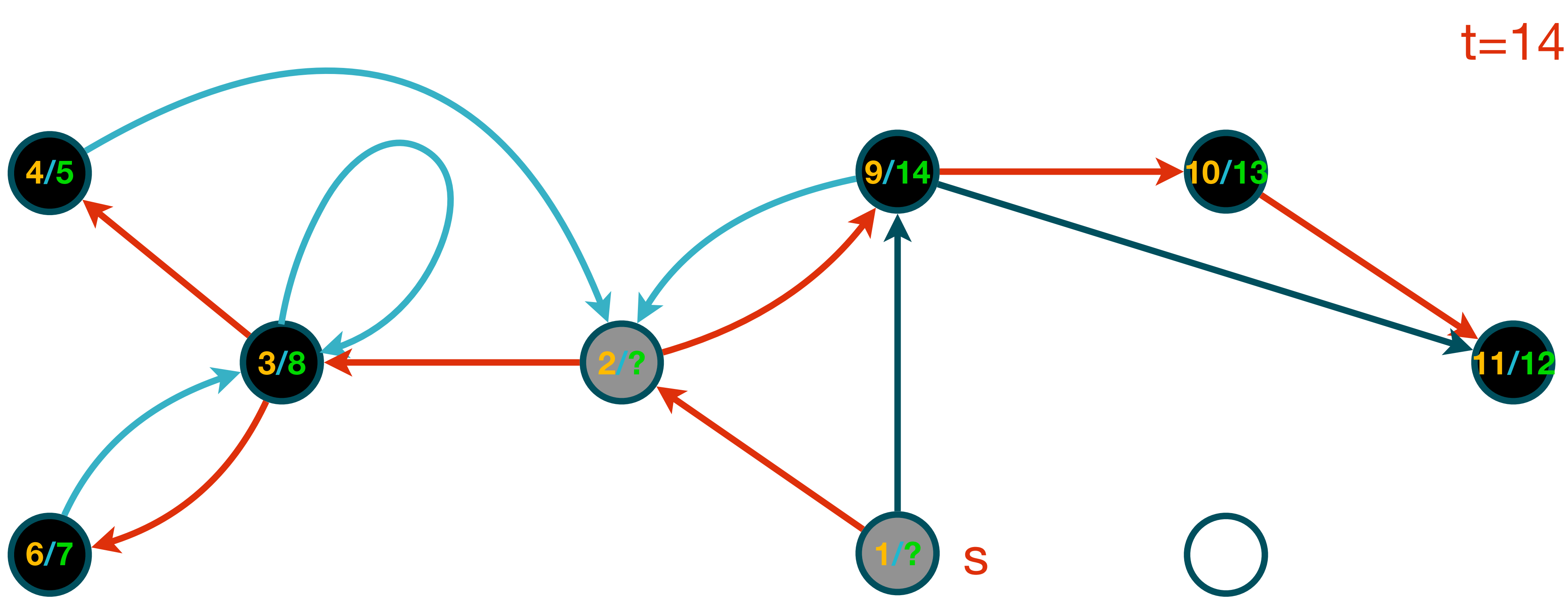

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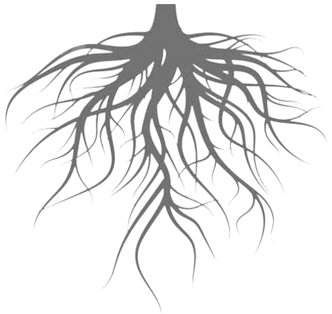

