28 Aprile

Thm $\quad f : X \subseteq \mathbb{R}^d \longrightarrow Y \subseteq \mathbb{R}^N$

$\qquad g : Y \subseteq \mathbb{R}^N \longrightarrow \mathbb{R}^M$

X, Y aperti

$x_0 \in X, \ y_0 \in Y \qquad y_0 = f(x_0)$

se f è diff in x_0

$\quad g$ " " " y_0

$\Rightarrow \quad g \circ f : X \longrightarrow \mathbb{R}^M$ è diff in x_0

con

$$\underbrace{D(g \circ f)(x_0)}_{M \times d} = \underbrace{Dg(y_0)}_{M \times N} \ \underbrace{Df(x_0)}_{N \times d}$$

Dim $\qquad y_0 = f(x_0)$

$$f(x_0+h) - f(x_0) = Df(x_0)h + o(|h|)$$

$$g(y_0+k) - g(y_0) = Dg(y_0)k + o(|k|)$$

$$g(f(x_0+h)) - g(f(x_0)) =$$

$$= g(\underbrace{f(x_0+h) - y_0}_{k} + y_0) - g(y_0)$$

$$= Dg(y_0)(f(x_0+h) - f(x_0)) +$$

$$+ o(|f(x_0+h) - f(x_0)|)$$

$$= Dg(y_0)(Df(x_0)h + o(|h|))$$

$$+ o(|Df(x_0) + o(|h|)|)$$

$$= Dg(y_0)Df(x_0)h +$$

$$+ \underbrace{Dg(y_0)o(|h|) + o(|Df(x_0)h + o(|h|)|)}_{o(|h|)}$$

$$g(f(x_0+h)) - g(f(x_0)) = Dg(y_0)Df(x_0)h + o(|h|)$$

$$\Longrightarrow \quad D(g\circ f)(x_0) = Dg(y_0)\,Df(x_0)$$

Ci resta da dimostrare

$$\underbrace{Dg(y_0)\,o(|h|)}_{o(|h|)} + \underbrace{o(|Df(x_0)h + o(h)|)}_{o(|h|)} = o(|h|)$$

$$0 \leq \frac{|Dg(y_0)\,o(|h|)|}{|h|} \leq |Dg(y_0)| \frac{|o(h)|}{|h|} \xrightarrow{h\to 0} 0$$

$$f(x_0+h) - f(x_0) = Df(x_0)h + o_1(|h|)$$

$$g(y_0+k) - g(y_0) = Dg(y_0)k + o_2(|k|)$$

$$o_2(|Df(x_0)h + o_1(h)|) = o(|h|)$$

$$o_j(|h|)$$

$$\forall \varepsilon > 0 \quad \exists\, \delta_\varepsilon^{(j)} > 0 \text{ t.c. } |h| < \delta_\varepsilon^{(j)}$$

$$\Longrightarrow \quad |o_j(|h|)| < \varepsilon |h|$$

$$|Df(x_0)h + o_1(h)| \leq \qquad h \in \mathbb{R}^d$$

$$|h| < \delta^{(1)}_{\frac{1}{2}\delta^{(2)}_\varepsilon} \quad , \quad |h| < \frac{\frac{1}{2}\delta^{(2)}_\varepsilon}{|Df(x_0)|} \quad , \quad |h| < 1$$

$$|Df(x_0)|\,|h| + \frac{1}{2}\delta^{(2)}_\varepsilon |h| =$$

$$= |Df(x_0)|\,|h| + \frac{1}{2}\delta^{(2)}_\varepsilon |h|$$

$$< \delta^{(2)}_\varepsilon$$

$$o_2(|Df(x_0)h + o_1(|h|)|) < \varepsilon\,|Df(x_0)h + o_1(|h|)|$$

$$\leq \varepsilon\left(|Df(x_0)| + \frac{\delta^1_\varepsilon}{2}\right)|h|$$

$$\leq \varepsilon\left(|Df(x_0)| + 1\right)|h|$$

se $\delta^{(3)}_\varepsilon = \min\left\{1,\ \delta^{(1)}_{\frac{1}{2}\delta^{(2)}_\varepsilon},\ \frac{\frac{1}{2}\delta^{(2)}_\varepsilon}{|Df(x_0)|}\right\}$

