26 Aprile

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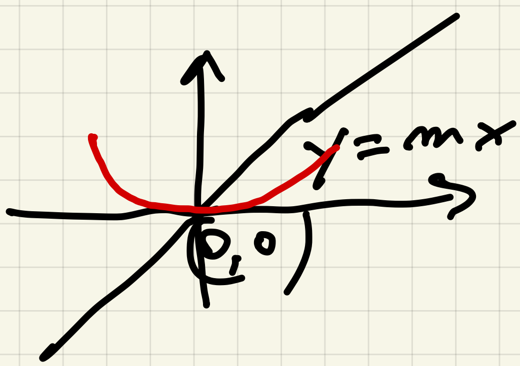
$$f(x,y)=\begin{cases} 0 & \text{se} \quad x=0 \\ \dfrac{e^{-\frac{1}{x^2}}\,y}{e^{-\frac{2}{x^2}}+y^2} & \quad x\neq 0 \end{cases}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) \quad \text{non esiste}$$



$$f(x,mx)=\frac{e^{-\frac{1}{x^2}}\,mx}{e^{-\frac{2}{x^2}}+m^2x^2}$$

$$\lim_{x\to 0} f(x,mx)=0$$

$$=\lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}\,mx}{m^2x^2}=$$

$$=\frac{1}{m}\lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}}{x}=0$$

$$y = e^{-\frac{1}{x^2}}$$

$$f\left(x, e^{-\frac{1}{x^2}}\right) = \frac{e^{-\frac{1}{x^2}}\, e^{-\frac{1}{x^2}}}{e^{-\frac{2}{x^2}} + e^{-\frac{2}{x^2}}} = \frac{1}{2}$$

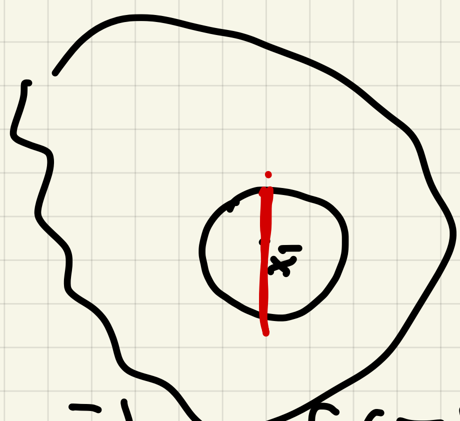
Derivate

Derivate parziali

<u>Def</u> $f : A \subseteq \mathbb{R}^d \to \mathbb{R}^N$

A un aperto e sia $\bar{x} \in A$

$i \in \{1, \dots, d\}$



se esiste il limite, ed è finito

$$\lim_{x_i \to \bar{x}_i} \frac{f(\bar{x}_1, \dots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \dots, \bar{x}_d) - f(\bar{x})}{x_i - \bar{x}_i} = \partial_i f(\bar{x}) = \frac{\partial}{\partial x_i} f(\bar{x}) = f_{x_i}(\bar{x})$$

$N = 1$

Se esistono tutte $\partial_1 f(\bar{x}), \dots, \partial_d f(\bar{x})$

$$\nabla f(\bar{x}) = \operatorname{grad} f(\bar{x}) =$$
$$= \left(\partial_1 f(\bar{x}), \dots, \partial_d f(\bar{x}) \right)$$

<u>Es</u> $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) = \sin(xy)$$

Per ogni fissato y,

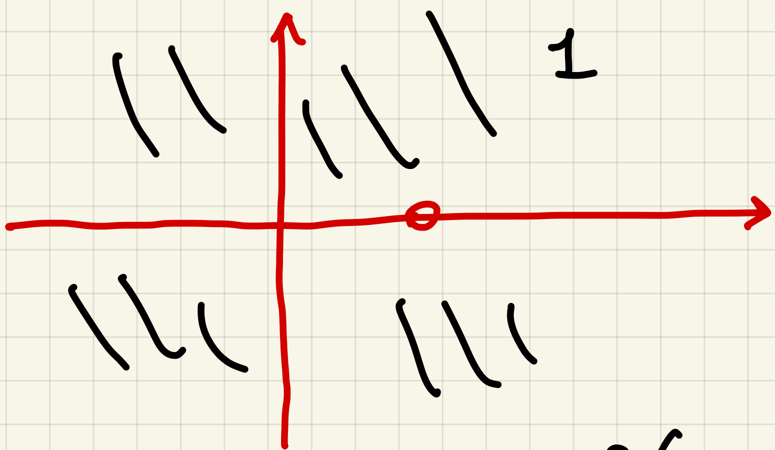
$x \longrightarrow \sin(xy)$ è derivabile $\forall x$