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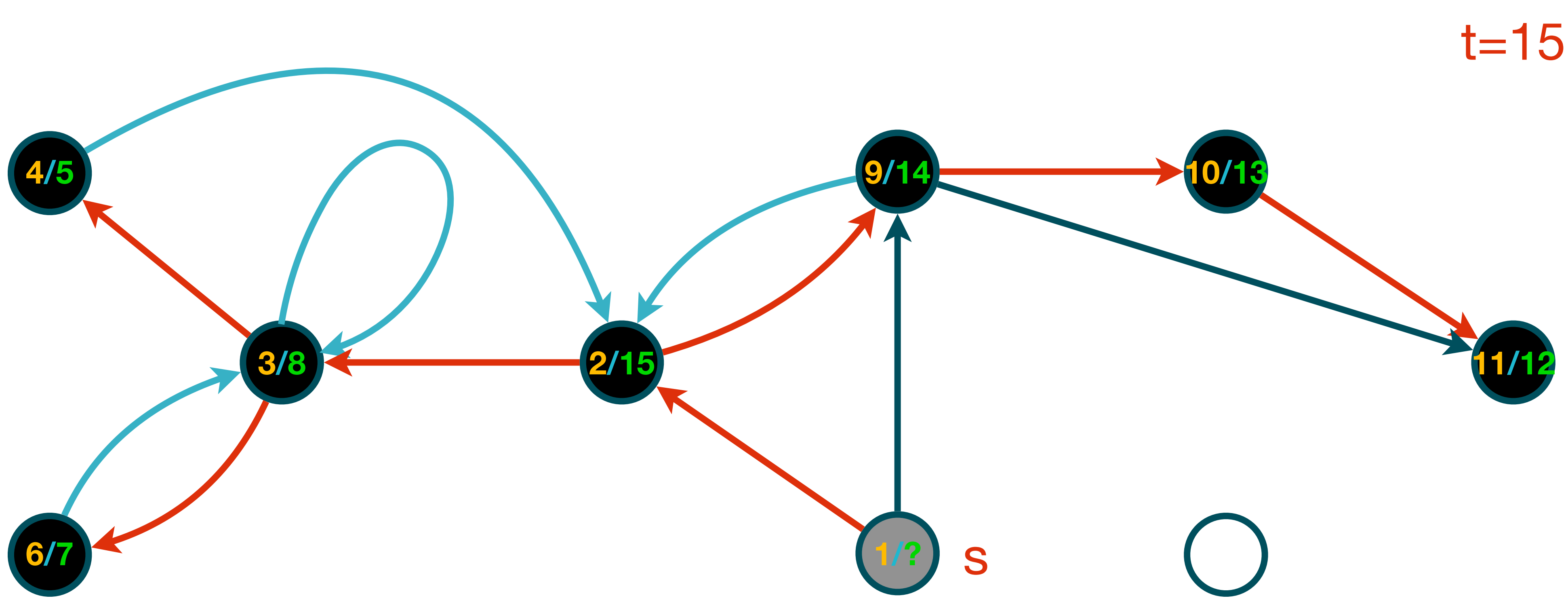

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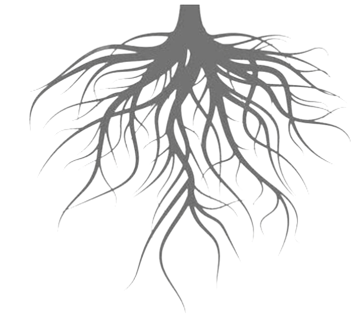

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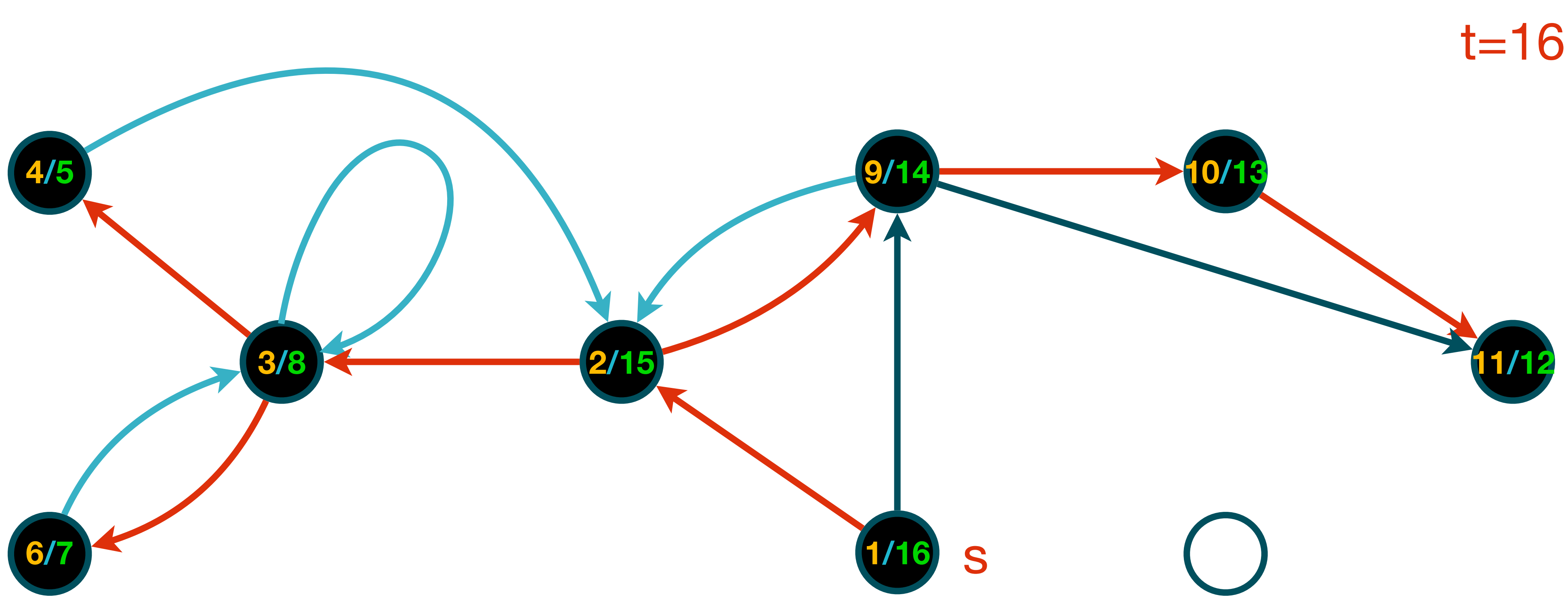

