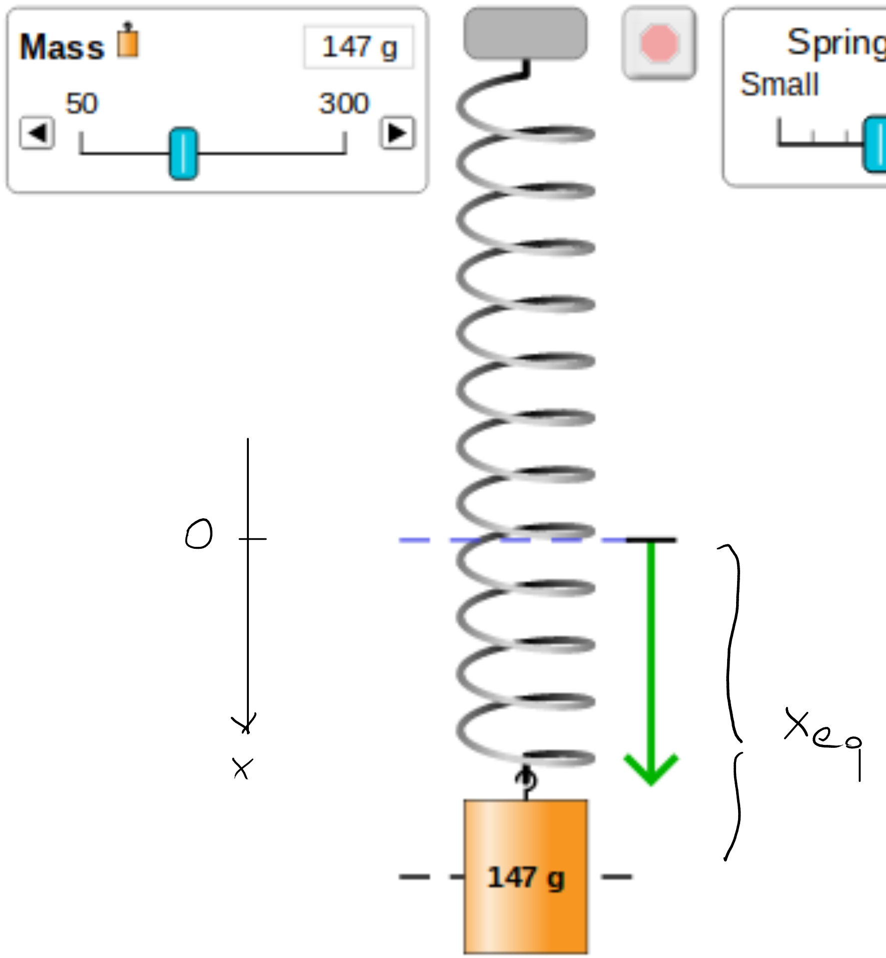
OSCILLAZIONI

⚠ $\frac{N}{M}$ $\quad \frac{ML}{LT^2} = \frac{M}{T^2}$



$$k x_{eq} = mg$$

$$k = \frac{mg}{x_{eq}} \qquad SI \ ⚠$$

Analisi dimensionale

$$\tau, m, k, l_0$$

$$[\tau] = [m]^\alpha [k]^\beta [l_0]^\gamma$$

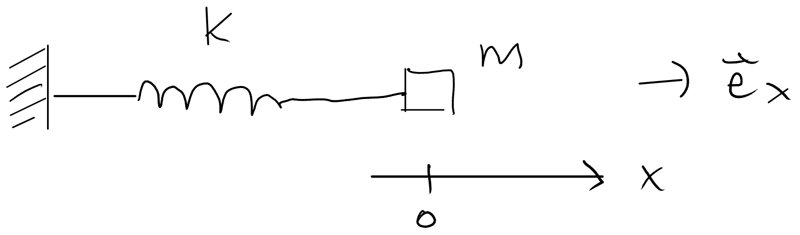
$$T = M^\alpha M^\beta L^\gamma T^{-2\beta}$$

$$\begin{cases} \alpha + \beta = 0 \Rightarrow \alpha = 1/2 \\ -2\beta = 1 \Rightarrow \beta = -1/2 \\ \gamma = 0 \Rightarrow \gamma = 0 \end{cases}$$

$$[\tau] = [k]^{-1/2} [m]^{1/2}$$

$$\tau \sim \left(\frac{m}{k}\right)^{1/2} \sim \sqrt{\frac{m}{k}}$$

Oscillatore armonico



II Newton : $m \dfrac{d^2x}{dt^2}\,\vec{e}_x = -kx\,\vec{e}_x$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$\frac{d^2x}{dt^2} = \frac{F}{m} = \text{cost}$$

