5 maggio

Derivate di ordine superiore

Def Dato $f : A \subseteq \mathbb{R}^d \to \mathbb{R}^N$

supponiamo che esista

$$\partial_j f \text{ in } A$$

$$\partial_j f : A \longrightarrow \mathbb{R}^N$$

se questa funzione in un punto x_0 ammette $\partial_i \partial_j f(x_0)$

otteniamo la una derivata di 2° ordine di f in x_0

In generale se $i \neq j$

e se esistono

$\partial_i \partial_j f(x_0)$ e

$\partial_j \partial_i f(x_0)$ non sono

uguali.

<u>Es</u> $f(x,y) = \begin{cases} xy \dfrac{x^4 - y^2}{x^4 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$\partial_x f(0,0) = 0$

$\partial_y f(0,0) = 0$

$$\partial_x f(x,y) = y \, \frac{x^4 - y^2}{x^4 + y^2} + \frac{8x^4 y^3}{(x^4 + y^2)^2}$$

$$\partial_y f(x,y) = x \, \frac{x^4 - y^2}{x^4 + y^2} - \frac{4x^5 y^2}{(x^4 + y^2)^2}$$

$\partial_y \partial_x f(0,0) = -1$ $\qquad$ $\partial_x \partial_y f(0,0) = 1$

$$\lim_{y\to 0} \frac{\partial_x f(0,y) - \overbrace{\cancel{\partial_x f(0,0)}}^{0}}{y} =$$

$$= \lim_{y\to 0} \frac{\cancel{y}\;\frac{-\cancel{y^2}}{\cancel{y^2}}}{\cancel{y}} = -1$$

$$\lim_{x\to 0} \frac{\partial_y f(x,0) - \overbrace{\partial_y f(0,0)}^{0}}{x}$$

$$= \lim_{x\to 0} \frac{x \;\frac{x^4}{x^4}}{x} = 1$$

__Def__ $f : A \subseteq \mathbb{R}^d \to \mathbb{R}^N$ differenziabile in A

in modo che resti definita

$$Df : A \subseteq \mathbb{R}^d \longrightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^N)$$

se $Df(x)$ è differenziabile in $x_0 \in A$

allora poniamo

$$D^2 f(x_0) = D(Df)(x_0)$$

Osservazione

$$Df(x) = \begin{pmatrix} \partial_1 f_1(x) & \cdots\cdots & \partial_d f_1(x) \\ \vdots & & \\ \partial_1 f_N(x) & \cdots\cdots & \partial_d f_N(x) \end{pmatrix}$$

è una funzione a valori vettoriali con coordinate

$$\partial_\alpha f_j \qquad j = 1, \dots, N$$
$$\alpha = 1, \dots, d$$

$Df(x)$ è differenziabile in x^0 $\iff$

$\partial_\alpha f_j(x)$ è diff. in x^0

per ogni $\alpha = 1, \dots, d$

$j = 1, \dots, N$

$\iff$ $\partial_\alpha f(x)$ è diff in x^0

$\forall \quad \alpha = 1, \dots, d$

<u>Conseguenza</u> Se tutte le derivate seconde $\partial_\beta \partial_\alpha f(x)$ $\forall \alpha, \beta$ sono definite in un intorno di x^0 e sono continue in x^0 allora $\partial_\alpha f$ è diff in x^0 $\forall \alpha$

$\Longrightarrow \quad \exists \; D^2 f(x_0)$

$$Df : A \longrightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^N)$$

$\forall \; v \in \mathbb{R}^d$ resto definita

$$\underset{\in A}{x} \longrightarrow \underbrace{Df(x)\,v}_{g(x)} \in \mathbb{R}^N \qquad A \subseteq \mathbb{R}^d \xrightarrow{g} \mathbb{R}^N$$

se esiste $\quad D^2 f(x_0) = D(Df)(x_0)$

$$D^2 f(x_0) \in \mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^N))$$

$$Dg(x_0) \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^N)$$

$Dg(x_0)\,u \qquad u \in \mathbb{R}^d$

$$Dg(x_0)\,u = \left(D^2 f(x_0)\,u\right) v$$

$g(x) = Df(x)\,v$ è una funzione composta

Se pongo $\quad G : \mathcal{L}(\mathbb{R}^d, \mathbb{R}^N) \longrightarrow \mathbb{R}^N$

$$L \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^N) \xrightarrow{G} Lv \in \mathbb{R}^N$$