ho $|h| < \delta^{(3)}_\varepsilon$

$$\Rightarrow \underbrace{o_2\left(|Df(x_0)h + o_1(|h|)|\right)}^{o_3(|h|)} < \varepsilon\left(|Df(x_0)| + 1\right)|h|$$

$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad$ t.c. $\quad |h| < \delta_\varepsilon$

$$\Rightarrow \quad |o_3(|h|)| < \varepsilon \, |h|$$

$$\delta_\varepsilon = \delta^{(3)}_{\frac{\varepsilon}{|Df(x_0)|+1}}$$

Cor Sia $g : \underset{\text{aperto}}{Y} \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}^N$

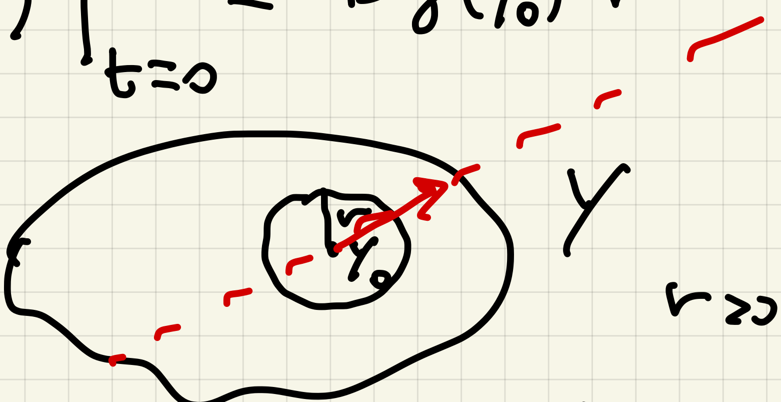
$y_0 \in Y$ $\quad g$ diff in y_0

Sia $v \in \mathbb{R}^d$ non nullo

Allora

$$\frac{d}{dt} g(y_0 + tv)\Big|_{t=0} = Dg(y_0)\, v$$

Dim



$\exists \, r > 0$ t.c. $\quad D(y_0, r) \subset Y$

$$f(t) = y_0 + tv \quad \in \mathbb{R}^d$$

per $\quad \boxed{|t| < \frac{r}{|v|}}$

risulta che $f(t) \in D(y_0, r)$

$$|f(t) - y_0| = |t\,v| \leq |t|\,|v| < \frac{r}{\cancel{|v|}}\,\cancel{|v|}$$

$$g \circ f(t) = g(y_0 + tv)$$

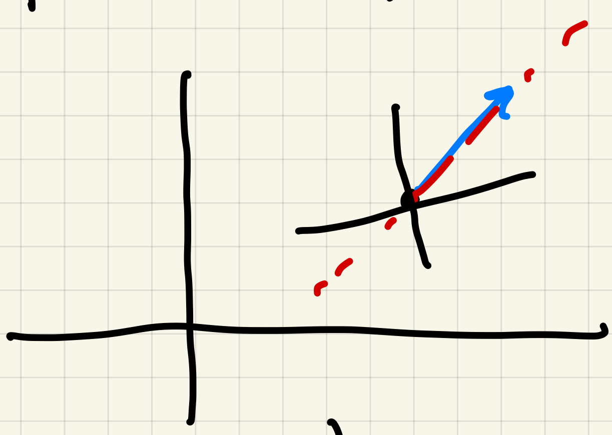
$$\frac{d}{dt}\, g(y_0 + tv)\Big|_{t=0} = Dg(y_0)\, f'(0) =$$
$$= Dg(y_0)\, v$$