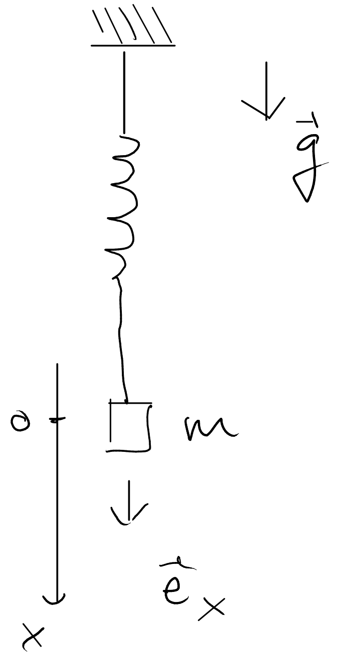
moto uniformemente accelerato

$$\frac{d^2x}{dt^2} = -\frac{\xi}{m}\frac{dx}{dt}$$

moto smorzato

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

moto armonico / oscillazioni armoniche



II Newton: $m \dfrac{d^2x}{dt^2}\vec{e}_x = -Kx\vec{e}_x + mg\vec{e}_x$

$$\frac{d^2x}{dt^2} = -\frac{K}{m}x + g$$

$$x_{eq} = \frac{mg}{K}$$

$$X = x - x_{eq} \quad \Rightarrow \quad x = X + x_{eq}$$

$$\frac{d^2X}{dt^2} = -\frac{K}{m}X - \cancel{\frac{K}{m}}\,\cancel{\frac{mg}{K}} + g$$

$$\boxed{\frac{d^2X}{dt^2} = -\frac{K}{m}X}$$ oscillatore armonico

Soluzione

$$\frac{d^2x}{dt^2} = -x \qquad x = A\sin t + B\cos t \qquad x = A\cos(t+\phi)$$

costanti → condizioni iniziali

costanti

$$\frac{d^2x}{dt^2} = -\frac{K}{m}x \quad \Rightarrow \quad \frac{d^2x}{dt^2} = -\omega^2 x \quad \text{con} \quad \omega^2 = \frac{K}{m} \qquad \omega = \sqrt{\frac{K}{m}}$$

$$\boxed{x(t) = A\cos(\omega t + \phi)}$$ → soluzione dell'oscillatore armonico

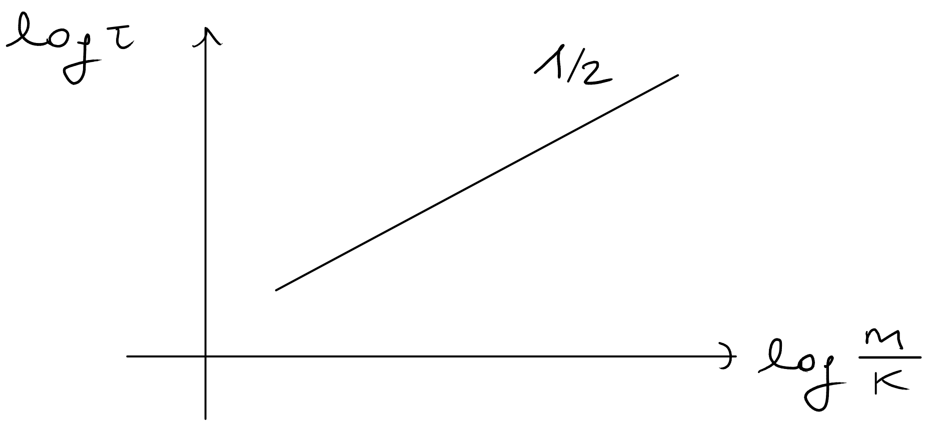
$$\frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$$

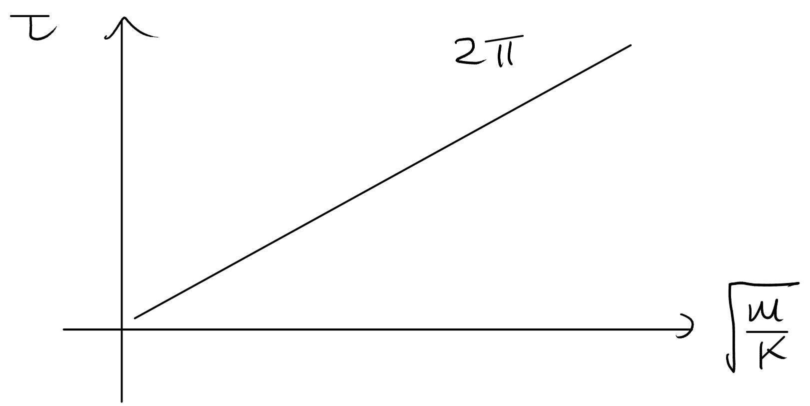
$$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi) = -\omega^2 \overbrace{A\cos(\omega t + \phi)}^{x(t)}$$

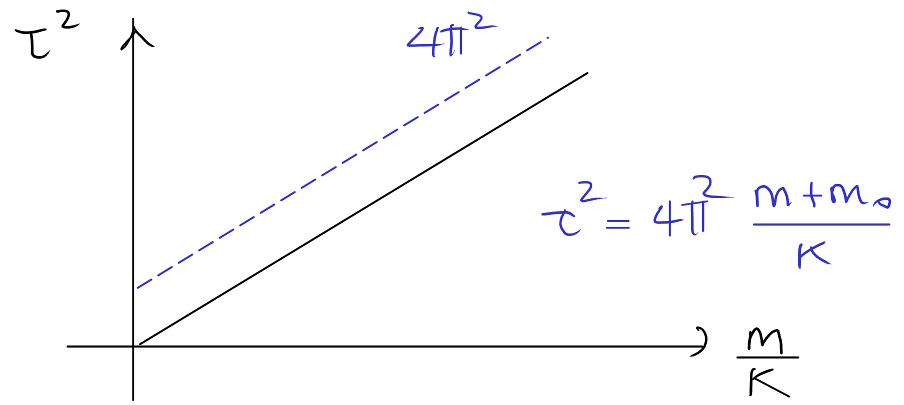
$f(t)$ è periodica se $\exists\, T > 0$ tale che $f(t+T) = f(t)$ → periodo = più piccolo T !