G è lineare $\Longrightarrow DG = G$

$$g(x) = (G \circ Df)(x) \qquad G(L) = Lv$$

$$Dg\big|_{x=x_0} u = D\big(G \circ Df(x)\big)\big|_{x=x_0} u$$

$$= DG \; D(Df)(x_0) \; u$$

$$= G \; D(Df)(x_0) \; u$$

$$= G \underbrace{\left(D^2 f(x_0) \quad u\right)}_{\mathcal{L}(\mathbb{R}^d, \mathbb{R}^N)} = \left(D^2 f(x_0) u\right) v$$

$$\left(D^2 f(x_0) \; u\right) v$$

<u>Teor</u> Sia $f : A \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}^N$, diff in A (Df continua in A) e due volte diff in $x^0 \in A$, allora

$$\left(D^2 f(x_0) \; u\right) v = \left(D^2 f(x_0) \; v\right) u \qquad \forall u, v \in \mathbb{R}^d$$

<u>Dim</u> Sia $D(x_0, r) \subseteq A$

e siano $u, v \in \mathbb{R}^d$ t.c.

$$|u| < \frac{r}{2} \qquad |v| < \frac{r}{2}$$

$$[-1, 1] \ni t \longrightarrow f(x_0 + u + tv) \in \mathbb{R}^N$$

$$g(t) \doteq f(x_0 + u + tv) - f(x_0 + tv)$$

$$g(1)-g(0)-g'(0) = \int_0^1 g'(t)\,dt - g'(0)$$

$$= \int_0^1 \left(g'(t)-g'(0)\right) dt$$

$$|g(1)-g(0)-g'(0)| = \left| \int_0^1 \left(g'(t)-g'(0)\right) dt \right|$$

$$\le \int_0^1 |g'(t)-g'(0)|\, dt$$

$$\le \sup_{t\in[0,1]} |g'(t)-g'(0)|$$

$$g(t) := f(x_0+u+tv)-f(x_0+tv)$$

$$g'(t) = \left(Df(x_0+u+tv)-Df(x_0+tv)\right)v$$

$$= \left[\left(Df(x_0+u+tv)-Df(x_0)\right)+\left(Df(x_0)-Df(x_0+tv)\right)\right]v$$

$$= \left(Df(x_0+u+tv)-Df(x_0)-D^2f(x_0)(u+tv)\right]v -$$

$$-\left(Df(x_0+tv)-Df(x_0)-D^2f(x_0)\,tv\right]v$$

$$+ \underbrace{\left(D^2f(x_0)(u+\cancel{tv})\right)v - \cancel{\left(D^2f(x_0)\,tv\right)v}}_{\left(D^2f(x_0)u\right)v}$$

$$g'(t) - (D^2 f(x_0)\, u)\, v =$$

$$\left(Df(x_0+u+tv) - Df(x_0) - D^2 f(x_0)(u+tv)\right]v -$$

$$-\left(Df(x_0+tv) - Df(x_0) - D^2 f(x_0)\, tv\right]v$$

$$\frac{Df(x_0+h) - Df(x_0) - D^2 f(x_0)h}{h} = o(1)$$

$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ t.c. $|h| < \delta_\varepsilon$

$$\Rightarrow |Df(x_0+h) - Df(x_0) - D^2 f(x_0)h| < \varepsilon |h|$$

Suppongo che $r < \delta_\varepsilon$

$$|g'(t) - (D^2 f(x_0)\, u)\, v| \leq$$

$$\left|\left(Df(x_0+u+tv) - Df(x_0) - D^2 f(x_0)(u+tv)\right]v\right| +$$

$$+\left|\left(Df(x_0+tv) - Df(x_0) - D^2 f(x_0)\, tv\right]v\right|$$

$$\leq |Df(x_0+u+tv) - Df(x_0) - D^2 f(x_0)(u+tv)|\, |v|$$

$$+ |Df(x_0+tv) - Df(x_0) - D^2 f(x_0)\, tv|\, |v|$$

$$\leq \varepsilon \, |u+tv| \, |v| +$$

$$+ \varepsilon \, |tv| \, |v|$$

$$= \varepsilon |v| \left(|u+tv| + |tv| \right)$$

$$\leq \varepsilon |v| \left(|u| + 2|tv| \right)$$

$$= \varepsilon |v| \left(|u| + 2|t| \, |v| \right)$$

$$\leq \varepsilon |v| \left(|u| + 2|v| \right)$$

$$\leq 2 \varepsilon |v| \left(|u| + |v| \right)$$

$$\left| g'(t) - \left(D^2 f(x_0) u \right) v \right| \leq 2 \varepsilon |v| \left(|u| + |v| \right)$$