<u>Teor</u> $f : X \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}$, $x_0 \in \mathring{X}$

f diff in x_0. Supponiamo che x_0 sia di max/min locale. Allora $Df(x_0) = 0$

<u>Dim</u> $v \in \mathbb{R}^d \setminus \{0\}$

$t \longrightarrow f(x_0 + tv)$ ha in 0 un max/min locale



$\forall\, v$

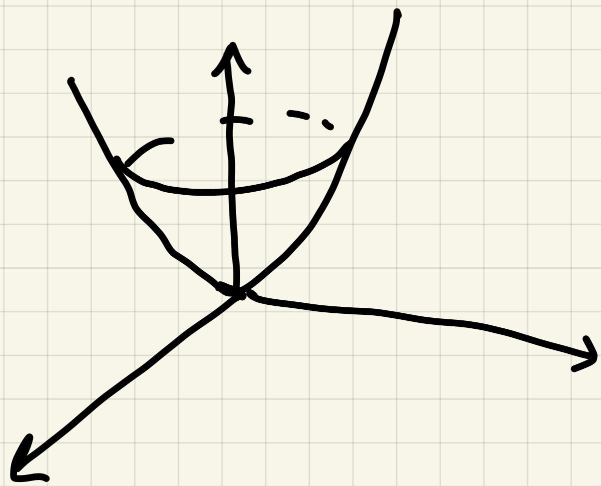
$$0 = \frac{d}{dt} f(x_0 + tv)\Big|_{t=0} = Df(x_0) \cdot v$$

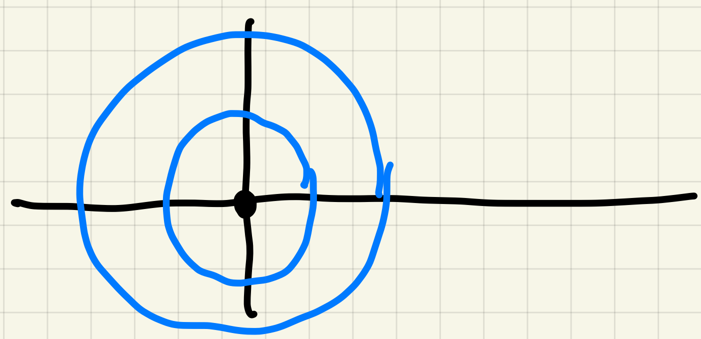
$$\Rightarrow \quad Df(x_0) = 0$$

<u>Esempi</u> $\quad f(x,y) = x^2 + y^2$

$$\partial_x f(x,y) = 2x = 0 \iff x = 0$$
$$\partial_y f(x,y) = 2y = 0 \qquad\qquad y = 0$$