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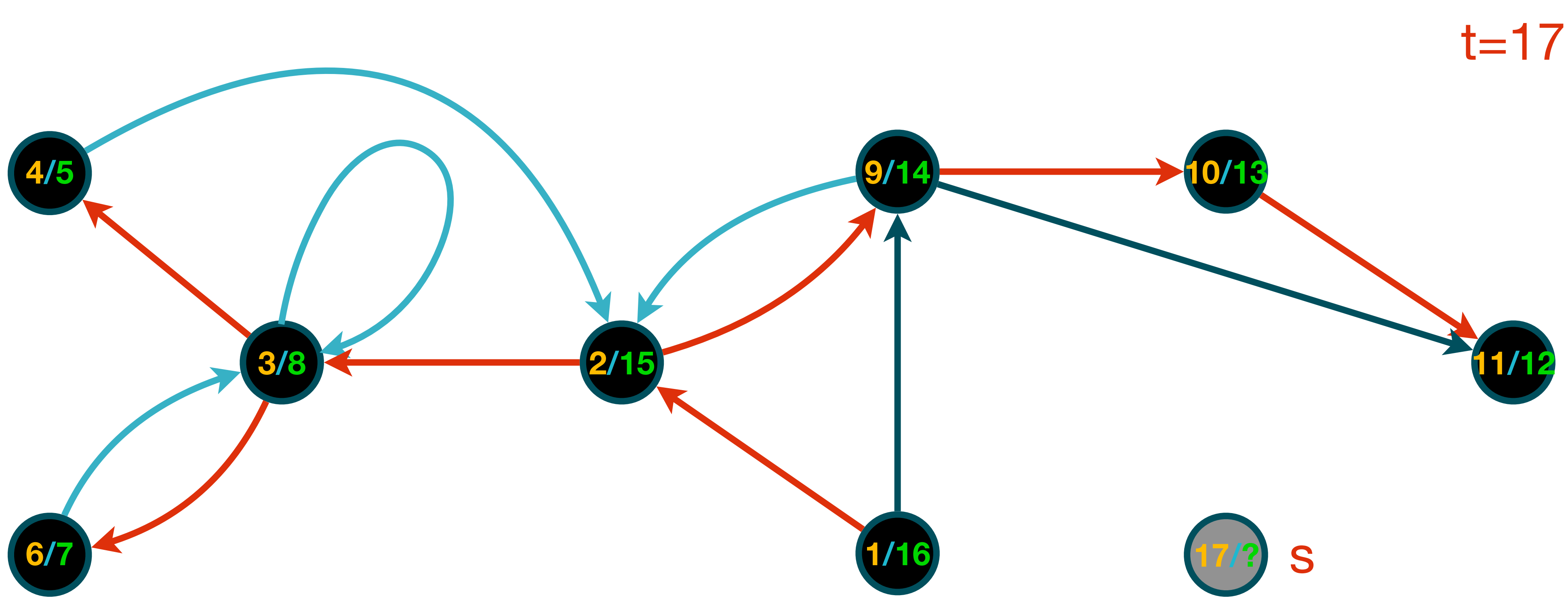
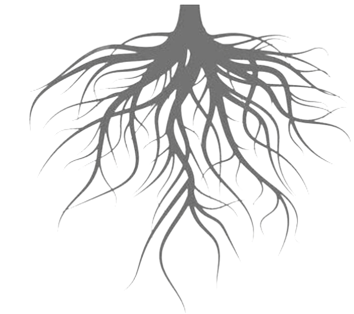