per ogni $t \in [0,1]$ e $|u|, |v| < \frac{\delta_\varepsilon}{2}$

Da qui ricaviamo che

$$\left| g(1) - g(0) - \left(D^2 f(x_0) u \right) v \right| \leq 6 \varepsilon |v| \left(|u| + |v| \right)$$

$$\left| g(1) - g(0) - \left(D^2 f(x_0) u \right) v \right| \leq$$

$$\leq \left| g(1) - g(0) - g'(0) \right| + \overbrace{\left| g'(0) - \left(D^2 f(x_0) u \right) v \right|}^{\leq 2\varepsilon |v| (|u| + |v|)}$$

$$|g(1)-g(0)-g'(0)| \leq \sup_{t\in[0,1]} |g'(t)-g'(0)|$$

$$\leq 4\varepsilon |v| (|u|+|v|)$$

$$\sup_{t\in[0,1]} \left(\left| g'(t) - (D^2 f(x_0) u) v \right| + \left| g'(0) - (D^2 f(x_0) u_0) v \right| \right)$$

$$|g'(t) - (D^2 f(x_0) u) v| \leq 2 \varepsilon |v| (|u| + |v|)$$

$$\leq 4 \varepsilon |v| (|u|+|v|)$$

Otteniamo

$$|g(1) - g(0) - (D^2 f(x_0) u) v| \leq 6 \varepsilon |v| (|u|+|v|)$$

dove

$$g(t) \doteq f(x_0 + u + tv) - f(x_0 + tv)$$

$$|\overbrace{f(x_0+u+v) - f(x_0+v)}^{g(1)-g(0)} - f(x_0+u) + f(x_0)$$
$$- (D^2 f(x_0) u) v| \leq 6 \varepsilon |v| (|u|+|v|)$$

se $|u| < \frac{\delta_\varepsilon}{2}$, $|v| < \frac{\delta_\varepsilon}{2}$

Vale anche

$$\left| f(x_0+u+v) - f(x_0+v) - f(x_0+u) + f(x_0) - \cancel{D} - \left(D^2 f(x_0) v\right) u \right| \leq 6\varepsilon |u| \left(|u|+|v|\right)$$

$$\left| \left(D^2 f(x_0) u\right) v - \left(D^2 f(x_0) v\right) u \right| \leq 6\varepsilon \left(|u|+|v|\right)^2$$

se $|u| < \frac{\delta_\varepsilon}{2}$, $|v| < \frac{\delta_\varepsilon}{2}$

$$\Big| \left(D^2 f(x_0) u\right) v - \left(g(1) - g(0)\right) + g(1) - g(0) - \left(D^2 f(x_0) v\right) u \Big|$$

$$\leq \left| g(1) - g(0) - \left(D^2 f(x_0) u\right) v \right| + \left| g(1) - g(0) - \left(D^2 f(x_0) v\right) u \right|$$

$$\leq 6\varepsilon |v| \left(|u|+|v|\right) + 6\varepsilon |u| \left(|u|+|v|\right)$$

$$= 6\varepsilon \left(|u|+|v|\right)^2$$

$$\left| \left(D^2 f(x_0) u\right) v - \left(D^2 f(x_0) v\right) u \right| \leq 6\varepsilon \left(|u|+|v|\right)^2$$

se $|u| < \frac{\delta_\varepsilon}{2}$, $|v| < \frac{\delta_\varepsilon}{2}$

$$\left| \left(D^2 f(x_0) \lambda u\right) \lambda v - \left(D^2 f(x_0) \lambda v\right) \lambda u \right|$$

$$= \lambda^2 \left| \left(D^2 f(x_0) u\right) v - \left(D^2 f(x_0) v\right) u \right|$$

$$6\varepsilon\,(|\lambda u|+|\lambda v|)^2 = |\lambda|\,6\varepsilon\,(|u|+|v|)^2$$

