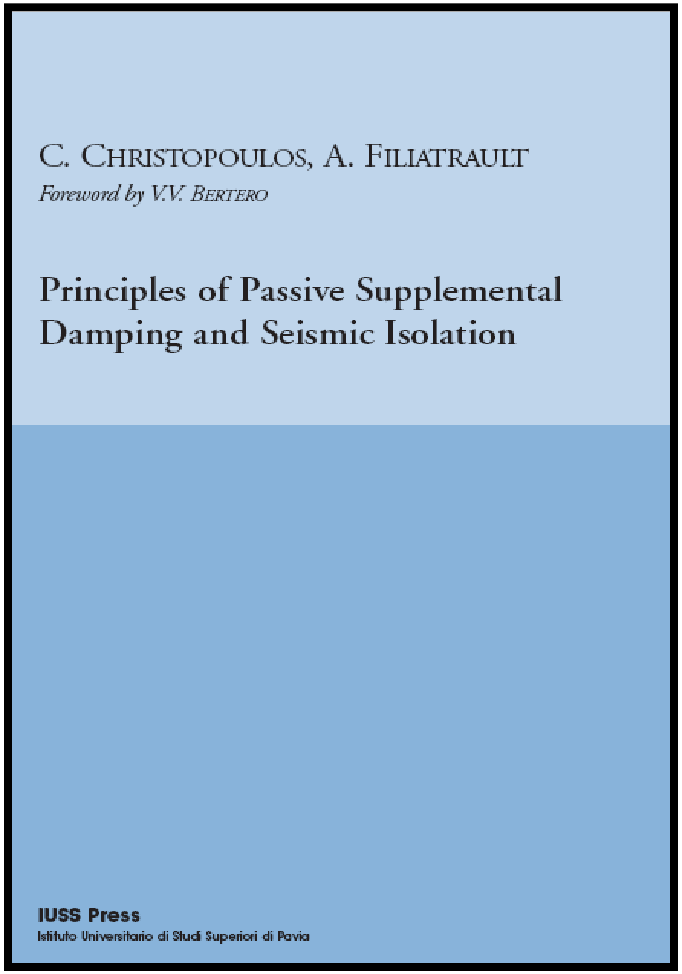
Chapter 13
Tuned-Mass Dampers

CONTENT

Major References

- Chapter 8
  - Sections 8.1 to 8.8



1. Introduction

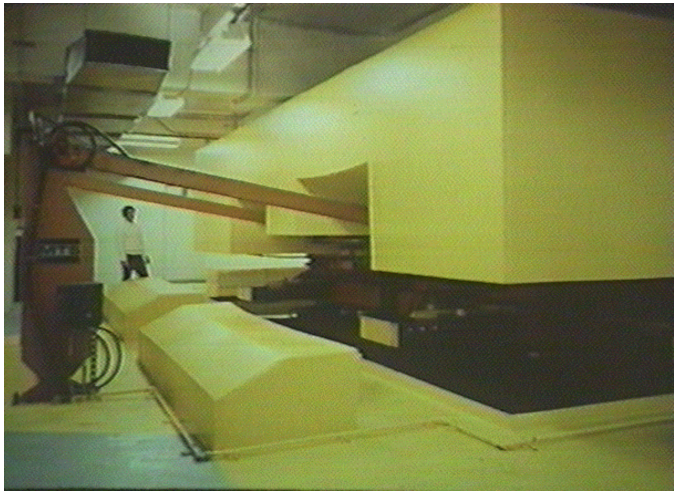
- Tuned-mass dampers (TMDs) or vibration absorbers:
  - First suggested by Hermann Frahm in 1909 (US Patent #989958).
  - Relatively small mass-spring-dashpot systems calibrated to be in resonance with a particular mode of vibration.
  - Usually installed on roofs of buildings.
- TMDs effective in reducing wind-induced vibrations in high-rise buildings and floor vibrations induced by occupant activity.
- More recently, TMDs considered for seismic protection of buildings.

1. Introduction

- Advantages of TMDs:
  - Capable of significantly reducing dynamic response of linear structures.
  - Construction is simple: assembly of a mass, a spring, and a viscous damper at a given point on the structure.
  - No need for external power source or sophisticated hardware.
- Disadvantages of TMDs:
  - Require a relatively large mass.
  - Require large space for installation.
  - Usually undergo large relative displacements and require large clearances.
  - Need to be mounted on a smooth surface to minimize friction and facilitate free motion.
- Basic theory of TMDs presented.
- Potential seismic applications of TMDs explored.

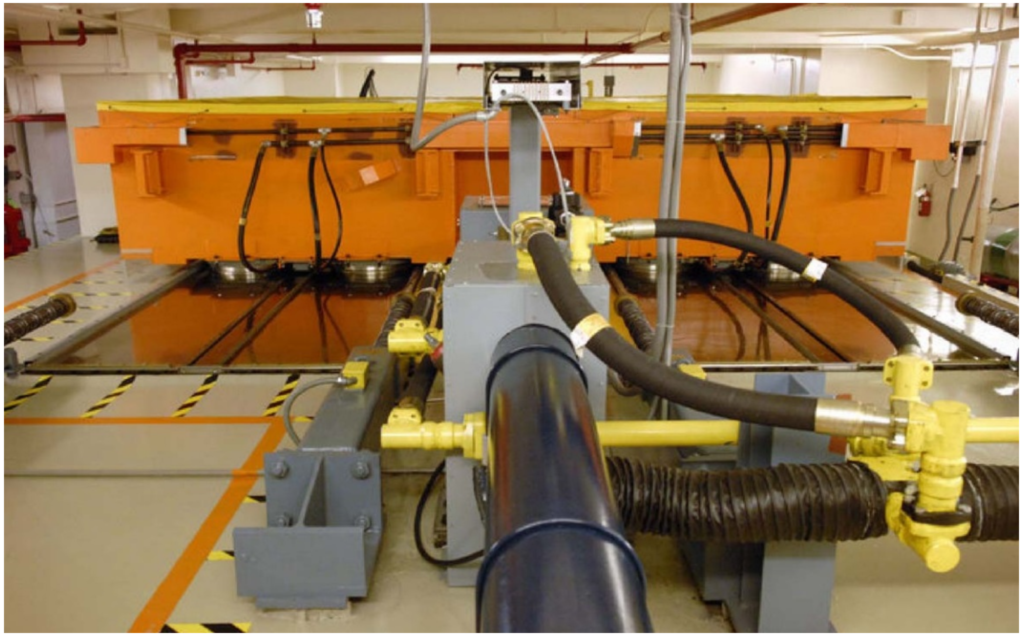
1. Introduction

- 373-ton TMD in Citycorp Center, New York City
  - First lateral natural frequency = 0.16 Hz.
  - 1% damping ratio in first mode.
  - TMD installed on 63rd floor.
  - TMD produces effective damping ratio of 4%.
  - Wind induced accelerations reduced by 50%.
  - Linear nitrogen charged springs, hydrostatic bearings, control actuators, power supply and electronic control.



1. Introduction

- 2-300-ton TMDs in John Hancock Tower, Boston
  - In-phase motions control lateral response.
  - Out-of-phase motions control torsional response.