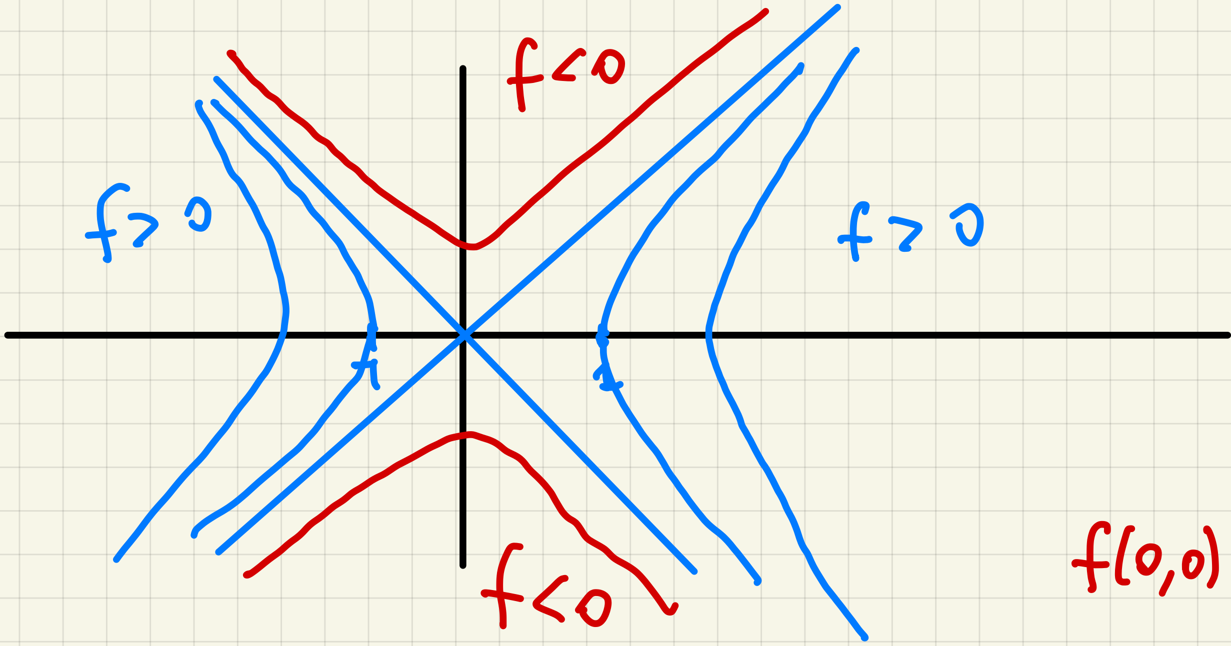
$$z = x^2 + y^2$$





$$f(x,y) = x^2 - y^2$$
$$\nabla f(x,y) = (\partial_x f, \partial_y f) = (2x, -2y)$$