$$x(t+T) = x(t) \rightarrow A\cos(\omega t + \omega T + \phi) = A\cos(\omega t + \phi)$$

$$\cancel{\omega t} + \omega\tau + \cancel{\phi} = \cancel{\omega t} + \cancel{\phi} + 2\pi \Rightarrow \omega\tau = 2\pi \Rightarrow \tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K}} \sim \sqrt{\frac{m}{K}}$$





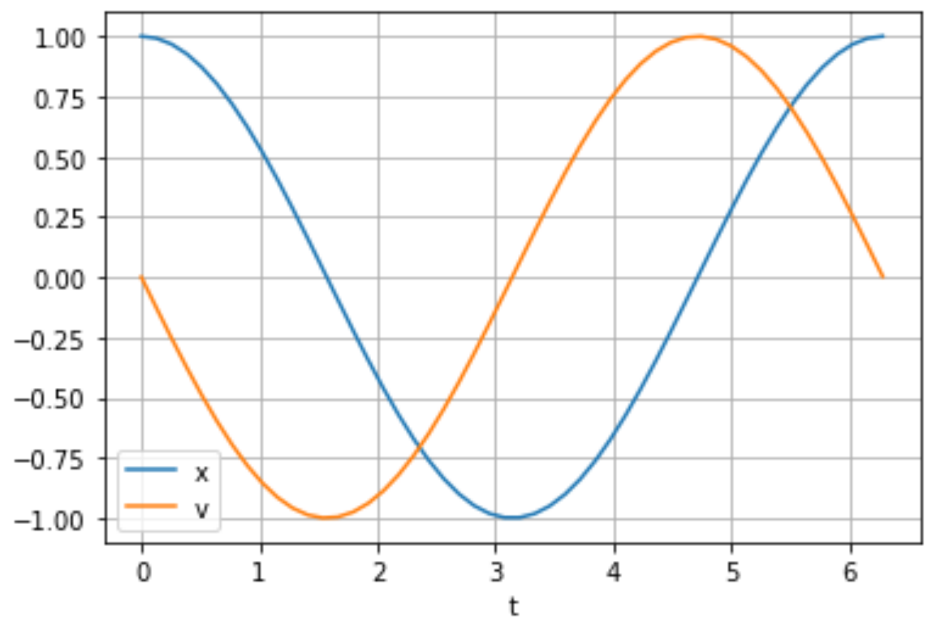


Condizioni iniziali : $t_i = 0$

$$\begin{cases} x(0) = x_i \\ v_x(0) = 0 \end{cases} \qquad \begin{cases} x_i = A\cos(\phi) & (1) \\ 0 = -A\omega\sin(\phi) & (2) \end{cases}$$

$$(2) \Rightarrow \phi = 0 \qquad (1)+(2) \Rightarrow A = x_i$$

$$\begin{cases} x(t) = x_i \cos\left(\sqrt{\frac{K}{m}}\, t\right) \\ v_x(t) = -x_i \sqrt{\frac{K}{m}} \sin\left(\sqrt{\frac{K}{m}}\, t\right) \end{cases}$$



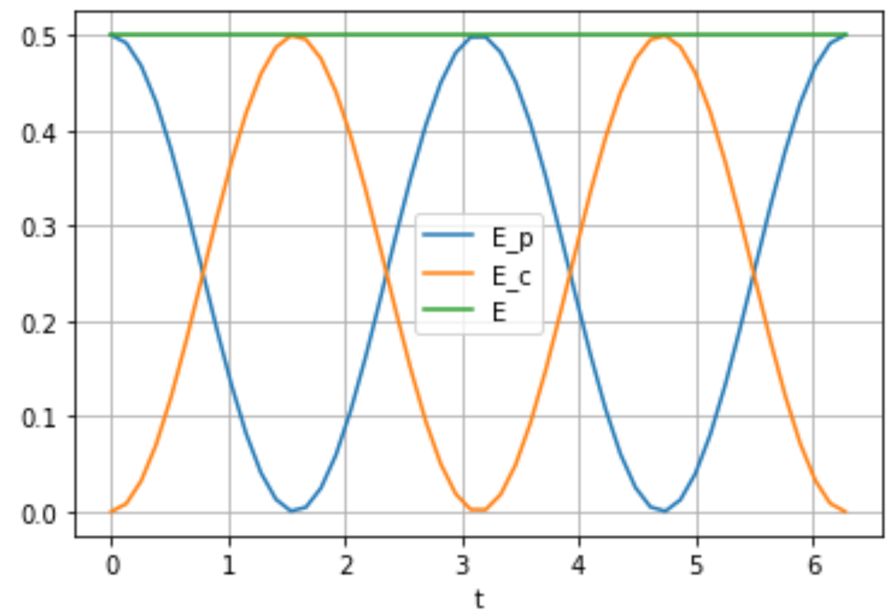
Energia meccanica

$$\omega^2 = \frac{k}{m}$$

$\{$corpo, molla, muro$\}$

$$E = E_c + E_p = \frac{1}{2} m A^2 \omega^2 \sin^2(\omega t + \phi) + \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