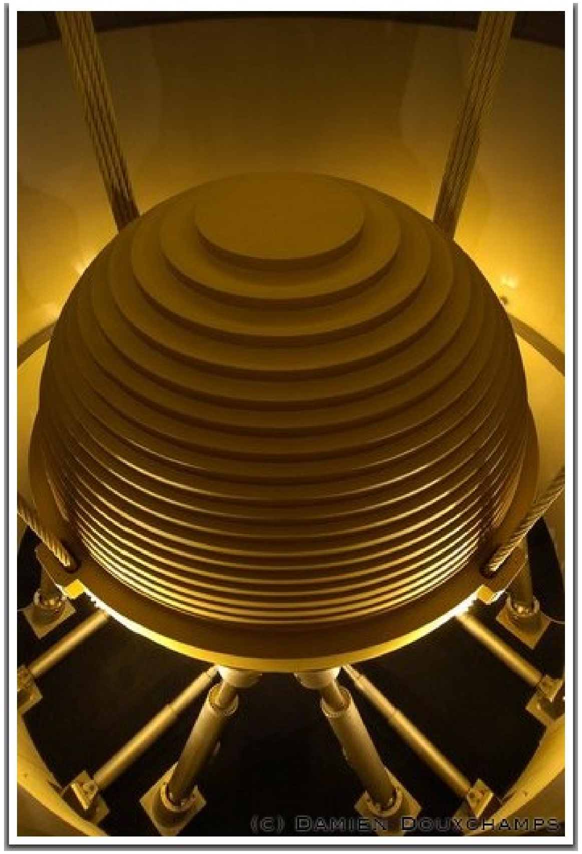
1. Introduction

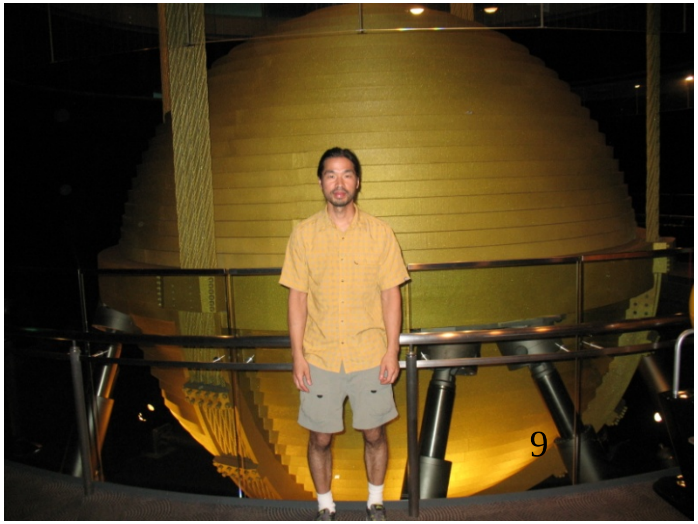
		John Hancock Boston, MA	Citicorp Center New York, NY
Typical floor size	(ft)	343×105	160×160
Floor area	(sq ft)	36,015	25,600
Building height	(ft)	800	920
Building modal weight	(tons)	47,000	20,000
Building period 1st mode	(sec)	7.00	6.25
Design wind storm	(years)	100	30
Mass block weight	(tons)	2×300	373
Mass block size	(ft)	18×18×3	30×30×8
Mass block material	(type)	lead/steel	concrete
TMD/AMD stroke	(ft)	±6.75*	±4.50*
Max spring force	(kips)	135	170
Max actuator force	(kips)	50	50
Max hydraulic supply	(gms)	145	190
Max operating pressure	(psi)	900	900
Operating trigger - acceleration	(g)	.002	.003
Max power	(HP)	120	160
Equivalent damping	(%)	4.0%	4.0%

* Including overtravel

1. Introduction

- 660-ton Pendulum TMD to reduce wind vibrations in 101 Taipei Building, Taiwan.





1. Introduction

- 5-ton Pendulum TMD to reduce wind vibrations of Marina Bay SkyPark in Singapore.

1. Introduction

- Seismic upgrade of LAX Airport Theme Building with 600-ton TMD and viscous

2. Theory of Undamped Tuned-mass Dampers Under Harmonic Loading

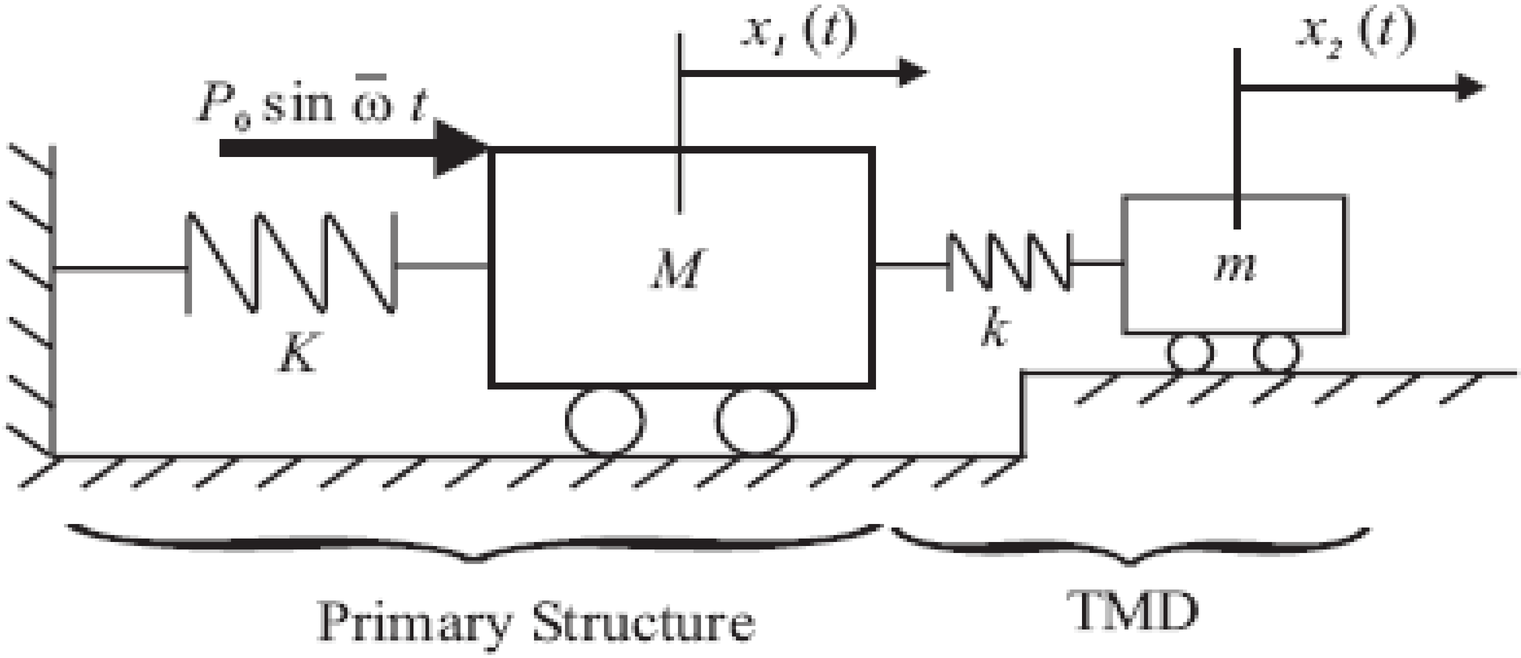
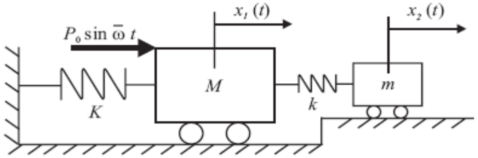


Figure 8.1 Main Structure and TMD

2. Theory of Undamped Tuned-mass Dampers Under Harmonic Loading



Applying Newton's second law on each mass yields the two equations of motion for this two-degree-of-freedom-system:

$$\begin{aligned} M\ddot{x}_1 + (K+k)x_1 - kx_2 &= P_0 \sin \bar{\omega} t \\ m\ddot{x}_2 + k(x_2 - x_1) &= 0 \end{aligned} \tag{8.2}$$

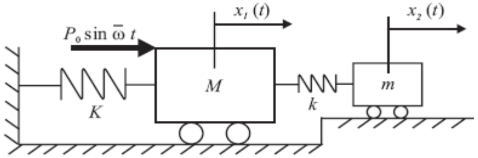
Since the system is undamped, the forced vibration response takes a simple form:

$$\begin{aligned} x_1(t) &= a_1 \sin \bar{\omega} t \\ x_2(t) &= a_2 \sin \bar{\omega} t \end{aligned} \tag{8.3}$$

where a_1 and a_2 are constants representing the amplitude of vibration of the main and the secondary mass respectively. Substituting Equation (8.3) into Equation (8.2) yields:

$$\begin{aligned} (-Ma_1\bar{\omega}^2 + (K+k)a_1 - ka_2)\sin \bar{\omega} t &= P_0 \sin \bar{\omega} t \\ (-ma_2\bar{\omega}^2 + k(a_2 - a_1))\sin \bar{\omega} t &= 0 \end{aligned} \tag{8.4}$$

2. Theory of Undamped Tuned-mass Dampers Under Harmonic Loading



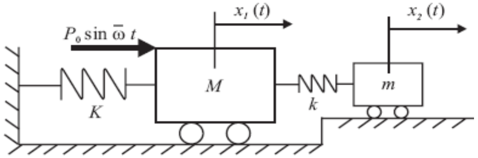
Since Equation (8.4) must be satisfied at all times:

$$\begin{aligned} a_1(-M\bar{\omega}^2 + K + k) - ka_2 &= P_0 \\ -ka_1 + a_2(-m\bar{\omega}^2 + k) &= 0 \end{aligned} \qquad (8.5)$$

For simplification, we introduce the following variables:

$$\begin{aligned} x_{st} &= \frac{P_0}{K} : \text{static displacement of the primary structure} \\ \Omega_n^2 &= \frac{K}{M} : \text{natural frequency of the primary structure} \\ \omega_a^2 &= \frac{k}{m} : \text{natural frequency of the TMD} \end{aligned} \qquad (8.6)$$

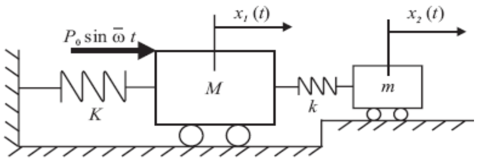
2. Theory of Undamped Tuned-mass Dampers Under Harmonic Loading



Dividing by K the first expression of Equation (8.5) yields:

$$\begin{cases} a_1\left(1 + \dfrac{k}{K} - \dfrac{\bar{\omega}^2}{\Omega_n^2}\right) - a_2\dfrac{k}{K} = x_{st} \\ a_1 = a_2\left(1 - \dfrac{\bar{\omega}^2}{\omega_a^2}\right) \end{cases} \qquad (8.7)$$

2. Theory of Undamped Tuned-mass Dampers Under Harmonic Loading



Solving for the amplitudes a_1 and a_2 we get:

$$\frac{a_1}{x_{st}} = \frac{\left(1-\dfrac{\bar{\omega}^2}{\omega_a^2}\right)}{\left(1-\dfrac{\bar{\omega}^2}{\omega_a^2}\right)\left(1+\dfrac{k}{K}-\dfrac{\bar{\omega}^2}{\Omega_n^2}\right)-\dfrac{k}{K}}$$

$$\frac{a_2}{x_{st}} = \frac{1}{\left(1-\dfrac{\bar{\omega}^2}{\omega_a^2}\right)\left(1+\dfrac{k}{K}-\dfrac{\bar{\omega}^2}{\Omega_n^2}\right)-\dfrac{k}{K}} \tag{8.8}$$

From the first of these expressions, it becomes clear that when the natural frequency $\omega_a = \sqrt{k/m}$ of the attached TMD is chosen to be equal to the frequency $\bar{\omega}$ of the disturbing force, the main mass M does not vibrate at all ($a_1 = 0$).