$$\partial_x f(x,y) = \partial_x [\sin(xy)] =$$
$$= \cos(xy) \; y$$

$$\partial_y f(x,y) = \cos(xy) \; x$$

$$\partial_x^2 f(x,y) = \partial_x [\partial_x f(x,y)] =$$
$$= \partial_x [\cos(xy) \; y] =$$
$$= -\sin(xy) \; y^2$$

$$f(x,y) = \begin{cases} 1 & \text{se} \quad xy \neq 0 \\ 0 & \text{se} \quad xy = 0 \end{cases}$$



$x \to f(x,0)$

$$\partial_x f(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x}$$

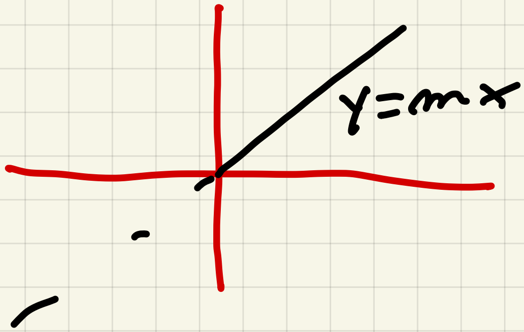
$$= \lim_{x\to 0} \frac{0}{x} = 0 = \partial_x f(0,0)$$

$$\partial_y f(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{0}{y} = 0 \Rightarrow$$

$$\partial_x f(0,0) = 0 = \partial_y f(0,0)$$

$\lim_{(x,y)\to(0,0)} f(x,y)$ non esiste



$m \neq 0$

$$\lim_{x\to 0} f(x, mx) = \lim_{x \to 0} 1 = 1$$

$$\lim_{x\to 0} f(x,0) = 0$$

$$f : A \subseteq \mathbb{R}^d \to \mathbb{R}^N$$

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_N(x) \end{pmatrix}$$

$$Df(x) = \begin{pmatrix} \partial_1 f_1(x) & \partial_2 f_1(x) & \cdots & \partial_d f_1(x) \\ \partial_1 f_2(x) & \partial_2 f_2(x) & \cdots & \partial_d f_2(x) \\ \vdots & & & \\ \partial_1 f_N(x) & \partial_2 f_N(x) & \cdots & \partial_d f_N(x) \end{pmatrix}$$