Depth-First Search

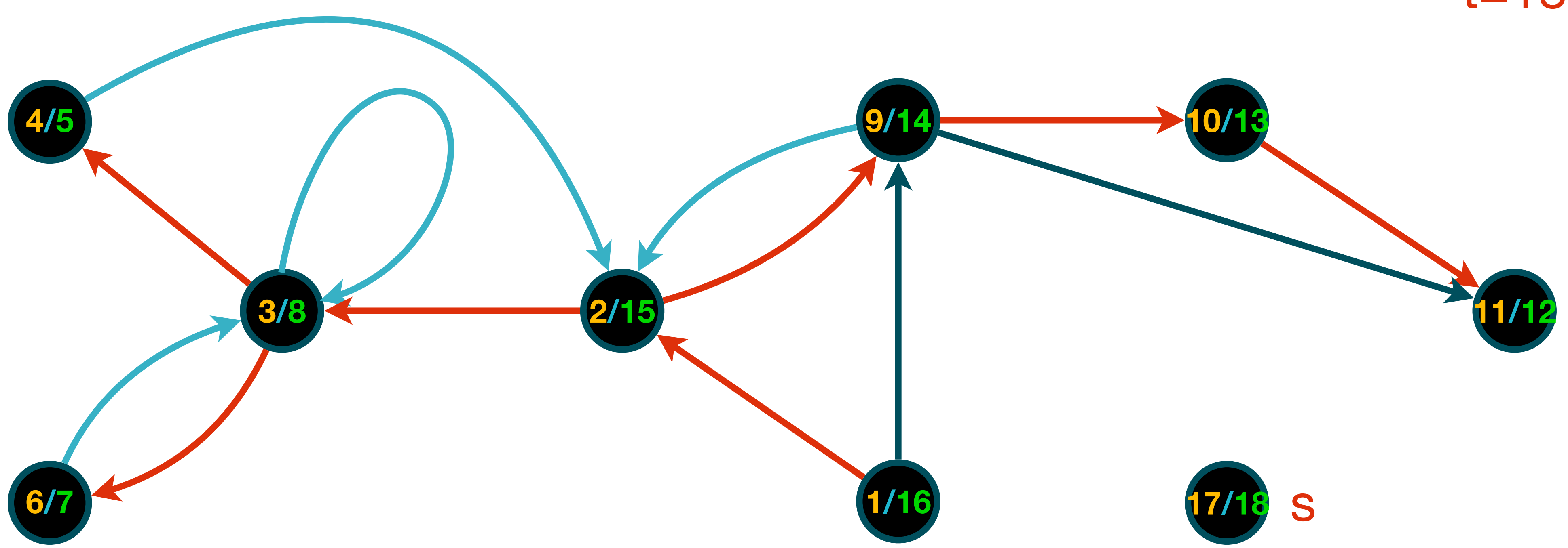


Much like BFS, DFS colors the nodes of G during the visit.

Again, white nodes have not been visited yet; gray nodes have been discovered but have undiscovered neighbours; black nodes have been discovered and their neighbours too.

DFS assigns two timestamps to each node *v*: *v.d* records when *v* becomes gray, *v.f* records when it becomes black.

