PARENTESI DI POISSON & MOMENTO ANGOLARE

Prendiamo un corpo che si muove in $\mathbb{R}^3$, con coord. cartesiane $\bar{q}$, allora $\bar{p}$ è la quantità di moto.

$$\bar{M} = \bar{q} \times \bar{p} \quad \leftarrow \text{funz. delle } p_h \text{ e delle } q_h$$

$$M_i = \sum_{m,k=1}^{3} \epsilon_{imk} q_m p_k$$

$$\{p_l, M_i\} = -\frac{\partial M_i}{\partial q_l} = -\frac{\partial}{\partial q_l} \sum_{mk} \epsilon_{imk} q_m p_k =$$

$$= -\sum_{mk} \epsilon_{imk} \delta_{ml} p_k = -\sum_k \epsilon_{ilk} p_k$$

$$= \sum_k \epsilon_{lik} p_k$$

$$\{p_l, \sum_{mk} \epsilon_{imk} q_m p_k\} \overset{\text{BILIN.}}{=} \sum_{mk} \epsilon_{imk} \{p_l, q_m p_k\} =$$

$$\overset{d}{=} \sum_{mk} \epsilon_{imk} \Big[\underbrace{\{p_l, q_m\}}_{-\delta_{lm}} p_k + \underbrace{\{p_l, p_k\}}_{0} q_m\Big] \overset{\text{P.P. fondam.}}{=}$$

$$= -\sum_k \epsilon_{ilk} p_k$$

$$\alpha\{p_l, M_z\} = \alpha\epsilon_{l3k} p_k$$

$$\hookrightarrow \{p_3, M_3\} = 0$$

$$\left\{\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, M_3\right\} = \underbrace{\begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}}_{\text{ROTAZIONE INFINITESIMA ATTORNO ASSE } z} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$\{q_l, M_i\} = \sum_m \epsilon_{lim} q_m$$

$$\boxed{\{M_i, M_j\} = \sum_{k=1}^{3} \epsilon_{ijk} M_k}$$

[M_i soddisfano l'algebra di Lie di $SU(2)$]

$$M_i = \sum_{m,s=1}^{3} \epsilon_{ims}\, q_m p_s$$

c)

$$\{M_i, M_j\} = \Big\{ \sum_{ms} \epsilon_{ims} q_m p_s, M_j \Big\} \overset{\text{BILIN.}}{=} \sum_{ms} \epsilon_{ims} \{q_m p_s, M_j\} =$$

$$= \sum_{ms} \epsilon_{ims} \Big[q_m \underbrace{\{p_s, M_j\}}_{=\sum_r \epsilon_{sjr} p_r} + \underbrace{\{q_m, M_j\}}_{=\sum_r \epsilon_{mjr} q_r} p_s \Big]$$

$$= \sum_{msr} \epsilon_{ims} \left(\epsilon_{sjr} p_r q_m + \epsilon_{mjr} p_s q_r \right) =$$

$$= \sum_{msr} \epsilon_{ims} \epsilon_{sjr} p_r q_m + \sum_{msr} \epsilon_{ims} \epsilon_{mjr} p_s q_r$$

$s \; m \quad s \quad m$

$$\sum_{smr} \epsilon_{ism} \epsilon_{sjr} p_m q_r = - \sum_{smr} \epsilon_{ims} \epsilon_{sjr} p_m q_r$$

$$= \sum_{msr} \epsilon_{ims} \underbrace{\epsilon_{sjr}}_{\epsilon_{jrs}} \left(p_r q_m - p_m q_r \right)$$

$$\delta_{ij}\delta_{mr} - \delta_{ir}\delta_{mj}$$

$$= \sum_{mr} \left(\delta_{ij} \delta_{mr} - \delta_{ir} \delta_{mj} \right) \left(p_r q_m - p_m q_r \right)$$

$$= \sum_m \delta_{ij} \left(\cancel{p_m q_m} - \cancel{p_m q_m} \right) - \left(p_i q_j - p_j q_i \right)$$

$$= q_i p_j - q_j p_i \qquad\qquad = \sum_k \epsilon_{ijk} M_k$$

$$\sum_k \epsilon_{ijk} \sum_{rs} \epsilon_{krs} q_r p_s =$$

$$= \sum_{rs} \left(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr} \right) q_r p_s =$$

$$= q_i p_j - q_j p_i \ //$$

$$\{M_i, M_j\} = \sum_k \epsilon_{ijk} M_k$$

$$\{M_1, M_2\} = \sum_k \epsilon_{12k} M_k = M_3$$

$$\{M_2, M_3\} = M_1 \qquad\qquad \{M_3, M_1\} = M_2$$

$$\{\bar{M}^2, M_i\} = 0$$

Dim. $\left\{ \sum_l M_l^2 , M_i \right\} \overset{b/c}{=} \sum_l \left[M_l \{M_l, M_i\} + \{M_l, M_i\} M_l \right] =$