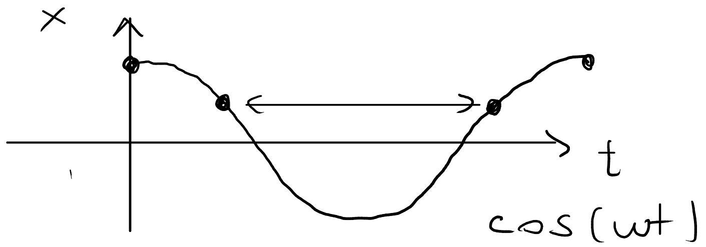
$$= \frac{1}{2} k A^2 \left[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right] = \frac{1}{2} k A^2$$



Oscillatore armonico come modello per diversi fenomeni fisici

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x}$$

oscillatore armonico



$$x(t) = \underset{\uparrow}{A}\cos(\omega t) + \underset{\uparrow}{B}\sin(\omega t)$$

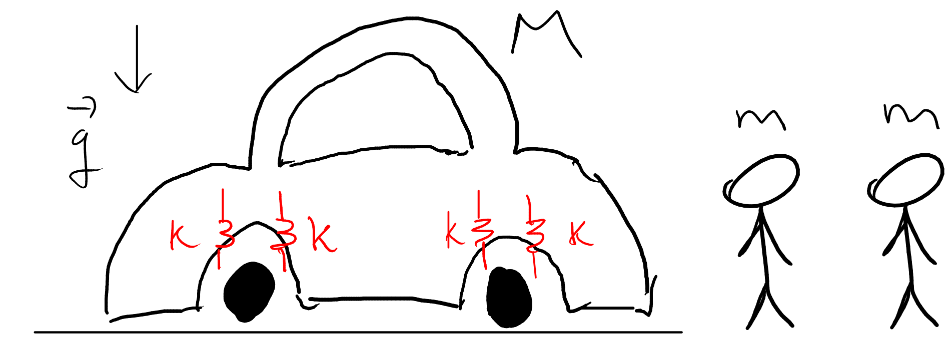
$$x(t) = \underset{\uparrow}{A}\cos(\omega t + \underset{\uparrow}{\phi})$$

Periodo: $\tau = \dfrac{2\pi}{\omega}$ $\quad [x(t+\tau) = x(t)]$

Frequenza: $f = \dfrac{1}{\tau}$ $\quad$ SI: $s^{-1} = Hz$ (Hertz)

Frequenza angolare: $\omega = \dfrac{2\pi}{\tau}$ $\quad$ SI: $\dfrac{rad}{s}$

Es: oscillazioni di un automobile



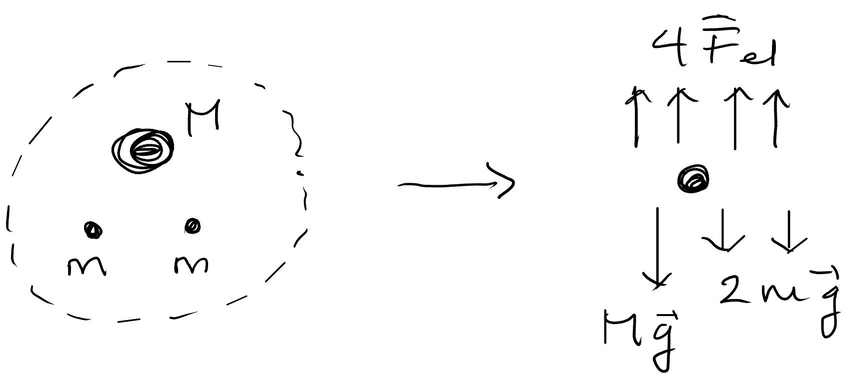
$M = 1300$ kg

$m = 80$ kg

$k = 20000$ N/m

f = frequenza delle oscillazioni verticali = ?