t=18

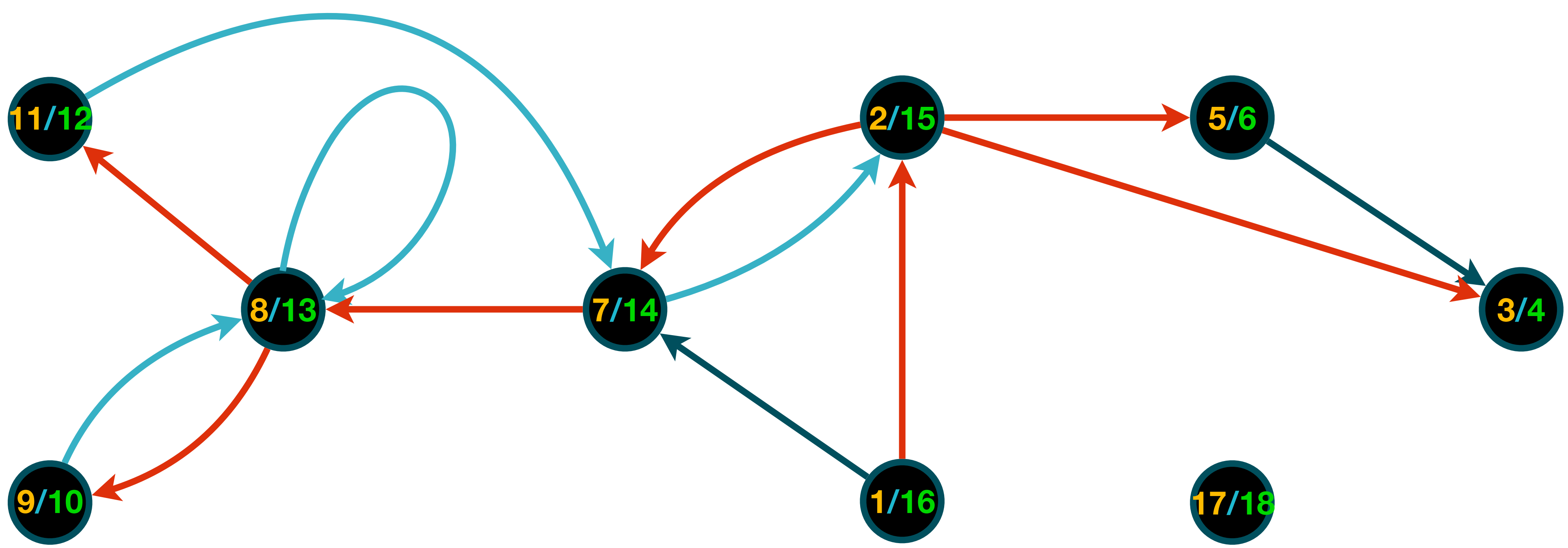


Depth-First Search

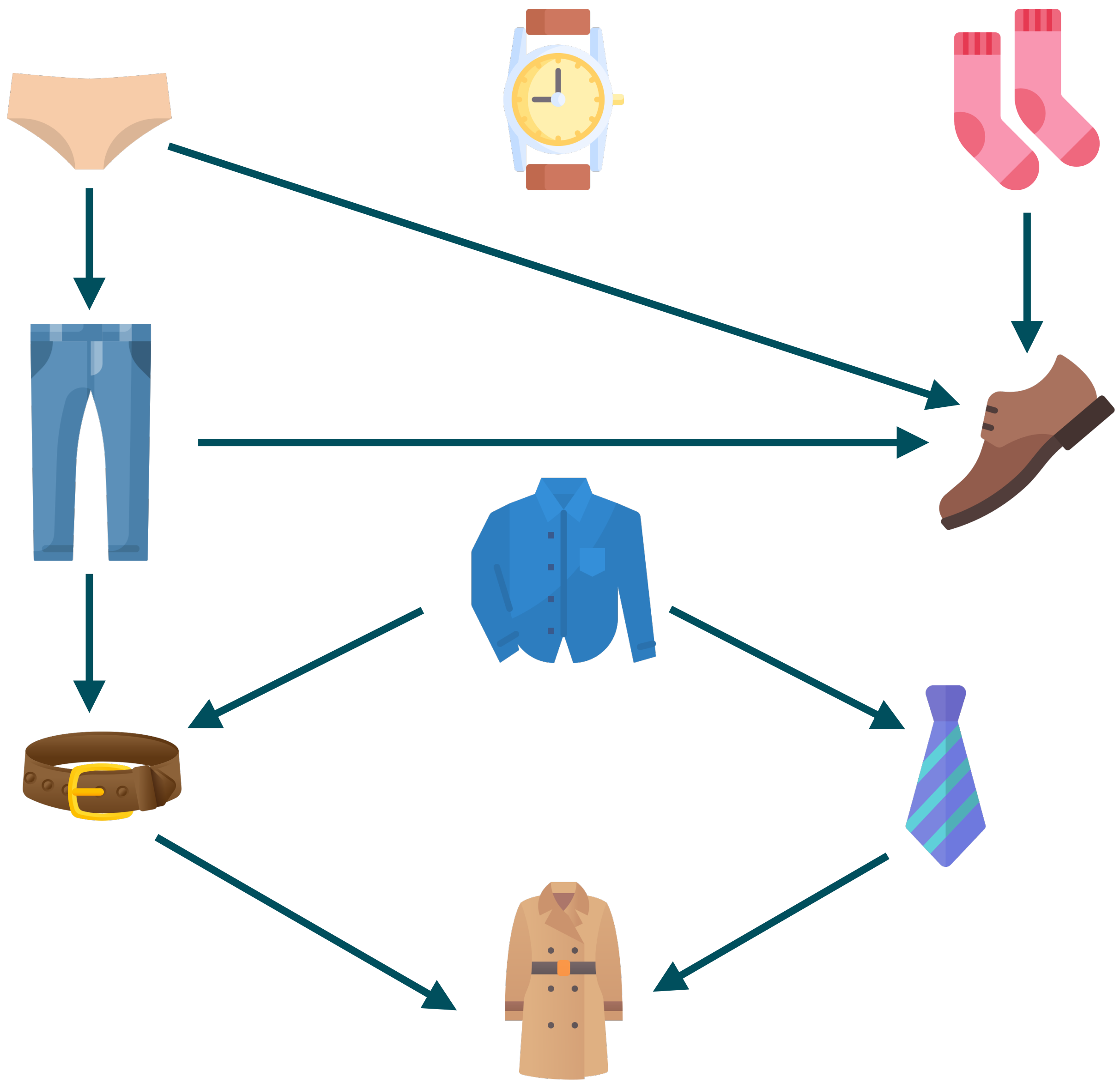
DFS produces a depth-first (DF) forest (a different tree for each source). Even for the same sources, this forest is not unique: it depends from the order in which the edges outgoing from each node are traversed. All the results are essentially equivalent.