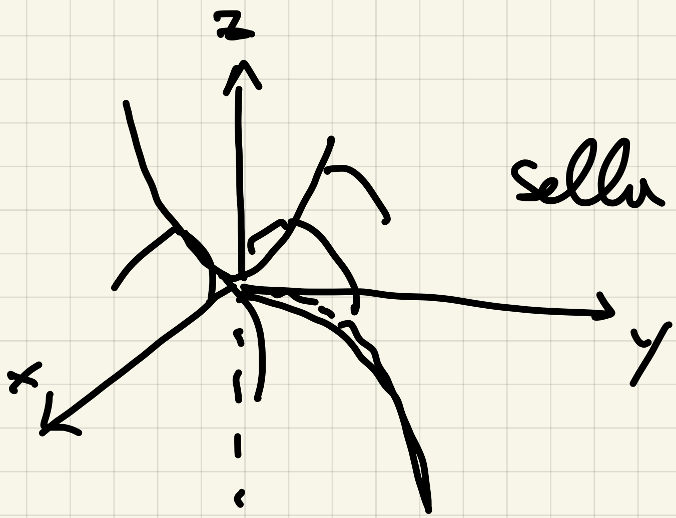
$f(x,y) = 0 \qquad (x-y)(x+y) = 0$



$f(0,0)=0$

$f(x,y)=1 \qquad x^2-y^2=1$

$z=x^2-y^2$



sella

$$f(x,y) = (y+x^2)\, e^{-x^2-y^2}$$

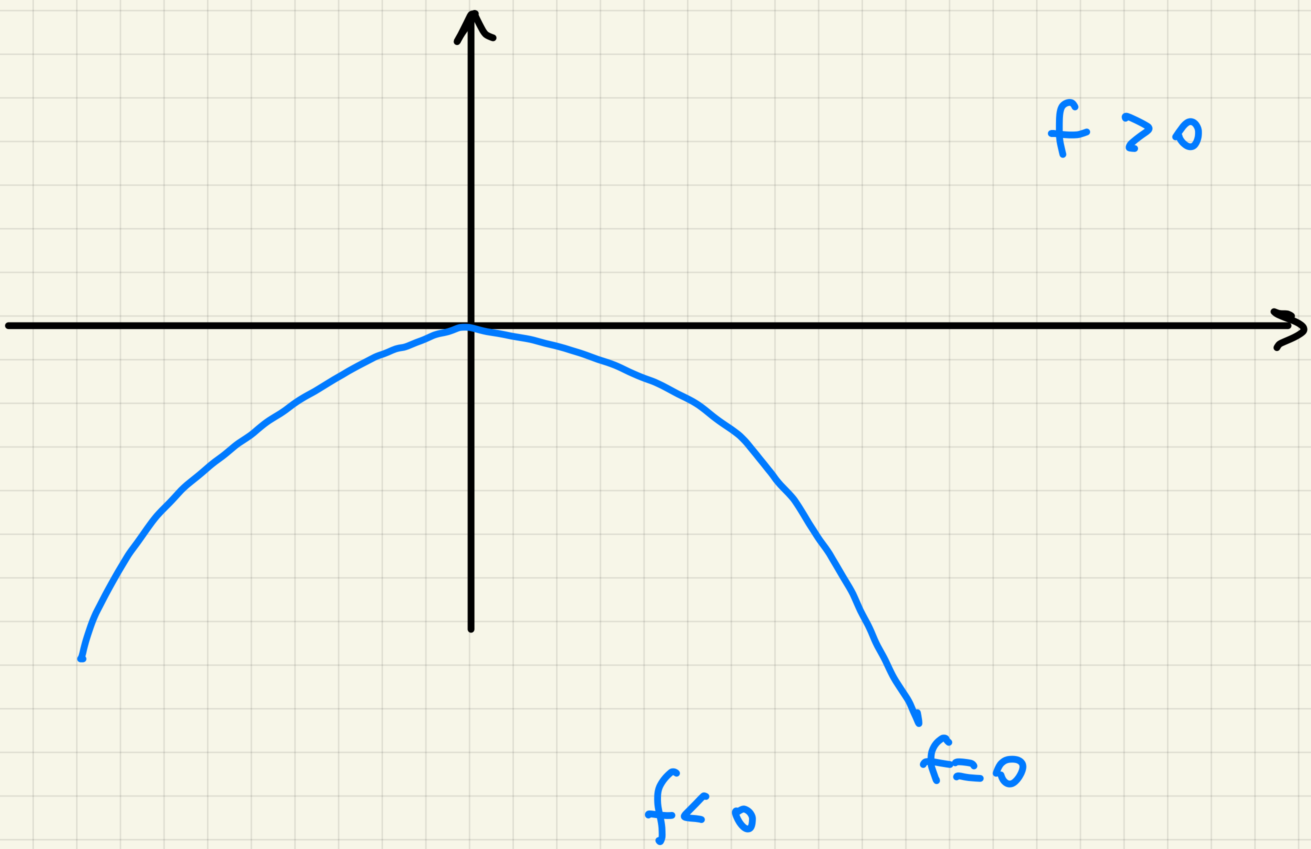
$\mathrm{Dom}\, f = \mathbb{R}^2$

$x = r\cos\vartheta$
$y = r\sin\vartheta$

$$\lim_{(x,y)\to\infty} f(x,y) = 0$$

$$|f(x,y)| \leq (|y|+x^2)\, e^{-x^2-y^2}$$

$$\le (r+r^2)\, e^{-r^2} \xrightarrow[r\to+\infty]{} 0$$



* Suppone $f>0$ in certi punti e $\lim\limits_{(x,y)\to\infty} f = 0$

~~$f<$~~ $\Rightarrow \exists$ punti di massimo assoluto

Suppone $f<0$ in certi punti e $\lim\limits_{(x,y)\to\infty} f = 0$

$\Rightarrow \exists$ punti di min assoluto.

$$S := \sup \{ \underbrace{f(x,y) : (x,y) \in \mathbb{R}^2}_{f(\mathbb{R}^2)} \} > 0$$

Si considera una successione $\{z_n\}$ in $f(\mathbb{R}^2)$ t.c. $z_n \xrightarrow{n\to+\infty} S$