L'omogeneità di ordine 2 consente di concludere

$$\left|\left(D^2 f(x_0)\,u\right)v - \left(D^2 f(x_0)\,v\right)u\right| \leq 6\varepsilon\,(|u|+|v|)^2$$

$$\forall\, u, v \in \mathbb{R}^d \quad e \quad \forall\, \varepsilon > 0$$

$$\Longrightarrow \quad \left(D^2 f(x_0)\,u\right)v = \left(D^2 f(x_0)\,v\right)u$$

$$D^2 f(x_0)\,(u,v) := \left(D^2 f(x_0)\,u\right)v$$

<u>Corollario</u> Nelle ipotesi del precedente teorema, risulta che in x_0

$$\partial_\alpha \partial_\beta f(x_0) = \partial_\beta \partial_\alpha f(x_0) \qquad \forall\, \alpha, \beta.$$

<u>Dim</u> $\quad Df : A \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^N)$

$$\frac{d}{dt} f(x+tv)\Big|_{t=0} = Df(x)\,v = \sum_{\alpha=1}^{d} \partial_\alpha f(x)\, v_\alpha$$

$$\forall\, x \in A,\ \forall\, v \in \mathbb{R}^d$$

$$\frac{d}{dt} Df(x^0+tu)\,v\Big|_{t=0} = \left(D^2 f(x_0)\,u\right)v$$

$$= \frac{d}{dt} \sum_{\alpha=1}^{d} \partial_\alpha f(x^0 + tu) v_\alpha \Big|_{t=0}$$

$$= \sum_{\alpha=1}^{d} D\partial_\alpha f(x^0) u \quad v_\alpha$$

$$= \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \partial_\beta \partial_\alpha f(x^0) \; u_\beta \; v_\alpha$$

$$\left(D^2 f(x_0) \, u\right) v = \sum_{\alpha,\beta=1}^{d} \partial_\beta \partial_\alpha f(x^0) \, u_\beta v_\alpha$$

$$\partial_\beta \partial_\alpha f(x^0) = \qquad\qquad u = e_\beta$$

$$= \left(D^2 f(x_0) \, e_\beta\right) e_\alpha \qquad\qquad v = e_\alpha$$

$$= \left(D^2 f(x_0) \, e_\alpha\right) e_\beta = \partial_\alpha \partial_\beta f(x_0) \qquad \forall \alpha, \beta \text{ in } \{1, \dots, d\}$$

<u>Def</u> Sia $f : A \subseteq \mathbb{R}^d \to \mathbb{R}^N$

Supponiamo di avere definito $D^{k-1} f(x)$

e che esista su tutto x

$$D^{k-1} f : A \subseteq \mathbb{R}^d \longmapsto \mathcal{L}^{k-1}(\mathbb{R}^d, \mathbb{R}^N)$$

Se $D^{k-1} f$ è differenziabile in $x_0 \in A$

allora poniamo

$$D^k f(x_0) = D\left(D^{k-1} f\right)(x_0)$$

$$\mathcal{L}\left(\mathbb{R}^d, \mathcal{L}^{k-1}(\mathbb{R}^d, \mathbb{R}^N)\right)$$

$$= \mathcal{L}^k(\mathbb{R}^d, \mathbb{R}^N)$$

<u>Teor</u> Supponiamo $f : A \subseteq \mathbb{R}^d \to \mathbb{R}^N$

$D^{k-1} f : A \to \mathcal{L}^{k-1}(\mathbb{R}^d, \mathbb{R}^N)$

e $D^{k-1} f \in C^0(A, \mathcal{L}^{k-1}(\mathbb{R}^d, \mathbb{R}^N))$

e che esista $D^k f(x^0)$ in un punto $x^0 \in A$. Allora

$D^k f(x_0) \in \mathcal{L}^k(\mathbb{R}^d, \mathbb{R}^N)$ è simmetrica

$$D^k f(x_0)(v_1, \dots, v_k) \in \mathbb{R}^N$$

$$v_1, \dots, v_k \in \mathbb{R}^d$$

$$= D^k f(x_0)(v_{\sigma_1}, \dots, v_{\sigma_k})$$

per ogni permutazione σ di $\{1, \dots, k\}$

$A \subseteq \mathbb{R}^d$

$C^k(A, \mathbb{R}^N)$ è l'insieme delle $f : A \to \mathbb{R}^N$ che hanno tutte le derivate parziali fino all'ordine k che sono definite in A ~~&~~ sono continue

$\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ $\qquad \mathbb{N}_0 = \{0, 1, 2, \dots\}$

$|\alpha| = \alpha_1 + \dots + \alpha_d$

$|\alpha| \leq k$

$\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$

$\partial^\alpha f(x) = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} f(x)$