The red edges are tree edges; the light blue edges are back edges, linking a node with one of its ancestors in the DF forest.

You can verify yourself that the result below is another possible outcome of DFS with the same two sources.

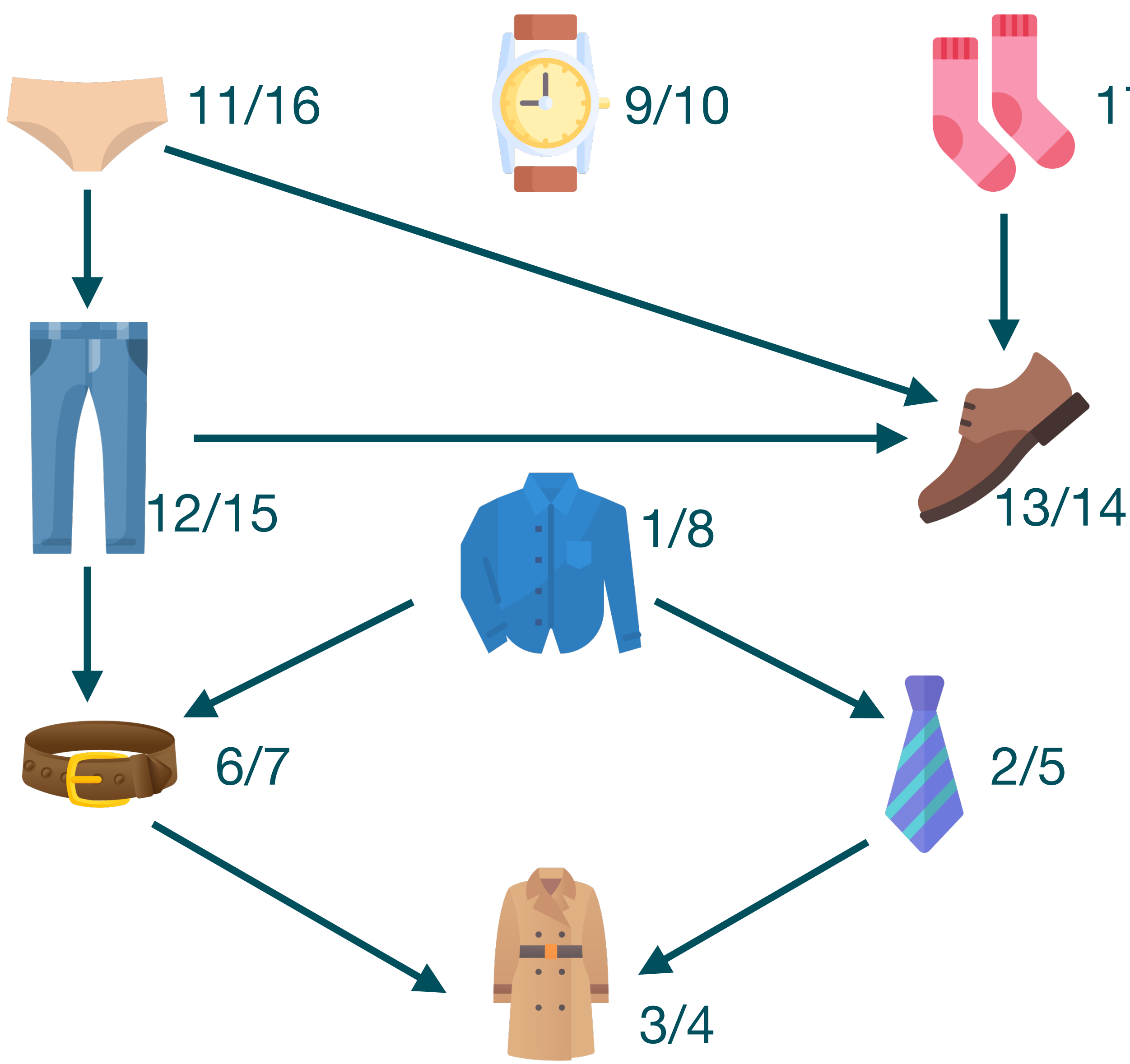


An application: Topological Sort



An edge (u,v) indicates that item u must be worn before item v.