2. Theory of Undamped Tuned-mass Dampers Under Harmonic Loading

Examine now the second equality of Equation (8.8) when $\omega_a = \bar{\omega}$. The first term of the denominator is then zero, and this equation reduces to:

$$a_2 = -\frac{K}{k}x_{st} = -\frac{P_0}{k} \qquad (8.9)$$

With the main mass standing still and the TMD having a motion $-(P_0/k)\sin\bar{\omega}t$, the force in the TMD varies as $-P_0\sin\bar{\omega}t$, which is actually equal and opposite to the external force, as illustrated in Figure 8.2.

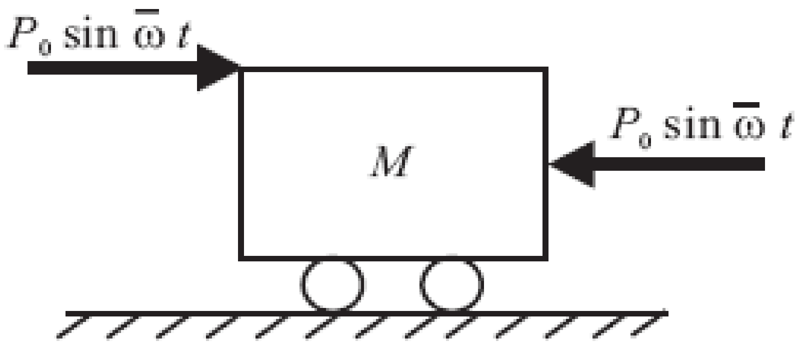
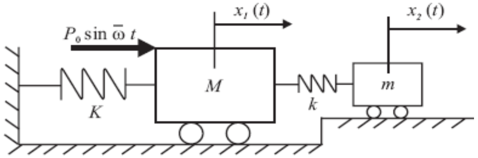


Figure 8.2 Free Body Diagram of Main Mass for Optimum Tuning Conditions of TMD

2. Theory of Undamped Tuned-mass Dampers Under Harmonic Loading

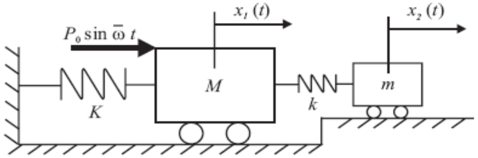


Now consider the case in which the TMD is in resonance with the primary structure, with $\omega_a = \Omega_n$, which can also be expressed as:

$$\frac{k}{m} = \frac{K}{M}$$
$$\text{or } \frac{k}{K} = \frac{m}{M} = \mu \tag{8.10}$$

where μ is defined as the ratio of the mass of the TMD to the mass of the primary structure.

2. Theory of Undamped Tuned-mass Dampers Under Harmonic Loading

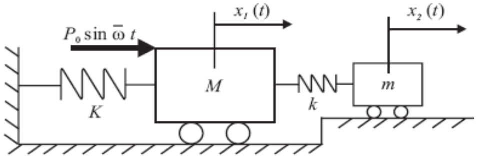


For this special case, Equation (8.8) becomes:

$$\frac{x_1(t)}{x_{st}} = \frac{\left(1-\frac{\bar{\omega}^2}{\omega_a^2}\right)}{\left(1-\frac{\bar{\omega}^2}{\omega_a^2}\right)\left(1+\mu-\frac{\bar{\omega}^2}{\omega_a^2}\right)-\mu}\sin\bar{\omega}t$$

$$\frac{x_2(t)}{x_{st}} = \frac{1}{\left(1-\frac{\bar{\omega}^2}{\omega_a^2}\right)\left(1+\mu-\frac{\bar{\omega}^2}{\omega_a^2}\right)-\mu}\sin\bar{\omega}t \qquad (8.11)$$

2. Theory of Undamped Tuned-mass Dampers Under Harmonic Loading



The two denominators of Equation (8.11) are identical and are quadratic, with two roots in $(\bar{\omega}^2/\omega_a^2)$. Thus, for two values of the excitation frequency $\bar{\omega}$, both denominators become zero, and consequently x_1 and x_2 become infinitely large. Obviously, these two frequencies are the natural frequencies of the two-degrees-of-freedom system. These natural frequencies are determined by setting the denominators equal to zero:

$$\left(\frac{\bar{\omega}}{\omega_a}\right)^4 - \left(\frac{\bar{\omega}}{\omega_a}\right)^2 (2+\mu) + 1 = 0 \qquad (8.12)$$

with the solutions:

$$\left(\frac{\bar{\omega}}{\omega_a}\right)^2 = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}} \qquad (8.13)$$

2. Theory of Undamped Tuned-mass Dampers Under Harmonic Loading

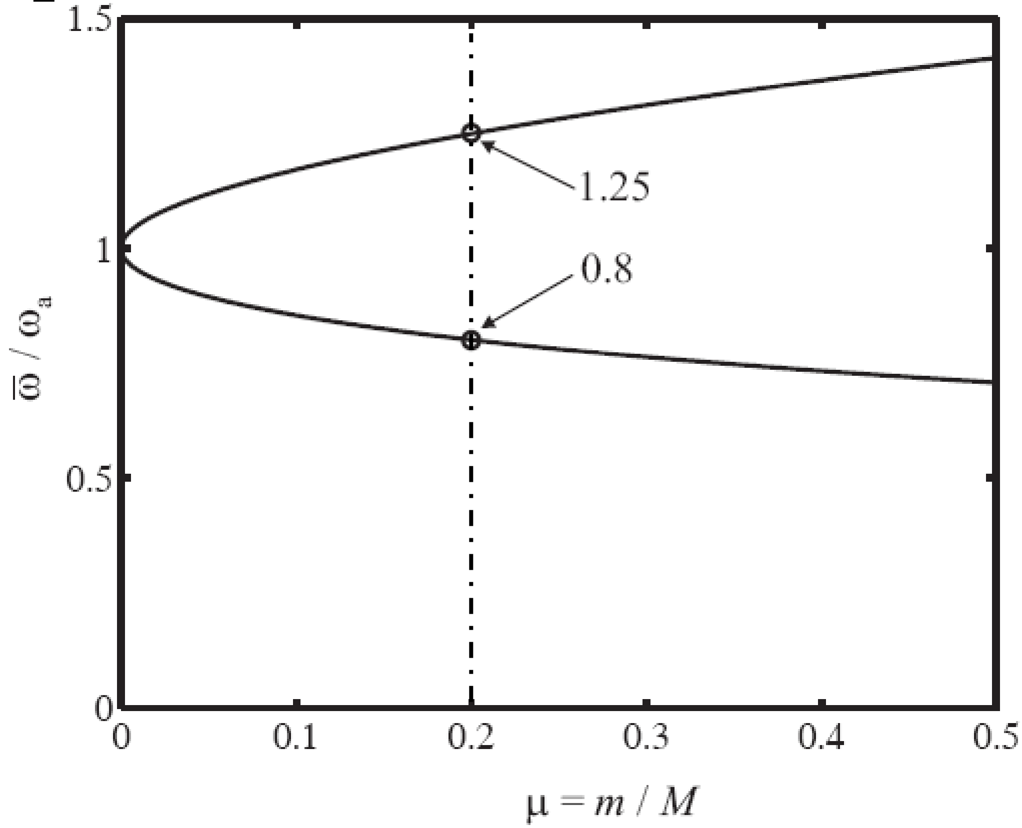


Figure 8.3 Combined Natural Frequencies for TMDs Tuned to the Main Structure: $\omega_a = \Omega_n$, $\mu = 0.2$ (after Den Hartog 1985)