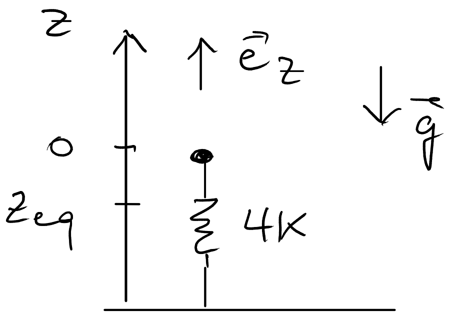
Sistema: $\{M, m, m\}$



Forze:
- peso
- forze elastiche

II Newton: $\Sigma \vec{F} = (M + 2m)\vec{a}$

$$(M+2m)\vec{g} + 4\vec{F}_{el} = (M+2m)\vec{a}$$



Equilibrio meccanico

$$(M+2m)\vec{g} + 4\vec{F}_{el} = \vec{0}$$

$$-(M+2m)g\,\vec{e}_z - 4k\,z_{eq}\vec{e}_z = \vec{0} \Rightarrow z_{eq} = -\frac{(M+2m)g}{4k} < 0$$

Eq. del moto:

$$(M+2m)\frac{d^2z}{dt^2}\vec{e}_z = -(M+2m)g\vec{e}_z - 4Kz\vec{e}_z$$

$$Z = z - z_{eq} \qquad z = Z + z_{eq}$$

$$(M+2m)\frac{d^2Z}{dt^2} = -(M+2m)g - 4K(Z + z_{eq})$$

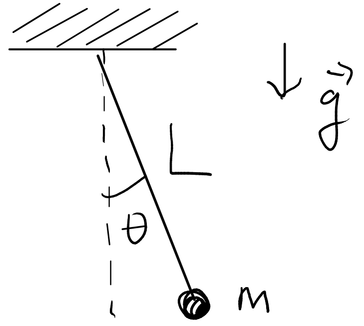
$$= -4KZ - \cancel{(M+2m)g} + \cancel{4K\frac{M+2m}{4K}}g = -4KZ$$

$$\frac{d^2Z}{dt^2} = -\underbrace{\frac{4K}{M+2m}}_{\omega^2}Z \quad \rightarrow \text{ oscillatore armonico}$$

Periodo: $\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M+2m}{4K}}$

Frequenza: $f = \frac{1}{\tau} = \frac{1}{2\pi}\sqrt{\frac{4K}{M+2m}} = \ldots = 1.18\ \text{Hz}$ $\square$

Pendolo semplice



- corpo = particella
- filo ideale (inestensibile, senza massa)
- no attrito

Sistema : $\{m\}$

Forze : peso, tensione

II Newton :

$$m\vec{a} = m\vec{g} + \vec{T}$$

$$\vec{a} = a_{\parallel}\vec{e}_{\parallel}$$

$$\theta = \frac{s}{L}$$

Proietto su $\vec{e}_{\parallel}$

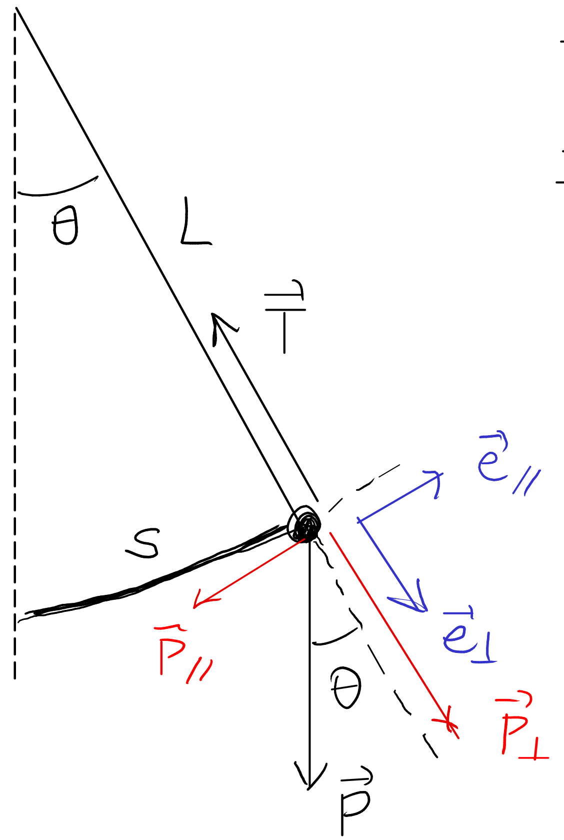
$$a_{\parallel} = \frac{d^2 s}{dt^2}$$

$$s = L\theta$$

↑ costante

$$-\cancel{m}g\sin\theta = \cancel{m}\frac{d^2 s}{dt^2}$$

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta \quad \rightarrow \quad \boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta}$$



Piccole oscillazioni : $\theta \ll 1$

$\sin\theta \approx \theta$ (Taylor I ordine) $\theta < 10^\circ \Rightarrow \sim 1\%$