$N \times d$ matrice Jacobiana

$J f(x)$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

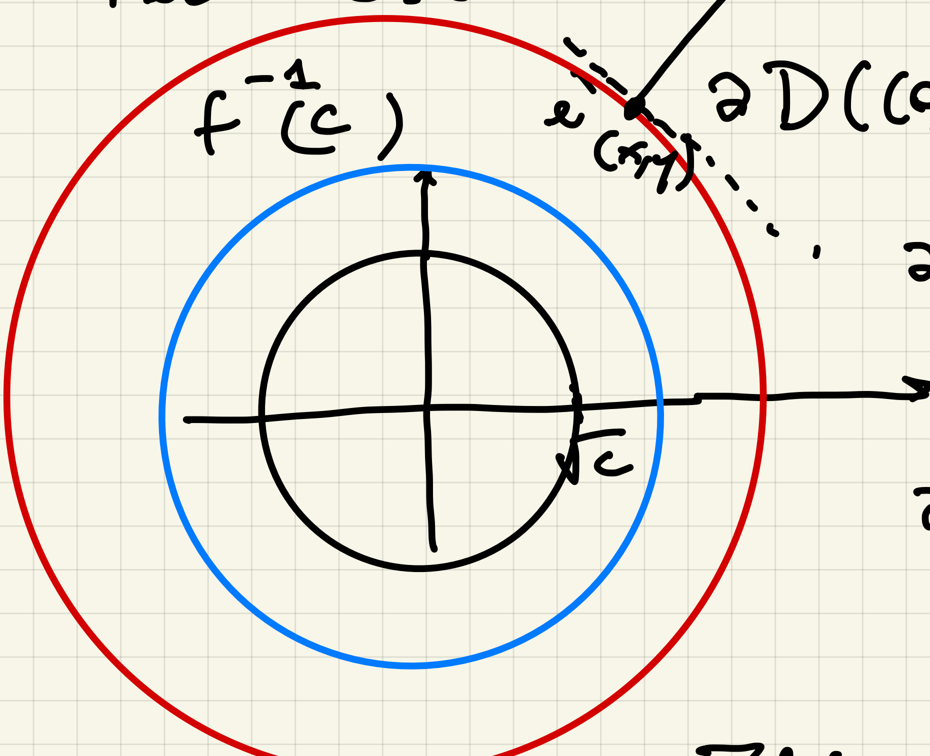
$$f(x,y) = x^2 + y^2$$

$\forall\ c \in \mathbb{R}$ la curva di livello c di f è $f^{-1}(c) =$

$= \{ (x,y) : \ x^2 + y^2 = c \}$

Per $c \geq 0$

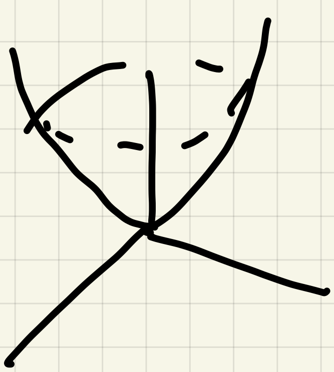
$f^{-1}(c)$ è $\partial D((0,0), \sqrt{c})$



$$\partial_x f(x,y) = \partial_x (x^2 + y^2) = 2x$$

$$\partial_y f(x,y) = \partial_y (x^2 + y^2) = 2y$$

$$\nabla f(x,y) = (2x, 2y) = 2(x,y)$$



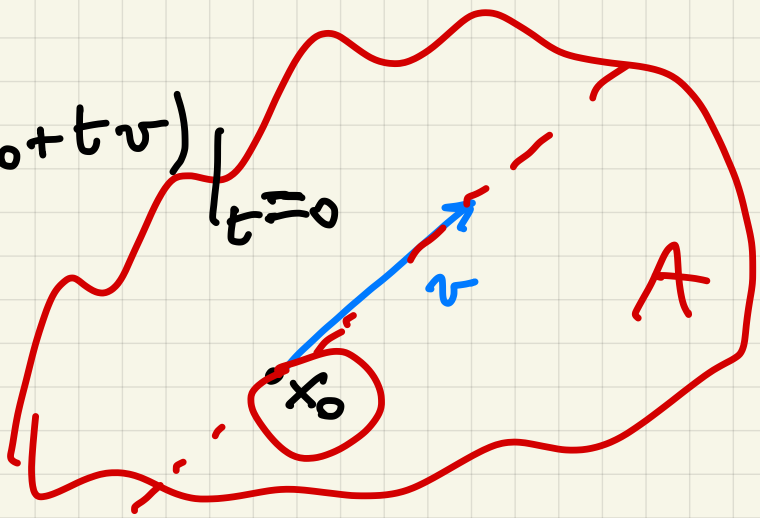
$z = f(x,y)$

$x_0 \in \mathbb{R}^d \qquad r > 0$

$$D_{\mathbb{R}^d}(x_0, r) = \{ x \in \mathbb{R}^d : |x - x_0| < r \}$$

<u>Def</u> $f : A \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}^N$ A aperto, $x_0 \in A$, $0 \neq v \in \mathbb{R}^d$. Diciamo che f ammette in x_0 la derivata direzionale in direzione v se esiste