2. Theory of Undamped Tuned-mass Dampers Under Harmonic Loading

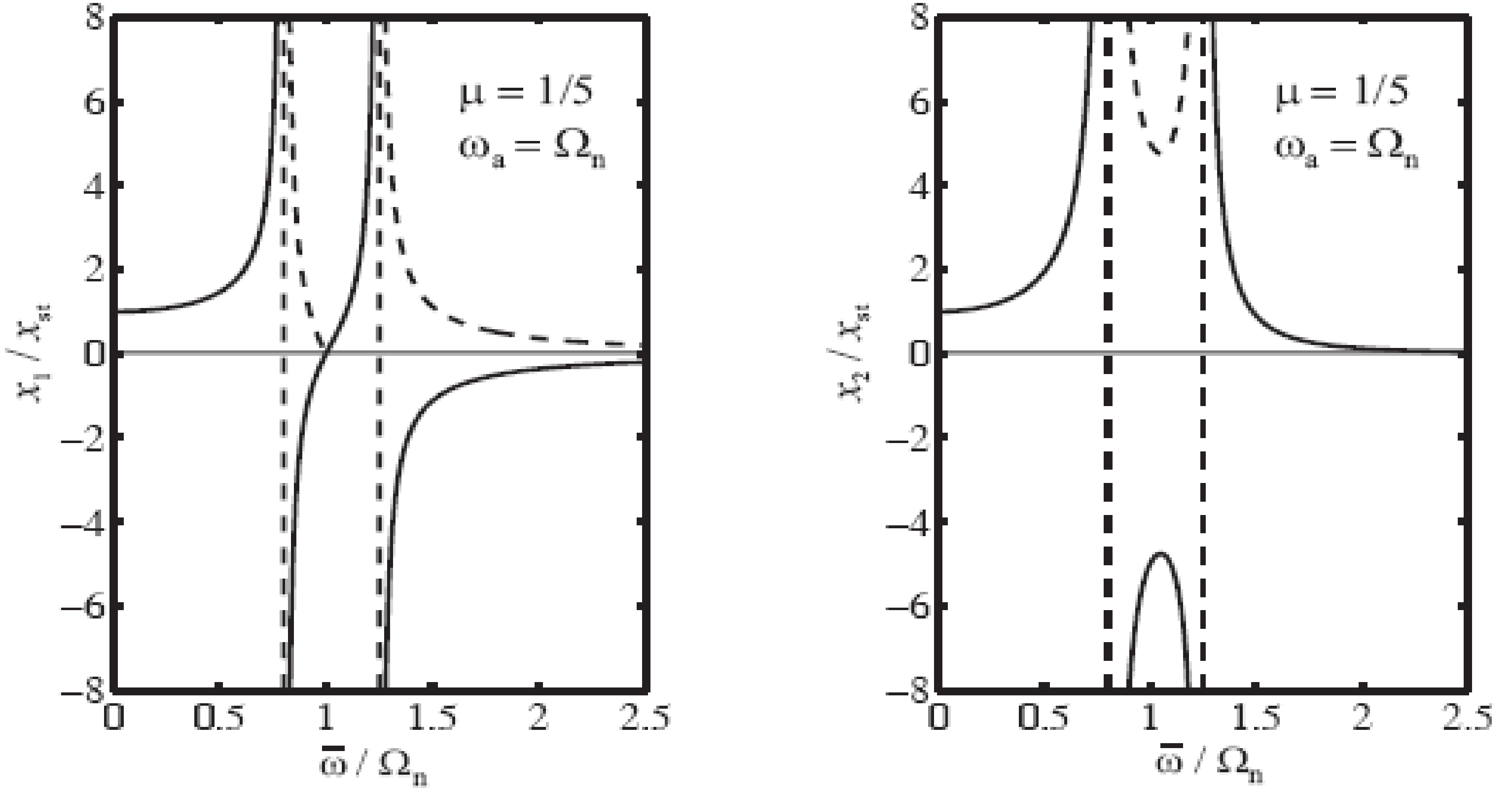


Figure 8.4 Amplitude Spectrum for TMDs Tuned to the Primary Structure: $\omega_a = \Omega_n$, (after Den Hartog 1985)

3. Theory of Undamped Tuned-mass Dampers Under Harmonic Base Motion

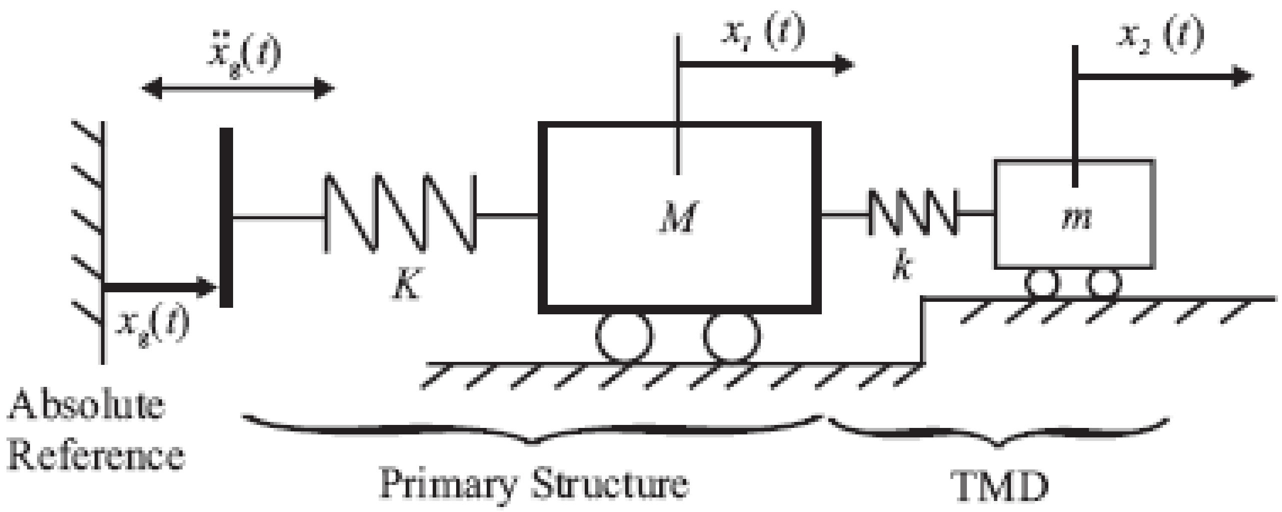
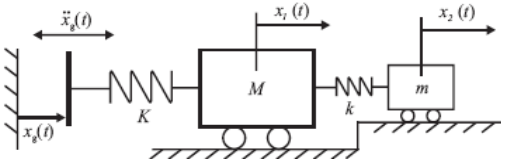


Figure 8.5 Primary Structure with TMD Subjected to a Base Excitation

$$x_g(t) = x_0 \sin \bar{\omega} t$$

3. Theory of Undamped Tuned-mass Dampers Under Harmonic Base Motion



Applying Newton's second law on each mass now yields the two equations of motion for this two-degrees-of-freedom-system:

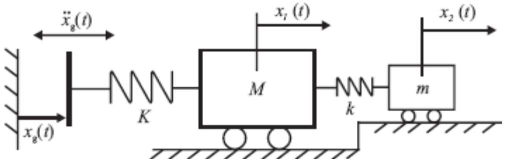
$$
\begin{aligned}
M\ddot{x}_1 + (K+k)x_1 - kx_2 &= M\bar{\omega}^2 x_0 \sin\bar{\omega}t \\
m\ddot{x}_2 + k(x_2 - x_1) &= m\bar{\omega}^2 x_0 \sin\bar{\omega}t
\end{aligned}
\qquad (8.15)
$$

Recalling the form of the solution given in Equation (8.3), we get:

$$
\begin{aligned}
x_1(t) &= a_1 \sin\bar{\omega}t \\
x_2(t) &= a_2 \sin\bar{\omega}t
\end{aligned}
$$

$$
\begin{aligned}
(-Ma_1\bar{\omega}^2 + (K+k)a_1 - ka_2)\sin\bar{\omega}t &= M\bar{\omega}^2 x_0 \sin\bar{\omega}t \\
(-ma_2\bar{\omega}^2 + k(a_2 - a_1))\sin\bar{\omega}t &= m\bar{\omega}^2 x_0 \sin\bar{\omega}t
\end{aligned}
\qquad (8.16)
$$

3. Theory of Undamped Tuned-mass Dampers Under Harmonic Base Motion



Since Equation (8.16) must be satisfied at all time:

$$
\begin{aligned}
a_1(-M\bar{\omega}^2 + K + k) - ka_2 &= M\bar{\omega}^2 x_0 \\
-ka_1 + a_2(-m\bar{\omega}^2 + k) &= m\bar{\omega}^2 x_0
\end{aligned}
\tag{8.17}
$$

Using the same variables defined in Equation (8.6), Equation (8.17) becomes:

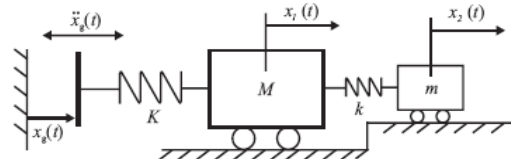
$$
\begin{aligned}
a_1\left(1 + \frac{k}{K} - \frac{\bar{\omega}^2}{\Omega_n^2}\right) - a_2\frac{k}{K} &= \frac{\bar{\omega}^2}{\Omega_n^2}x_0 \\
-a_1 + a_2\left(1 - \frac{\bar{\omega}^2}{\omega_a^2}\right) &= \frac{\bar{\omega}^2}{\omega_a^2}x_0
\end{aligned}
\tag{8.18}
$$