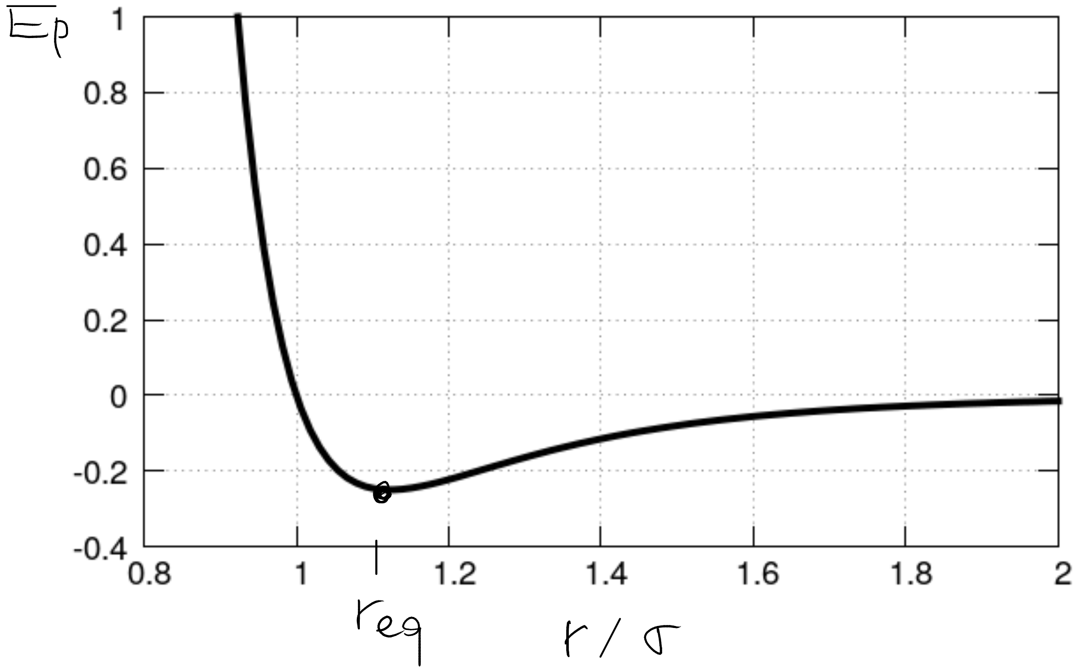
$$\frac{d^2\theta}{dt^2} \approx -\frac{g}{L}\theta \qquad \omega^2 = \frac{g}{l} \qquad \tau = 2\pi\sqrt{\frac{L}{g}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

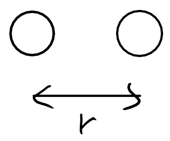
$$\theta(t) = \theta_0 \cos(\omega t + \phi) \qquad \tau \sim \sqrt{\frac{L}{g}}$$

Oscillazioni arbitrarie :

$$\tau = f(\theta_0)\sqrt{\frac{L}{g}}$$

Interazioni tra atomi neutri : gas rari Xe, Ne, Ar





Energia potenziale d'interazione

$$E_p(r) = 4\epsilon \left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6} \right]$$

$$\frac{dE_p}{dr}(r_{eq}) = 0$$

$$\frac{dE_p}{dr} = 4\epsilon \left[-12 \frac{\sigma^{12}}{r^{13}} + 6 \frac{\sigma^6}{r^7} \right]$$

$$\cancel{12}_{\,2} \frac{\sigma^{12}}{r_{eq}^{13}} = \cancel{6} \frac{\sigma^6}{r_{eq}^{7}} \rightarrow 2 \frac{\sigma^6}{r_{eq}^6} = 1$$

$$r_{eq}^6 = 2\sigma^6 \Rightarrow r_{eq} = \sqrt[6]{2}\,\sigma \approx 1.122\sigma$$

Taylor II ordine di E_p attorno a r_{eq}

$$E_p(r) = E_p(r_{eq}) + \frac{dE_p}{dr}(r_{eq})\,(r - r_{eq}) + \frac{1}{2}\frac{d^2E_p}{dr^2}(r - r_{eq})^2 + O((r-r_{eq})^3)$$

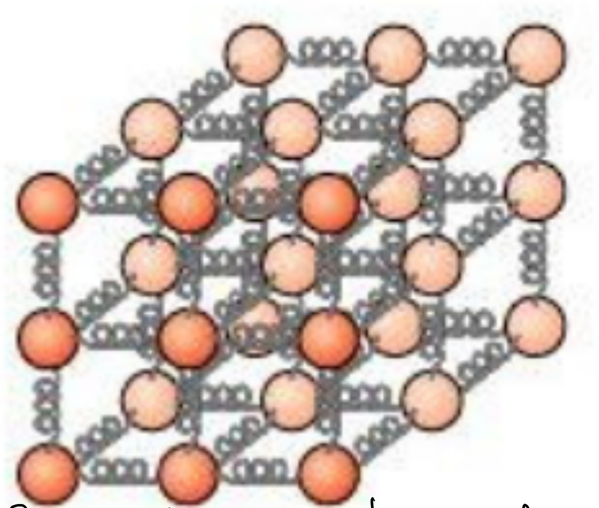
$$E_p(r) \approx E_p(r_{eq}) + \frac{1}{2}\frac{d^2E_p}{dr^2}(r_{eq})\,(r - r_{eq})^2$$

$$m \frac{d^2 r}{dt^2} = -\frac{dE_p}{dr} = -\frac{d^2 E_p}{dr^2}(r_{eq})\,(r - r_{eq}) \qquad x = r - r_{eq}$$