$$= \sum_{l,k} \left(M_l \epsilon_{lik} M_k + \epsilon_{lik} M_k M_l \right) = - \sum_{lk} \underbrace{\epsilon_{ilk}}_{\substack{\text{antisym.} \\ \text{in } l \leftrightarrow k}} \big(\underbrace{M_l M_k + M_k M_l}_{\substack{\text{simm.} \\ \text{in } l \leftrightarrow k}} \big)$$

$$\Big[\sum_{lk} \underset{\text{antisim}}{a_{lk}} \underset{\text{sim.}}{S_{lk}} = \sum_{lk} (-a_{kl}) S_{kl} = - \sum_{ij} a_{ij} S_{ij} = - \sum_{lk} a_{lk} S_{lk}$$

$$\Rightarrow \sum_k a_{lk} S_{lk} = 0 \ \Big]$$

Vettore di Laplace-Runge-Lenz e Parentesi di Poisson

$$\bar{A} = \bar{p} \times \bar{M} - mk\,\frac{\bar{q}}{r} \qquad r = \sqrt{q_1^2 + q_2^2 + q_3^2}$$

indip. da t

$$A_i = \sum_{mh} \epsilon_{imh}\, p_m M_h - mk\,\frac{q_i}{r}$$

$$H = \frac{\bar{p}^2}{2m} - \frac{k}{r} = \frac{1}{2m}\sum_{j=1}^{3} p_j^2 - \frac{k}{r}$$

Kepl.

→ verifichiamo che $\bar{A}$ è cost. del moto, cioè $\{A_i, H\} = 0$

$$\{A_i, H\} = \sum_{mh} \epsilon_{imh} \left\{ p_m M_h, \frac{1}{2m}\sum_j p_j^2 - \frac{k}{r} \right\}$$

$$- mk \left\{ \frac{q_i}{r}, \frac{1}{2m}\sum_j p_j^2 - \frac{k}{r} \right\} =$$

$\{f(\bar{p}), G(\bar{p})\} = 0$

$$= \sum_{mh} \epsilon_{imh} \Big[p_m \cancel{\Big\{ M_h, \frac{1}{2m}\sum_j p_j^2 \Big\}} + \cancel{\Big\{ p_m, \frac{1}{2m}\sum_j p_j^2 \Big\}} M_h$$

inv. rotaz.

$$- p_m \cancel{\Big\{ M_h, \frac{k}{r} \Big\}} - \Big\{ p_m, \frac{k}{r} \Big\} M_h \Big]$$

$\{f(\bar{q}), g(\bar{q})\} = 0$

$$- mk \Big[\Big\{ \frac{q_i}{r}, \frac{1}{2m}\sum_j p_j^2 \Big\} - \cancel{\Big\{ \frac{q_i}{r}, \frac{k}{r} \Big\}} \Big]$$

$$= \sum_{mh} \epsilon_{imh} \underbrace{\Big\{ p_m, -\frac{k}{r} \Big\}}_{-\frac{q_m}{r^3}} M_h - \frac{k}{2}\sum_j \underbrace{\Big\{ \frac{q_i}{r}, p_j^2 \Big\}}_{2\{p_j, -\frac{q_i}{r}\} p_j}$$

$$= \underbrace{-k \sum_{mn} \epsilon_{imn} \frac{q_m M_n}{r^3}}_{A} + \underbrace{k \sum_j p_j}_{B} \{ p_j , \underbrace{\tfrac{q_i}{r}}_{\frac{q_i q_j}{r^3} - \frac{\delta_{ij}}{r}} \} \quad - \frac{\partial}{\partial q_j}\left(\frac{q_i}{r}\right)$$

$$\frac{\partial}{\partial q_m}\left(\frac{1}{r}\right) = \frac{\partial}{\partial q_m} \frac{1}{\sqrt{q_1^2 + p_2^2 + p_3^2}} = -\frac{1}{r^2} \frac{1}{2r} 2 q_m = -\frac{q_m}{r^3}$$

$$A = -k \sum_{mn} \epsilon_{imn} \frac{q_m}{r^3} \sum_{ab} \epsilon_{nab} q_a p_b = -k \sum_{m\,ab} (\delta_{ia}\delta_{mb} - \delta_{ib}\delta_{ma}) \cdot \frac{1}{r^3} (q_m q_a p_b)$$

$$= -\frac{k}{r^3} \sum_m \left(q_m q_i p_m - \overbrace{q_m q_m}^{r^2} p_i \right) =$$

$$= k\left(\frac{1}{r} p_i - \frac{1}{r^3} (\bar{q} \cdot \bar{p}) q_i \right)$$

$$B = k \sum_j p_j \left(\frac{q_i q_j}{r^3} - \frac{\delta_{ij}}{r} \right) = k \frac{(\bar{p} \cdot \bar{q}) q_i}{r^3} - \frac{k}{r} p_i$$

$$A + B = 0 \quad //$$