$x_{st} = \dfrac{P_0}{K}$: static displacement of the primary structure

$\Omega_n^2 = \dfrac{K}{M}$: natural frequency of the primary structure

$\omega_a^2 = \dfrac{k}{m}$: natural frequency of the TMD

3. Theory of Undamped Tuned-mass Dampers Under Harmonic Base Motion

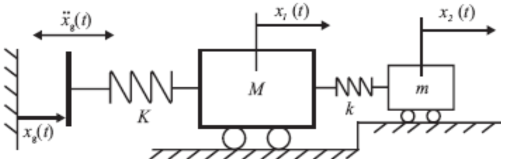


Solving for the amplitudes a_1 and a_2

$$\frac{a_1}{x_0} = \frac{\bar{\omega}^2\left(\frac{1}{\Omega_n^2}\left(1-\frac{\bar{\omega}^2}{\omega_a^2}\right)+\frac{1}{\omega_a^2}\left(\frac{k}{K}\right)\right)}{\left(1-\frac{\bar{\omega}^2}{\omega_a^2}\right)\left(1+\frac{k}{K}-\frac{\bar{\omega}^2}{\Omega_n^2}\right)-\frac{k}{K}}$$

$$\frac{a_2}{x_0} = \frac{\frac{\bar{\omega}^2}{\Omega_n^2}+\frac{\bar{\omega}^2}{\omega_a^2}\left(1+\frac{k}{K}-\frac{\bar{\omega}^2}{\Omega_n^2}\right)}{\left(1-\frac{\bar{\omega}^2}{\omega_a^2}\right)\left(1+\frac{k}{K}-\frac{\bar{\omega}^2}{\Omega_n^2}\right)-\frac{k}{K}} \qquad (8.19)$$

3. Theory of Undamped Tuned-mass Dampers Under Harmonic Base Motion



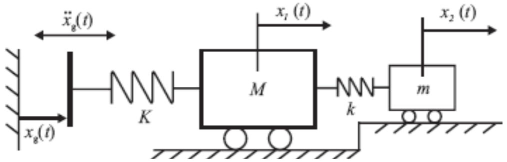
From Equation (8.19), two tuning conditions can be obtained. The first tuning condition is obtained when the displacement of the main mass M relative to the base is equal to zero. For this condition, the main mass moves rigidly with the base motion, experiencing an absolute acceleration equals $\ddot{x}_g(t)$, but with no force induced in the main spring K. This tuning condition exists when a_1 is equal to zero, which from Equation (8.19) corresponds to:

$$\frac{1}{\Omega_n^2}\left(1-\frac{\bar{\omega}^2}{\omega_a^2}\right)+\frac{1}{\omega_a^2}\left(\frac{k}{K}\right) = 0 \tag{8.20}$$

Equation (8.20) can be simplified to reveal the following tuning condition on the natural frequency of the TMD:

$$\omega_a = \frac{\bar{\omega}}{\sqrt{1+\mu}} \tag{8.21}$$

3. Theory of Undamped Tuned-mass Dampers Under Harmonic Base Motion



The second tuning condition is obtained when the absolute displacement of the main mass M is equal to zero. For this condition, the main mass remains immobile, experiencing an absolute acceleration equal to zero during the movement of the base, but a force $-Kx_g(t)$ is induced in the main spring. This tuning condition exists when a_1 is equal to $-x_0$, which, from Equation (8.19), corresponds to:

$$\frac{\bar{\omega}^2}{\Omega_n^2}\left(1-\frac{\bar{\omega}^2}{\omega_a^2}\right)+\frac{\bar{\omega}^2}{\omega_a^2}\left(\frac{k}{K}\right) = -\left(\left(1-\frac{\bar{\omega}^2}{\omega_a^2}\right)\left(1+\frac{k}{K}-\frac{\bar{\omega}^2}{\Omega_n^2}\right)-\frac{k}{K}\right) \tag{8.22}$$

Equation (8.22) can be simplified to reveal the following tuning condition on the natural frequency of the TMD:

$$\omega_a = \bar{\omega} \tag{8.23}$$

4. Theory of Damped Tuned-mass Dampers Under Harmonic Loading

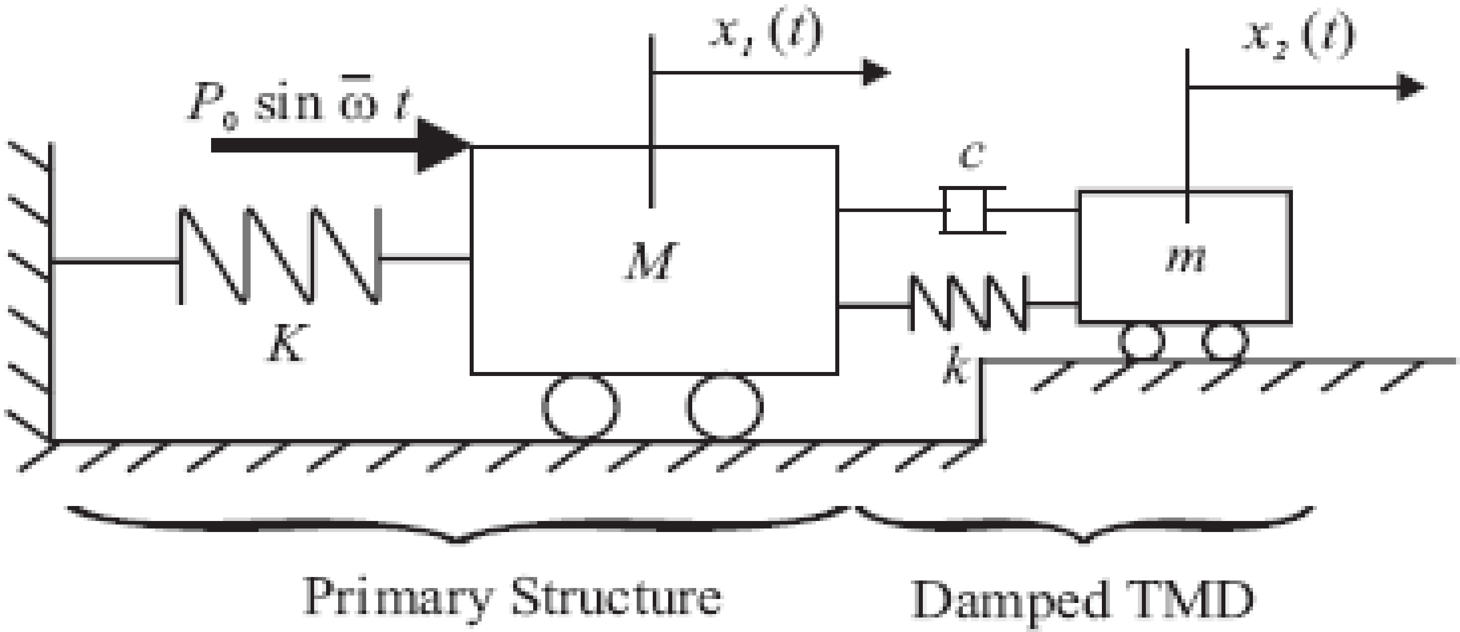
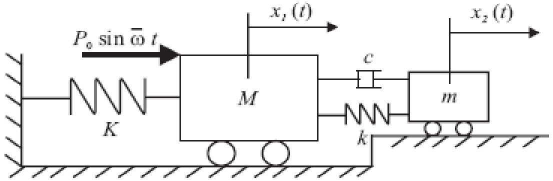


Figure 8.6 Primary Structure and Damped TMD

4. Theory of Damped Tuned-mass Dampers Under Harmonic Loading



Applying Newton's second law to the main mass M gives:

$$M\ddot{x}_1 + Kx_1 + k(x_1 - x_2) + c(\dot{x}_1 - \dot{x}_2) = P_0 \sin \bar{\omega} t \quad (8.24)$$

and to the secondary mass m yields:

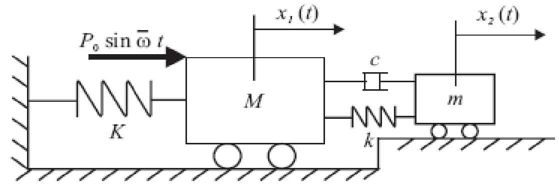
$$m\ddot{x}_2 + k(x_2 - x_1) + c(\dot{x}_2 - \dot{x}_1) = 0 \quad (8.25)$$

Again, we are interested in a solution of the forced vibrations only and do not consider the transient free vibration. Both x_1 and x_2 are harmonic motions at a frequency $\bar{\omega}$ and can be represented by complex numbers:

$$\begin{aligned} x_1(t) &= C_1 e^{i\bar{\omega}t} \\ x_2(t) &= C_2 e^{i\bar{\omega}t} \end{aligned} \quad (8.26)$$

where C_1 and C_2 are now unknown complex numbers with each an amplitude and a phase and $i = \sqrt{-1}$. We are now interested in finding the amplitude of the main mass a_1.