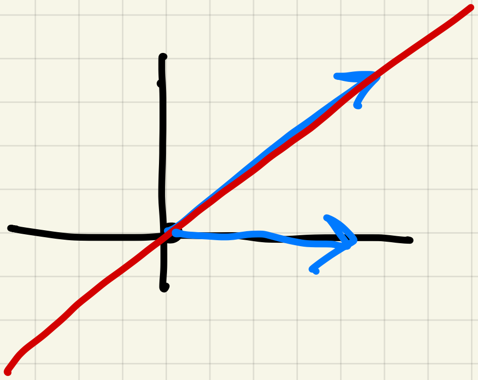
$$\frac{d}{dt} f(x_0 + tv) \Big|_{t=0}$$



$$= \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

Es 1 $\quad f(x,y) = \begin{cases} \dfrac{e^{-\frac{1}{x^2}}\, y}{e^{-\frac{2}{x^2}} + y^2} & \text{se } x \neq 0 \end{cases}$

se $x = 0$



$v = (a, b)$

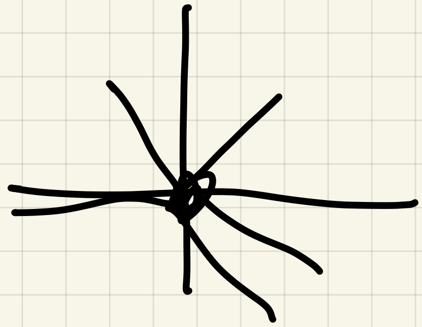
$a \neq 0 \qquad b \neq 0$

$$\frac{f(ta, tb)}{t} = \frac{e^{-\frac{1}{a^2 t^2}}\, tb}{e^{-\frac{2}{a^2 t^2}} + t^2 b^2}$$

$$\lim_{t \to 0} \frac{e^{-\frac{1}{a^2 t^2}}\, tb}{t^2 b^2} =$$

$$= \lim_{t \to 0} \frac{e^{-\frac{1}{a^2 t^2}}}{tb} = 0$$

Tutte le derivate direzionali in $(0,0)$ esistono e sono nulle senza che f sia continua in $(0,0)$



Def (Differenziabilità) Sia $f : A \subseteq \mathbb{R}^d \to \mathbb{R}^N$

$x_0 \in A$ $\qquad L : \mathbb{R}^d \to \mathbb{R}^N$

$$L \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^N)$$

matrice $N \times d$

Diremo che f è differenziabile in x_0 con differenziale L se

$$\lim_{\underset{\mathbb{R}^d}{h} \to \underset{\mathbb{R}^d}{0}} \frac{\overbrace{f(x_0+h) - f(x_0) - Lh}^{\in \mathbb{R}^N}}{|h|} = 0 \quad (*)$$

Osservazione $(*)$ è equivalente a scrivere che

$$f(x_0+h) - f(x_0) = Lh + o(|h|)$$

<u>Lemma</u> Se f è differenziabile in x_0 allora f è continua in x_0.

<u>Dim</u> vogliamo $\lim_{h \to 0} f(x_0+h) = f(x_0)$

$$\lim_{h \to 0} \left[f(x_0+h) - f(x_0) \right] = 0$$

$$f(x_0+h) - f(x_0) = L\,h + o(|h|)$$

$$\lim_{h \to 0} \left[f(x_0+h) - f(x_0) \right] =$$

$$\lim_{h \to 0} \left[L\,h + o(|h|) \right] = 0$$

$$= \lim_{h \to 0} L\,h + \underbrace{\lim_{h \to 0} \frac{o(|h|)}{|h|} \, |h|}$$

0 ← $\frac{o(|h|)}{|h|}$, 0 ← $|h|$

$$0 \leq |L\,h| \leq |L| \, \underbrace{|h|}_{\downarrow\, 0}$$

<u>Lemma</u> $f : A \to \mathbb{R}^N$ $\quad x_0 \in A$

f differenziabile con differenziale

$L \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^N)$. Allora in x_0

esistono $\partial_1 f(x_0), \dots, \partial_d f(x_0)$

con
$$L = Df(x_0) =$$
$$= \left(\partial_1 f(x_0), \dots, \partial_d f(x_0)\right)$$
$$= \begin{pmatrix} \partial_1 f_1(x_0), & \cdots & \partial_d f_1(x_0) \\ & & \\ \partial_1 f_N(x_0), & \cdots, & \partial_d f_N(x_0) \end{pmatrix}$$

<u>Dim</u> Supponiamo che $D(x_0, r_0) \subset A$ $\quad r_0 > 0$

$$f(x_0 + h) - f(x_0) - Lh = o(|h|)$$

$\forall\ |h| < r_0$

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \\ i \text{ esimo} \\ \\ \\ -d \end{array} \quad \text{in } \mathbb{R}^d$$