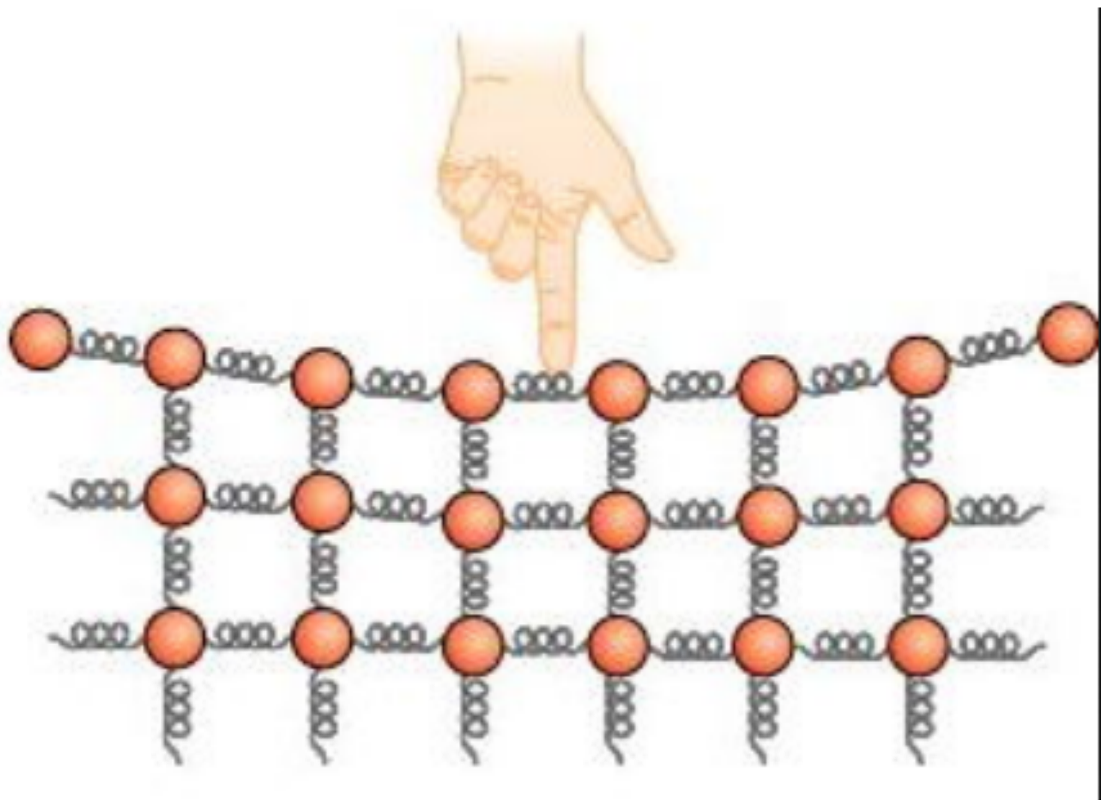
$$\frac{d^2 x}{dt^2} = -\frac{1}{m}\frac{d^2 E_p}{dr^2}(r_{eq})\,x \quad \rightarrow \text{ oscillatore armonico}$$

Vibrazioni e deformazioni nei solidi

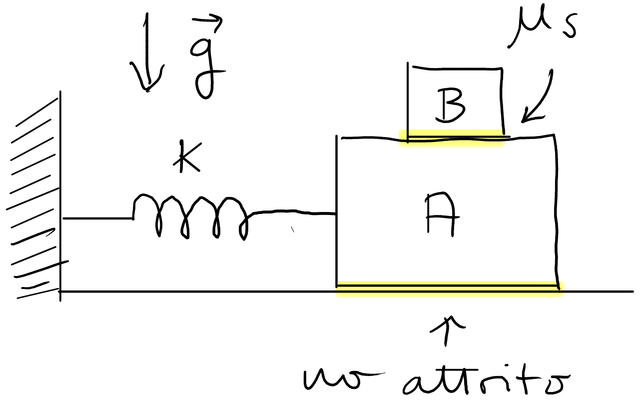
- oscillatori armonici indipendenti
- oscillatori armonici accoppiati → modi normali



Source: compadre.org



Problema di ricapitolazione (SJ 12.47)