4. Theory of Damped Tuned-mass Dampers Under Harmonic Loading



Substituting Equation (8.26) into Equations (8.24) and (8.25) yields:

$$
\begin{aligned}
-M\bar{\omega}^2 C_1 + KC_1 + k(C_1 - C_2) + i\bar{\omega}c(C_1 - C_2) &= P_0 \\
-m\bar{\omega}^2 C_2 + k(C_2 - C_1) + i\bar{\omega}c(C_2 - C_1) &= 0
\end{aligned}
\qquad (8.27)
$$

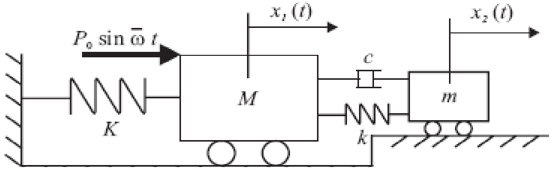
Rearranging Equation (8.27):

$$
\begin{aligned}
(-M\bar{\omega}^2 + K + k + i\bar{\omega}c)C_1 - (k + i\bar{\omega}c)C_2 &= P_0 \\
-(k + i\bar{\omega}c)C_1 + (-m\bar{\omega}^2 + k + i\bar{\omega}c)C_2 &= 0
\end{aligned}
\qquad (8.28)
$$

Solving these two equations for C_1:

$$
C_1 = P_0 \frac{(k - m\bar{\omega}^2) + i\bar{\omega}c}{[(-M\bar{\omega}^2 + K)(-m\bar{\omega}^2 + k) - m\bar{\omega}^2 k] + i\bar{\omega}c[-M\bar{\omega}^2 + K - m\bar{\omega}^2]} \qquad (8.29)
$$

4. Theory of Damped Tuned-mass Dampers Under Harmonic Loading



Since C_1 is complex, it can also be written as:

$$C_1 = P_0(A_1 + iB_1) \tag{8.30}$$

where A_1 and B_1 are real. The amplitude of C_1 can then be written as:

$$a_1 = |C_1| = P_0\sqrt{A_1^2 + B_1^2} \tag{8.31}$$

But Equation (8.29) is not in the form of Equation (8.30) but rather in the form:

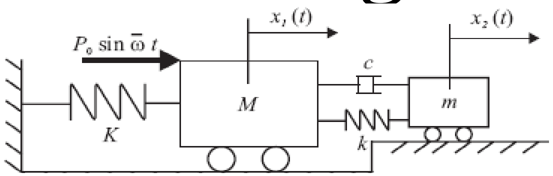
$$C_1 = P_0\frac{A + iB}{C + iD} \tag{8.32}$$

$$C_1 = P_0\frac{(k - m\bar{\omega}^2) + i\bar{\omega}c}{[(-M\bar{\omega}^2 + K)(-m\bar{\omega}^2 + k) - m\bar{\omega}^2 k] + i\bar{\omega}c[-M\bar{\omega}^2 + K - m\bar{\omega}^2]}$$

with:

$$\begin{aligned} A &= k - m\bar{\omega}^2 \\ B &= \bar{\omega}c \\ C &= (-M\bar{\omega}^2 + K)(-m\bar{\omega}^2 + k) - m\bar{\omega}^2 k \\ D &= \bar{\omega}c(-M\bar{\omega}^2 + K - m\bar{\omega}^2) \end{aligned} \tag{8.33}$$

4. Theory of Damped Tuned-mass Dampers Under Harmonic Loading



Now Equation (8.32) can be rewritten in the form of Equation (8.30):

$$C_1 = P_0 \frac{(A+iB)(C-iD)}{(C+iD)(C-iD)} = P_0 \frac{(AC+BD)+i(BC-AD)}{C^2+D^2} \qquad (8.34)$$

a_1 can then be evaluated:

$$\begin{aligned}\frac{a_1}{P_0} &= \sqrt{\left(\frac{(AC+BD)}{C^2+D^2}\right)^2 + \left(\frac{BC-AD}{C^2+D^2}\right)^2} \\ &= \sqrt{\frac{A^2C^2+B^2D^2+B^2C^2+A^2D^2}{(C^2+D^2)}} \\ &= \sqrt{\frac{(A^2+B^2)(C^2+D^2)}{(C^2+D^2)^2}} \\ &= \sqrt{\frac{(A^2+B^2)}{(C^2+D^2)}}\end{aligned} \qquad (8.35)$$

4. Theory of Damped Tuned-mass Dampers Under Harmonic Loading

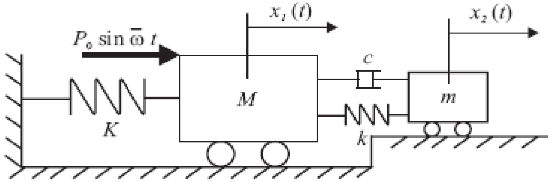
Substituting the values of the constants expressed in Equation (8.33) into Equation (8.35) yields an expression for the amplitude of the response of the main mass M:

$$\frac{a_1}{P_0} = \sqrt{\frac{(k - m\bar{\omega}^2) + \bar{\omega}^2 c^2}{[(-M\bar{\omega}^2 + K)(-m\bar{\omega}^2 + k) - m\bar{\omega}^2 k]^2 + \bar{\omega}^2 c^2 (-M\bar{\omega}^2 + K - m\bar{\omega}^2)^2}} \tag{8.36}$$

We can rewrite Equation (8.36) by defining the following variables:

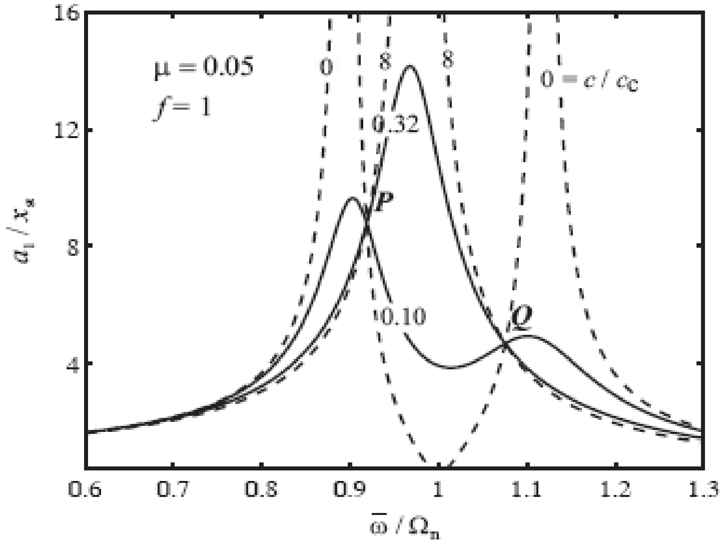
$$\begin{aligned}
\mu &= \frac{m}{M} = \frac{\text{TMD Mass}}{\text{main mass}} = \text{mass ratio} \\
\omega_a^2 &= \frac{k}{m} = \text{natural frequency of TMD} \\
\Omega_n^2 &= \frac{K}{M} = \text{natural frequency of primary structure} \\
f &= \frac{\omega_a}{\Omega_n} = \text{natural frequency ratio} \\
g &= \frac{\bar{\omega}}{\Omega_n} = \text{forcing frequency ratio} \\
c_c &= 2\omega_a m = \text{critical viscous damping constant of TMD}
\end{aligned} \tag{8.37}$$

4. Theory of Damped Tuned-mass Dampers Under Harmonic Loading

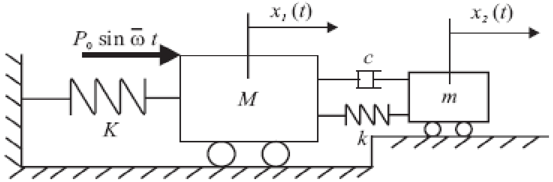


After some further algebraic manipulations, Equation (8.36) can be rewritten as:

$$\frac{a_1}{x_{st}} = \sqrt{\frac{\left(2\frac{c}{c_c}g\right)^2 + (g^2 - f^2)^2}{\left(2\frac{c}{c_c}g\right)^2 (g^2 - 1 + \mu g^2)^2 + [\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)]^2}} \qquad (8.38)$$



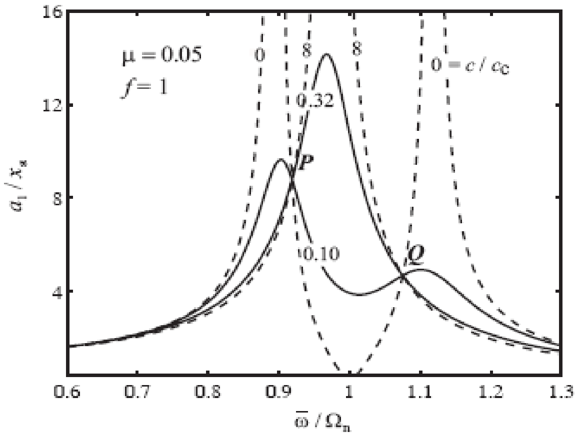
4. Theory of Damped Tuned-mass Dampers Under Harmonic Loading



First the locations of the two points P and Q are found. We can rewrite Equation (8.38) as:

$$\frac{a_1}{x_{st}} = \sqrt{\frac{\left(2\frac{c}{c_c}g\right)^2 + (g^2 - f^2)^2}{\left(2\frac{c}{c_c}g\right)^2 (g^2 - 1 + \mu g^2)^2 + [\mu f^2 g^2 - (g^2-1)(g^2-f^2)]^2}}$$

$$\frac{a_1}{x_{st}} = \sqrt{\frac{A\left(\frac{c}{c_c}\right)^2 + B}{C\left(\frac{c}{c_c}\right)^2 + D}} \quad (8.39)$$



with:

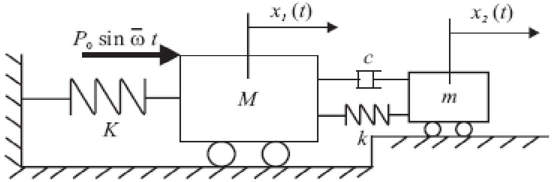
$$A = (2g)^2$$

$$B = (g^2 - f^2)^2$$

$$C = (2g)^2 (g^2 - 1 + \mu g^2)^2$$

$$D = (\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2))^2 \quad (8.40)$$

4. Theory of Damped Tuned-mass Dampers Under Harmonic Loading



If $A/C = B/D$, Equation (8.39) becomes independent of damping. This condition is given by:

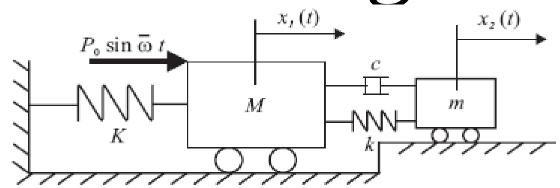
$$\left(\frac{1}{g^2 - 1 + \mu g^2}\right)^2 = \left(\frac{g^2 - f^2}{\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)}\right)^2 \qquad (8.41)$$

To remove the square sign on each side of Equation (8.41), a $\pm$ must be introduced in front of one side of the equation. With the minus sign, the solution becomes trivial since we find $g^2 = 0$, meaning that the static response is independent of damping.