Resta definita una successione (x_n, y_n) nel dominio di f

t.c. $\quad z_n = f(x_n, y_n)$

Questa successione è limitata perché altrimenti esisterebbe una sottosuccessione $\{(x_{n_k}, y_{n_k})\}_{k\in\mathbb{N}}$

t.c. $\quad \lim\limits_{k\to+\infty} \sqrt{x_{n_k}^2 + y_{n_k}^2} = +\infty$

$$0 < S = \lim_{n\to+\infty} f(x_n, y_n) = \lim_{k\to+\infty} f(x_{n_k}, y_{n_k}) = 0$$

Assurdo da $\Rightarrow \quad \exists \; R > 0$

t.c. $\quad \sqrt{x_n^2 + y_n^2} \le R \quad \forall \, n \in \mathbb{N}$

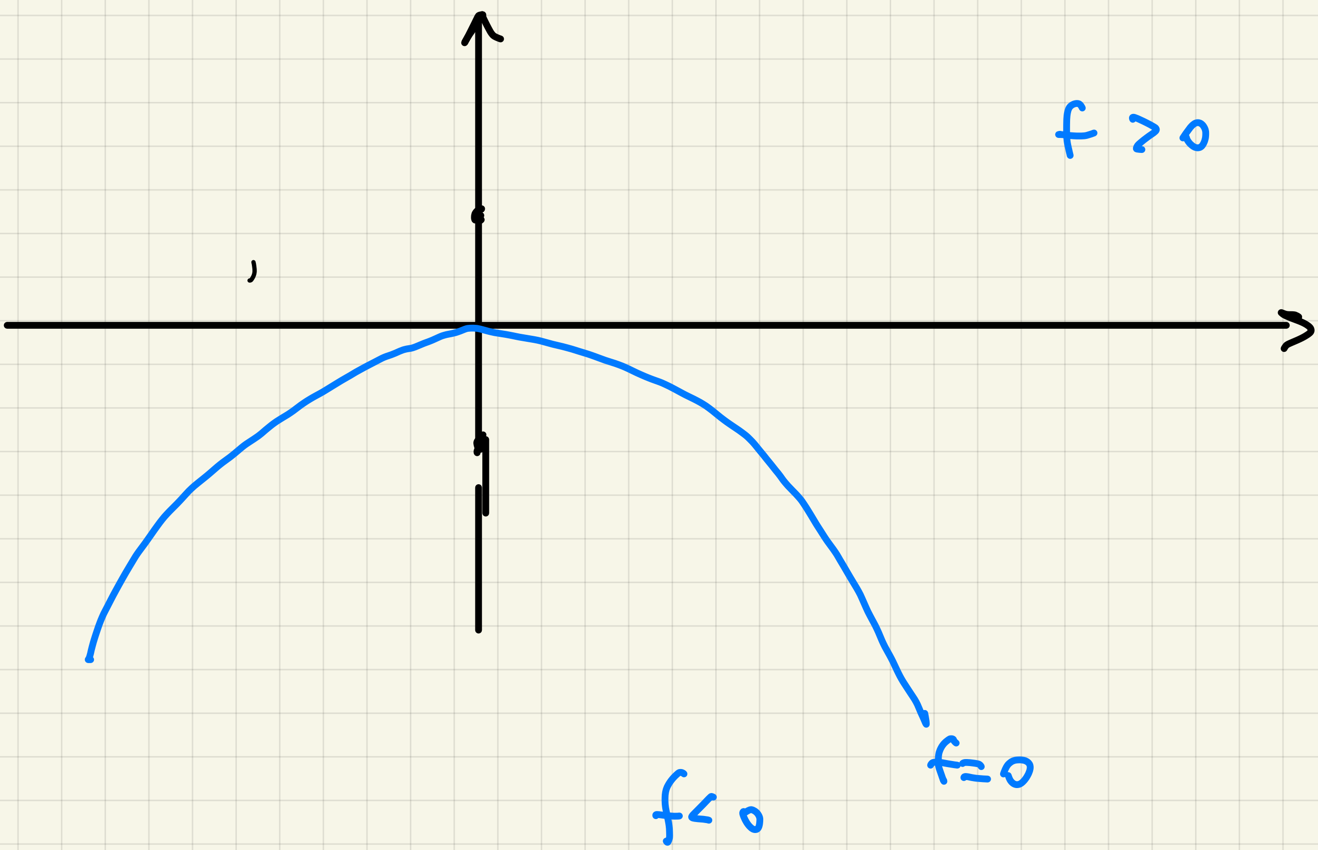
$\{(x_n, y_n)\}_{n\in\mathbb{N}} \quad$ in $\quad \overline{D(0,R)}$