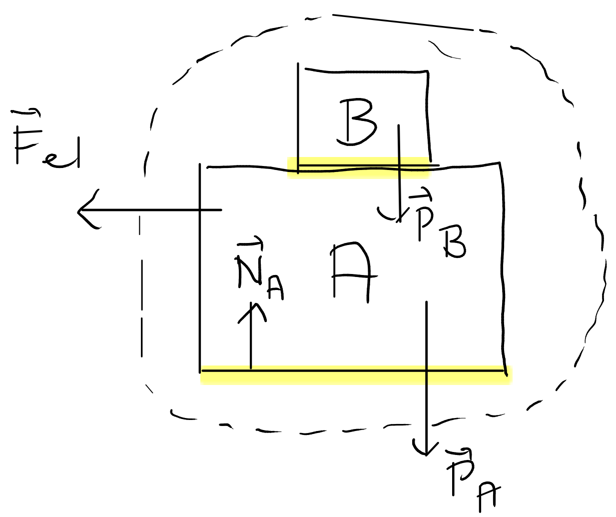


Altrito tra A e B ($\mu_s = 0.6$), non tra A e il suolo

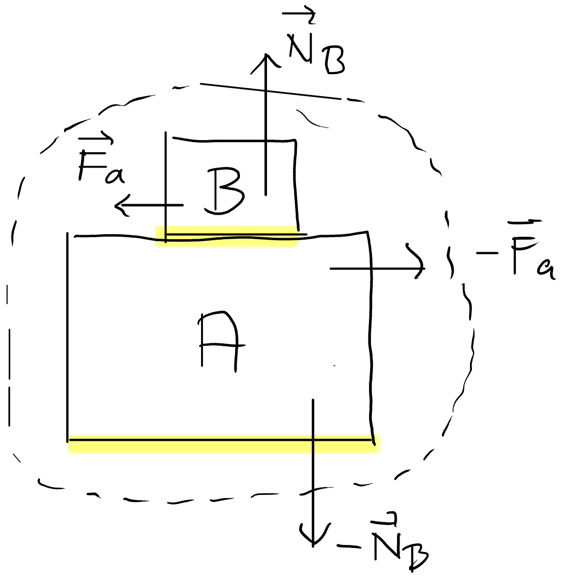
Massima ampiezza Δx delle oscillazioni senza che B scivoli su A?

Oscillazioni armoniche di A+B : $f = 1.5$ Hz

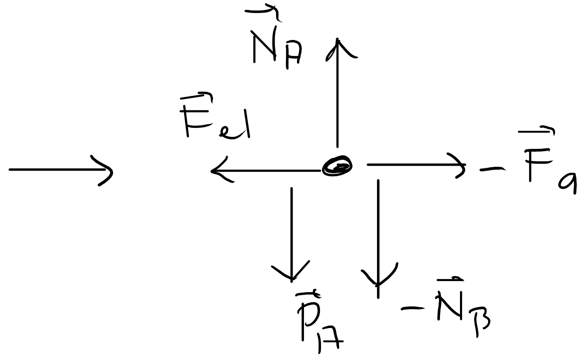
Sistema : $\{A, B\}$



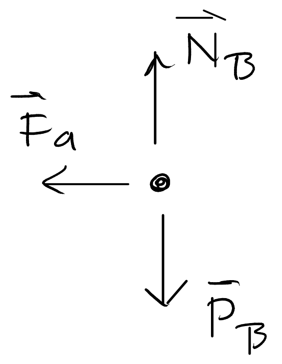
Forze esterne



Forze interne



A



B

II Newton :

$$\begin{cases} m_A \vec{a}_A = \vec{N}_A + \vec{P}_A + \vec{F}_{el} - \vec{F}_a - \vec{N}_B \\ m_B \vec{a}_B = \vec{N}_B + \vec{P}_B + \vec{F}_a \end{cases}$$

Corpi solidali : $\vec{a}_A = \vec{a}_B = \vec{a}$

Equilibrio lungo direzione verticale :

$$\begin{cases} \vec{N}_A + \vec{P}_A - \vec{N}_B = \vec{0} \\ \vec{N}_B + \vec{P}_B = \vec{0} \end{cases} \Rightarrow \begin{cases} \vec{N}_A = -\vec{P}_A - \vec{P}_B = -(\vec{P}_A + \vec{P}_B) \\ \vec{N}_B = -\vec{P}_B \end{cases}$$

$$\begin{cases} m_A \vec{a} = -\vec{P}_B + \vec{F}_{el} - \vec{F}_a + \vec{P}_B = \vec{F}_{el} - \vec{F}_a \\ m_B \vec{a} = \vec{F}_a \end{cases}$$

$\longrightarrow \vec{e}_x$ $\qquad \vec{a} = a_x \vec{e}_x$

 $\rightarrow \Delta x > 0$ $\qquad \vec{F}_a = F_a \vec{e}_x$

$$\begin{cases} m_A a_x = -K \Delta x - F_a \\ m_B a_x = F_a \end{cases}$$

Non è banale anticipare il segno di Fa
(dovrà essere opposto al moto imminente)
Il risultato finale è indipendente da ciò

$$\begin{cases} a_x = -\dfrac{K}{m_A}\Delta x - \dfrac{F_a}{m_A} \\ a_x = \dfrac{F_a}{m_B} \end{cases} \Rightarrow -\frac{k}{m_A}\Delta x = \frac{F_a}{m_A} + \frac{F_a}{m_B} = \left(\frac{1}{m_A} + \frac{1}{m_B}\right) F_a$$

$$F_a = -\frac{K \Delta x}{\cancel{m_A}} \frac{\cancel{m_A}\, m_B}{m_A + m_B} = -K \Delta x \frac{m_B}{m_A + m_B}$$

Regime attrito statico

$$|F_a| \leq \mu_s |\vec{N}_B| = \mu_s m_B g$$

$$\Delta x \; K \frac{\cancel{m_B}}{m_A + m_B} \leq \mu_s \cancel{m_B} g$$

$$\Delta x \leq \frac{\mu_s (m_A + m_B) g}{K}$$

$$= 6{,}63 \text{ cm}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{K}{m_A + m_B}} \Rightarrow 4\pi^2 f^2 = \frac{K}{m_A + m_B}$$

$$\Delta x \leq \frac{\mu_s g}{f}$$ $\rightarrow$ indipendente da m_A, m_B

$\square$