The other alternative is the plus sign which leads to:

$$g^4 - 2g^2 \frac{1 + f^2 + \mu f^2}{2 + \mu} + \frac{2f^2}{2 + \mu} = 0 \qquad (8.42)$$

4. Theory of Damped Tuned-mass Dampers Under Harmonic Loading

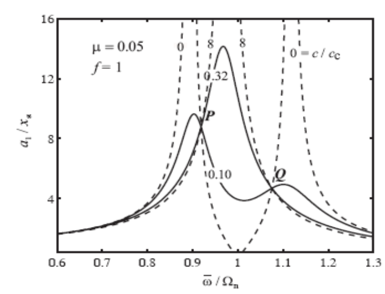


Equation (8.42) is a quadratic function in g^2, giving two roots (g_1 and g_2) representing the coordinates of the fixed points P and Q. These roots are still function of μ and f.

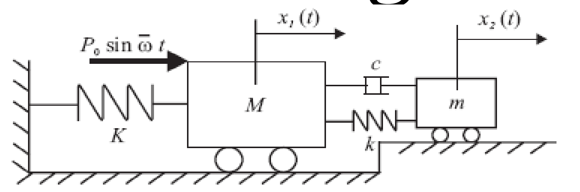
To adjust the frequency tuning such that the amplitudes of points P and Q are equal, the roots of Equation (8.42) are found and substituted into Equation (8.38). When the expressions for P and Q are equated, a simple relation between μ and f is obtained:

$$f = \frac{1}{1+\mu} \qquad (8.43)$$

Note that c/c_c cancels out since the amplitudes of points P and Q are independent of damping.



4. Theory of Damped Tuned-mass Dampers Under Harmonic Loading

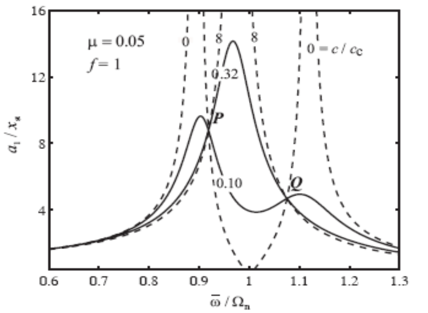


Now to find the optimum damping $(c/c_c)_{opt}$ Equation (8.43) is substituted into Equation (8.38). The resulting equation is differentiated with respect to g and set equal to zero while one of the two roots obtained in Equation (8.42) is also replaced in Equation (8.38). From this calculation, we obtain for an optimum at point P:

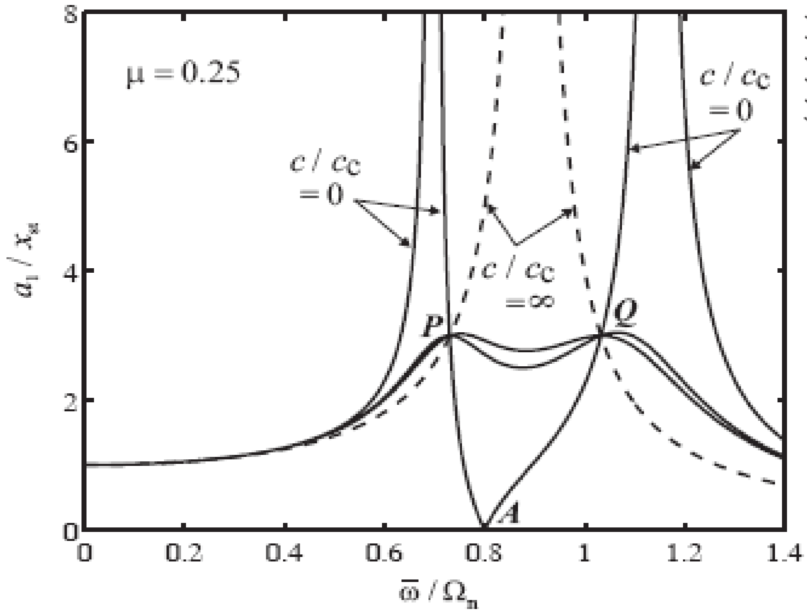
$$\left(\frac{c}{c_c}\right)^2_{opt-P} = \frac{\mu\left(3 - \sqrt{\frac{\mu}{\mu+2}}\right)}{8(1+\mu)^3} \qquad (8.44)$$

Alternatively, if the derivative is set to zero at point Q, we also get:

$$\left(\frac{c}{c_c}\right)^2_{opt-Q} = \frac{\mu\left(3 + \sqrt{\frac{\mu}{\mu+2}}\right)}{8(1+\mu)^3} \qquad (8.45)$$



4. Theory of Damped Tuned-mass Dampers Under Harmonic Loading



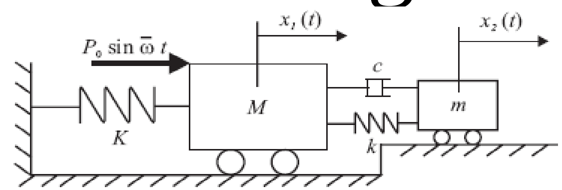


Figure 8.8 Resonance Curves for Optimum Frequency and Damping Tuning (after Den Hartog 1985)

In practice, for optimum tuning, the mean value of Equations (8.44) and (8.45) is used:

$$\left(\frac{c}{c_c}\right)_{opt} = \sqrt{\frac{(3\mu)}{8(1+\mu)^3}} \qquad (8.46)$$

5. Seismic Application of Tuned-Mass Dampers

Table 8-1: Optimum Tuning Conditions for damped TMDs Attached to Undamped Primary Structure (after Constantinou et al. 1998)

Loading Case	Optimization Criteria	Optimum Tuning Conditions	
		Frequency, f	Damping, c/c_c
1) Harmonic Load Applied to Primary Structure	Minimum Relative Displacement Amplitude of Primary Structure	$\frac{1}{1+\mu}$ [Equation (8.43)]	$\sqrt{\frac{3\mu}{8(1+\mu)^3}}$ [Equation (8.46)]
2) Harmonic Load Applied to Primary Structure	Minimum Relative Acceleration Amplitude of Primary Structure	$\frac{1}{\sqrt{1+\mu}}$	$\sqrt{\frac{3\mu}{8\left(1+\frac{\mu}{2}\right)}}$
3) Harmonic Base Acceleration	Minimum Relative Displacement Amplitude of Primary Structure	$\frac{\sqrt{1-\frac{\mu}{2}}}{1+\mu}$	$\sqrt{\frac{3\mu}{8(1+\mu)\left(1-\frac{\mu}{2}\right)}}$
4) Harmonic Base Acceleration	Minimum Absolute Acceleration Amplitude of Primary Structure	$\frac{1}{1+\mu}$	$\sqrt{\frac{3\mu}{8(1+\mu)}}$
5) Random Load Applied to Primary Structure	Minimum Root Mean Square Value of Relative Displacement of Primary Structure	$\frac{\sqrt{1-\frac{\mu}{2}}}{1+\mu}$	$\sqrt{\frac{\mu\left(1+\frac{3\mu}{4}\right)}{4(1+\mu)\left(1-\frac{\mu}{2}\right)}}$
6) Random Base Acceleration	Minimum Root Mean Square Value of Relative Displacement of Primary Structure	$\frac{\sqrt{1-\frac{\mu}{2}}}{1+\mu}$	$\sqrt{\frac{\mu\left(1-\frac{\mu}{4}\right)}{4(1+\mu)\left(1-\frac{\mu}{2}\right)}}$