$\exists \quad (\bar{x}, \bar{y}) \in \overline{D(0,R)} \quad$ e un sottosucc con $\lim_{k \to +\infty} (x_{n_k}, y_{n_k}) = (\bar{x}, \bar{y})$

$$S := \sup f(\mathbb{R}^2) = \lim_{n \to +\infty} f(x_n, y_n) =$$

$$= \lim_{k \to +\infty} f(x_{n_k}, y_{n_k}) = f(\bar{x}, \bar{y})$$

$(\bar{x}, \bar{y})$ e' un punto di max assoluto



$$f(x,y) = (y + x^2)\, e^{-x^2-y^2}$$

$$f_x = e^{-x^2-y^2}\left(2x - 2x(y+x^2)\right)$$

$$f_x = e^{-x^2-y^2}\, 2x\,(1-y-x^2)$$

$$f_y = e^{-x^2-y^2}\,(1-2y\,(y+x^2))$$

$$= e^{-x^2-y^2}\,(1-2y^2-2yx^2)$$

$$f_x = e^{-x^2-y^2}\, 2x\,(1-y-x^2) = 0$$

$$f_y = e^{-x^2-y^2}\,(1-2y^2-2yx^2) = 0$$

$$x\,(1-y-x^2) = 0$$

$x = 0$ oppure $x \neq 0$.

$x = 0$ $\quad (1-2y^2) = 0 \Rightarrow y = \pm \frac{1}{\sqrt{2}}$

$$\left(0, \pm \frac{1}{\sqrt{2}}\right)$$

$x \neq 0$

$$\begin{cases} 1-y-x^2 = 0 & y+x^2 = 1 \\ 1-2y\,(y+x^2) = 0 \end{cases}$$

$$\begin{cases} 1-y-x^2 = 0 \\ 1-2y = 0 & y = \frac{1}{2} \end{cases}$$

$$x^2 = 1 - y = \frac{1}{2}$$

$$x = \pm \frac{1}{\sqrt{2}} \qquad \left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$$

$$f\left(0, \pm \frac{1}{\sqrt{2}}\right) = (y + x^2)\, e^{-x^2-y^2} \Big|_{\left(0, \pm \frac{1}{\sqrt{2}}\right)}$$

$$= \pm \frac{1}{\sqrt{2}}\, e^{-\frac{1}{2}}$$

$$f\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}\right) = (y + x^2)\, e^{-x^2-y^2} \Big|_{\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}\right)}$$

$$= e^{-\frac{1}{2}-\frac{1}{4}} = e^{-\frac{3}{4}} = e^{-\frac{1}{4}}\, e^{-\frac{1}{2}} > 0$$