$t > 0$

$$f(x_0+t e_i)-f(x_0)-L t e_i = o(|t|)$$

$$\frac{f(x_0+t e_i)-f(x_0)-L t e_i}{t} = \frac{o(t)}{t}$$

$$\lim_{t\to 0^+}\left[\frac{f(x_0+t e_i)-f(x_0)}{t} - L e_i\right] =$$

$$\lim_{t\to 0^+} \frac{o(t)}{t} = 0$$

$$\lim_{t\to 0^+} \frac{f(x_0+t e_i)-f(x_0)}{t} = L e_i$$

$t<0$

$$\frac{f(x_0+te_i)-f(x_0)-L t e_i}{|t|} = \frac{o(|t|)}{|t|}$$

$$\frac{f(x_0+te_i)-f(x_0)}{-t} + L e_i = \frac{o(|t|)}{|t|}$$

$$\lim_{t \to 0^-} \frac{f(x_0 + t e_i) - f(x_0)}{-t} = -L e_i$$

$$\lim_{t \to 0^-} \frac{f(x_0 + t e_i) - f(x_0)}{t} = L e_i$$

$$\lim_{t \to 0^+} \frac{f(x_0 + t e_i) - f(x_0)}{t} = L e_i$$

$$\lim_{t \to 0} \frac{f(x_0 + t e_i) - f(x_0)}{t} = L e_i$$

$$=: \partial_i f(x_0)$$

$\partial_i f(x_0) = L e_i$ è la colonna i esimo di L

$$L = \left(\partial_1 f(x_0), \partial_2 f(x_0), \ .., \partial_d f(x_0) \right)$$

<u>Teor</u> Sia $f : A \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}^N$ $x_0 \in A$.

Supponiamo che $\partial_1 f(x), \ldots, \partial_d f(x)$ esistono

$\forall x \in A$ e che inoltre siano continue ~~rel~~ in x_0. Allora f è differenziabile in x_0

<u>Dim</u> Non è restrittivo considerare $N=1$.

$$f(x_0+h)-f(x_0)-Df(x_0)h = o(|h|)$$

$$f(x_{01}+h_1, x_{02}+h_2, .., x_{0d}+h_d) - f(x_{01}, .., x_{0d})$$

$$= \partial_1 f(x_0) h_1 + \cdot + \partial_d f(x_0) h_d + o(|h|)$$

$$f(x_{01}+h_1, x_{02}+h_2, .. x_{0d}+h_d) - f(x_{01}, x_{02}+h_2, .., x_{0d}+h_d)$$

$$+ f(x_{01}, x_{02}+h_2, .., x_{0d}+h_d) - f(x_{01}, x_{02}, x_{03}+h_3, .., x_{0d}+h_d)$$

$+$

$\vdots$

$$+ f(x_{01}, .., x_{0d-1}, x_{0d}+h_d) - f(x_0)$$

$$= \sum_{j=1}^{d} \Big[f(x_{01}, .., x_{0j}+h_j, x_{0j+1}+h_{j+1} .., x_{0d}+h_d) - f(x_{01}, .., x_{0j}, x_{0j+1}+h_{j+1}, .., x_{0d}+h_d) \Big]$$

$$= \sum_{j=1}^{d} h_j \partial_j f(x_{01}, .., x_{0j-1}, x_{0j}+t_j^* h_j, x_{0j+1}+h_{j+1}, .., x_{0d}+h_d)$$

$$-\sum_{j=1}^{d} h_j \partial_j f(x_0)) \overset{?}{=} \frac{o(|h|)}{|h|}$$

$$\left| \sum_{j=1}^{d} \frac{h_j}{|h|} [\partial_j f(x_{01}, \dots, x_{0j-1}, x_{0j} + t_j^* h_j, x_{0j+1} + h_{j+1} \dots) - \partial_j f(x_0)] \right|$$

$$t_j^* \in (0, 1)$$

$$\le \sum_{j=1}^{d} \underbrace{\frac{|h_j|}{|h|}}_{\le 1} \underbrace{|\partial_j f(x_{01}, \dots, x_{0j-1}, x_{0j} + t_j^* h_j, \dots) - \partial_j f(x_0)|}_{o(1)}$$