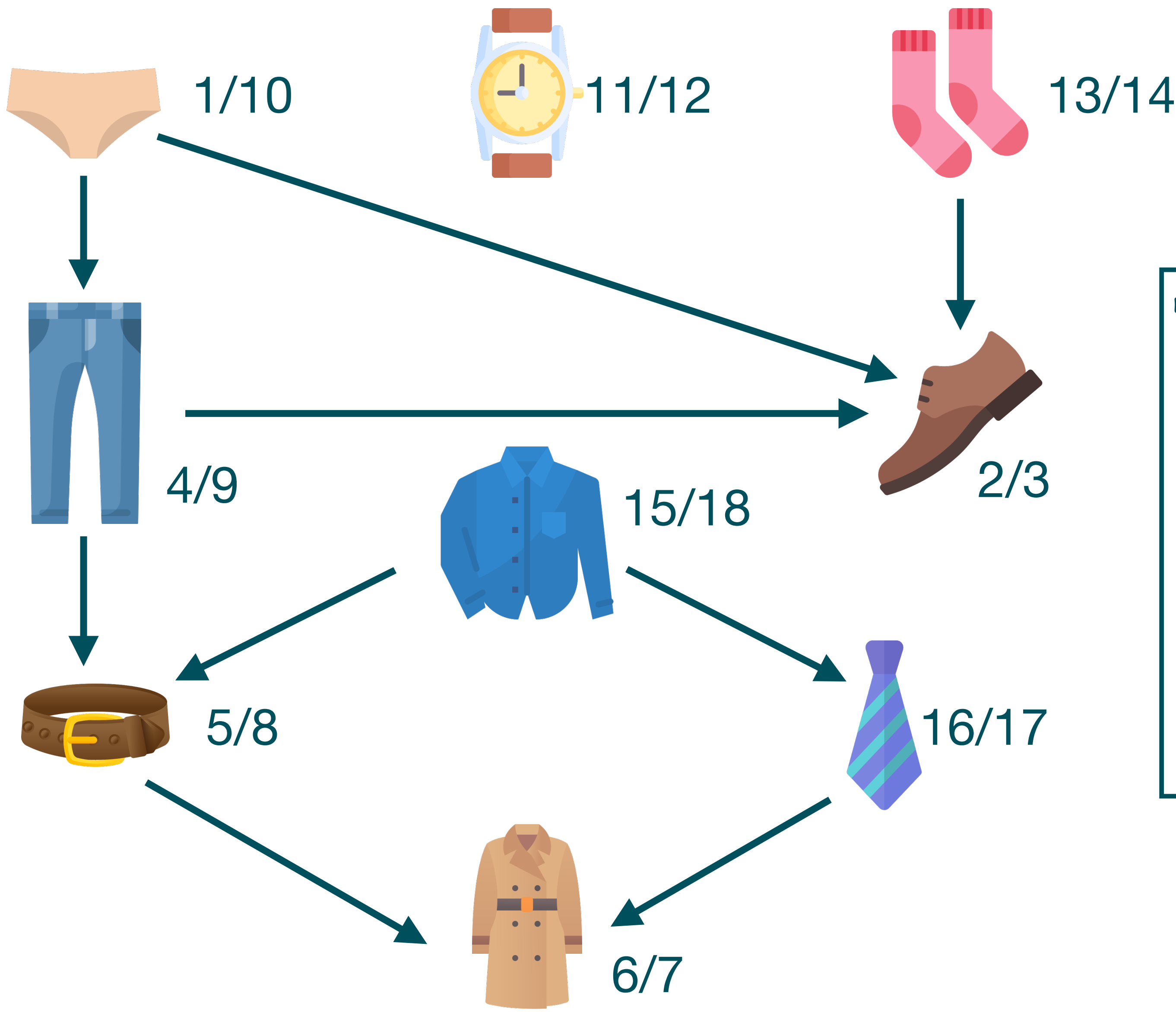
An application: Topological Sort



TopologicalSort(G)
  DFS(G);
  $\tilde{V}[1,\ldots,|V|] \leftarrow V$ sorted w.r.t finishing time
  TopOrder←empty_stack;
  for $i = 1\ldots|V|$
    TopOrder.push($\tilde{V}[i]$);
  return TopOrder;

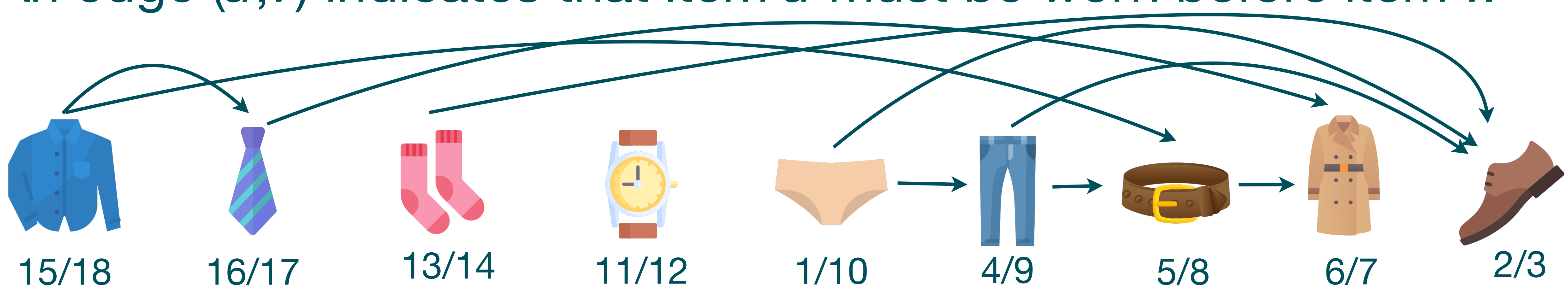
An edge (u,v) indicates that item u must be worn before item v.

An application: Topological Sort



TopologicalSort(G)
  DFS(G);
  $\tilde{V}[1,\ldots,|V|] \leftarrow V$ sorted w.r.t finishing time
  TopOrder←empty_stack;
  for $i = 1\ldots|V|$
    TopOrder.push($\tilde{V}[i]$);
  return TopOrder;

An edge (*u*,*v*) indicates that item *u* must be worn before item *v*.



Exercises

EX (Cormen 17.1-1): If the set of stack operations included a MULTIPUSH operation, which pushes k items onto the stack, would the O(1) bound on the amortized cost of stack operations continue to hold?

Exercises

EX1: Given a connected, undirected graph, design an algorithm that assigns one of two colors (say blue or green) to each vertex in such a way that no edge links two vertices of the same color; or return FAIL if no such coloring is possible.

Exercises

EX2: Give an $O(|V|)$-time algorithm that determines whether or not a given undirected graph contains a cycle. (*Hint: Think of the maximum number of edges that an acyclic undirected graph may have; use DFS and terminate it early when appropriate*).