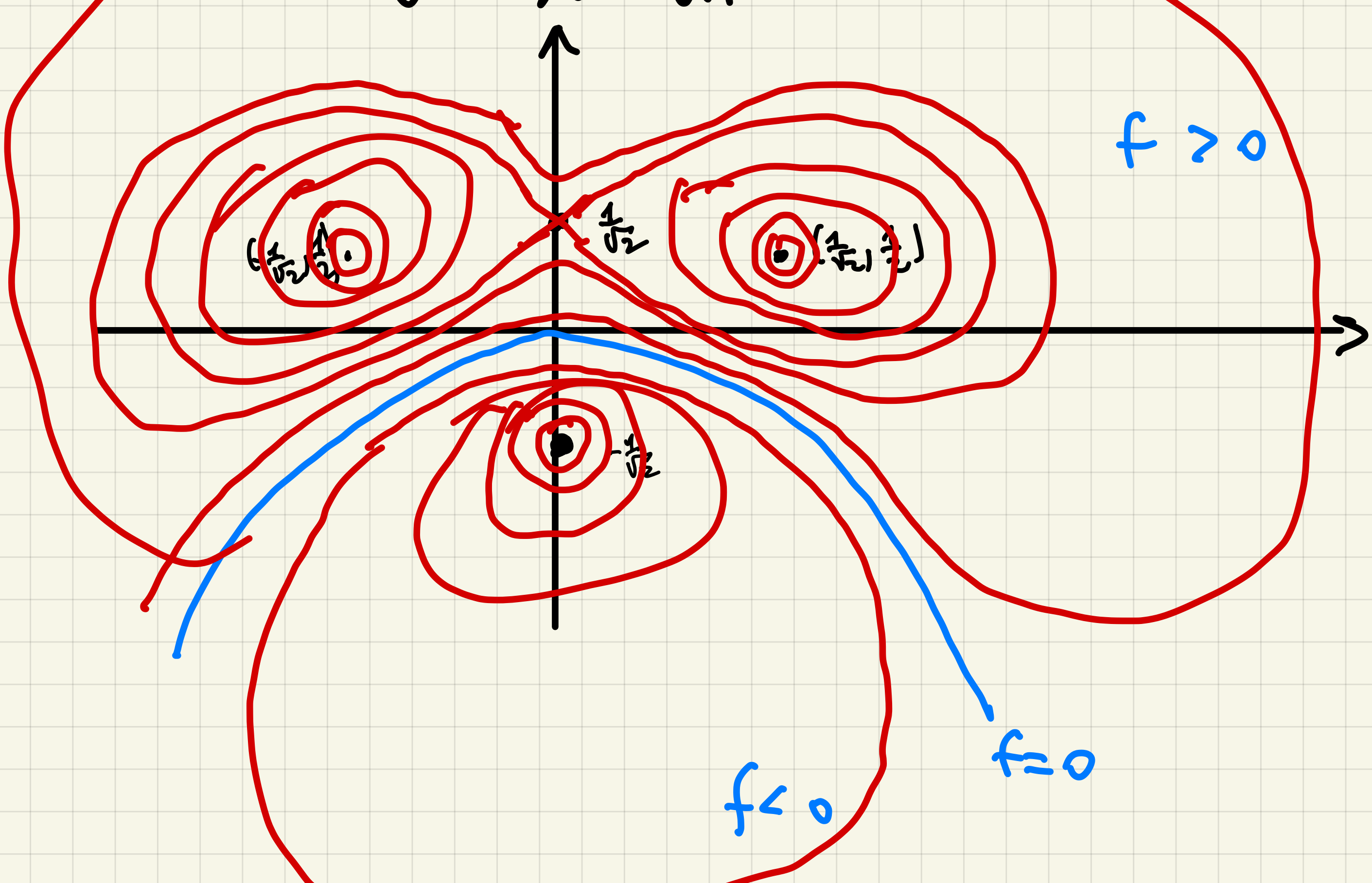
Da qui ricaviamo subito che $\left(0, -\frac{1}{\sqrt{2}}\right)$ e' l'unico punto di minimo assoluto

Confrontiamo $\frac{1}{\sqrt{2}}$ con $\frac{1}{e^{\frac{1}{4}}}$

Risulta $\sqrt[2]{2} = 4^{\frac{1}{4}} > e^{\frac{1}{4}}$

$\Rightarrow \quad \frac{1}{\sqrt{2}} < e^{-\frac{1}{4}}$

$\Rightarrow \left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ sono i due punti di massimo assoluto



$f(x,y) = x^2 - y^2$