5. Seismic Application of Tuned-Mass Dampers

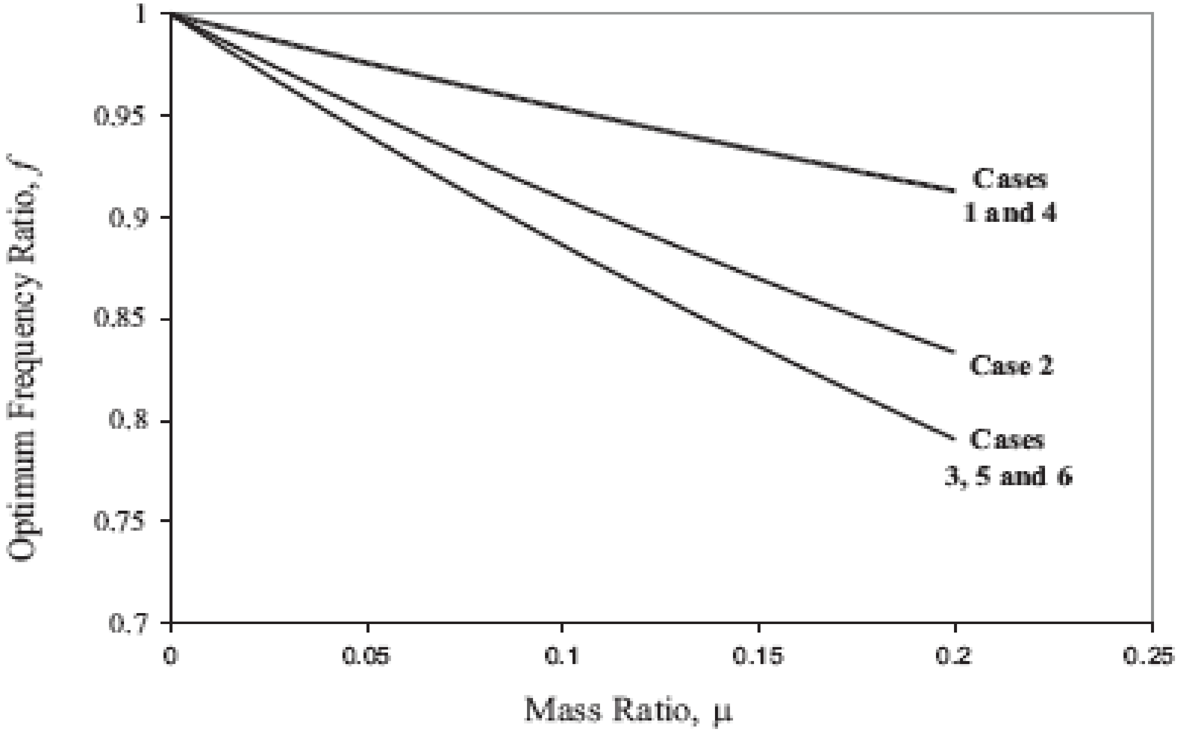


Figure 8.9 Optimum Frequency Tuning for Cases Listed in Table 8-1

5. Seismic Application of Tuned-Mass Dampers

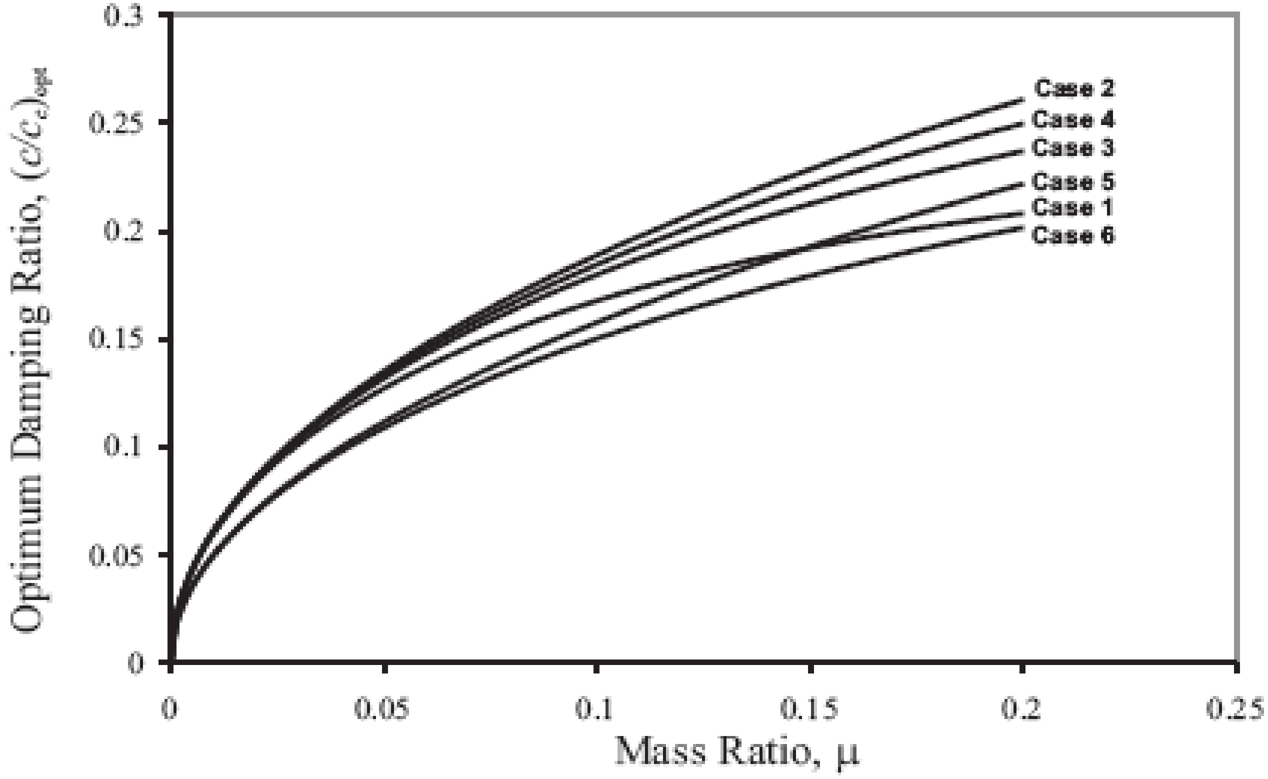
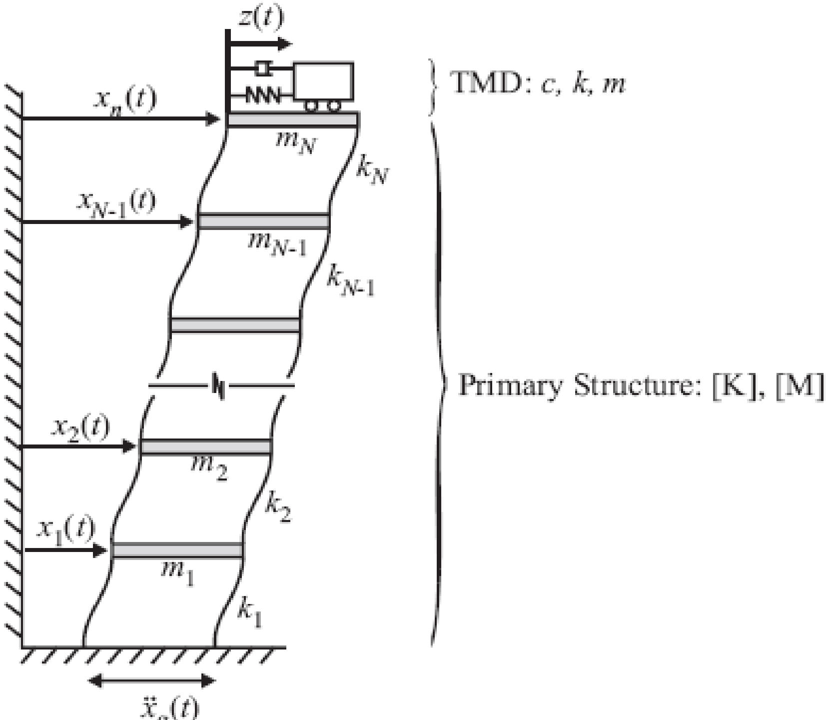


Figure 8.10 Optimum Damping Tuning for Cases Listed in Table 8-1

6. Analysis of Structures with Tuned-Mass Dampers

- TMD tuned to a single structural frequency.
- For seismic applications, TMDs usually tuned to fundamental mode of vibration.
- TMD often installed on building roof.



6. Analysis of Structures with Tuned-Mass Dampers

Neglecting the damping in the primary structure, the equations of motions for this coupled system are given by:

$$\begin{aligned} [M]\{\ddot{x}(t)\} + [K]\{x(t)\} &= -[M]\{r\}\ddot{x}_g(t) + \{P(t)\} \\ m\ddot{z} + c\dot{z} + kz &= -m\ddot{x}_N - m\ddot{x}_g \end{aligned} \qquad (8.47)$$

where N is the number of degrees of freedom (considering one degree-of-freedom per floor of the building) of the primary structure and represents the N^{th} level degree of freedom, $[M]$ and $[K]$ are the global mass and stiffness matrices of the main structure; $\{x\}$ and $(\ddot{x})$ are the displacements and accelerations of the structure relative to the ground; $\{r\}$ is the dynamic coupling vector; m, c, and k are the mass, damping constant and stiffness of the TMD; $z(t)$ is the displacement of the TMD relative to the roof; and $\{P(t)\} = \{0, ..., 0, c\dot{z} + kz\}^T$. From Equation (8.47), it is clear that the structural analysis needs to be carried in the $(N+1)$-dimensional space under general conditions.

6. Analysis of Structures with Tuned-Mass Dampers

Now consider the case where, under ground motion, the structure responds primarily in its first mode of vibration and where the response vector $\{x(t)\}$ can be approximated by:

$$\{x(t)\} = \{A^{(1)}\}x_N(t) \tag{8.48}$$

where $\{A^{(1)}\}$ is the first mode shape and $x_N(t)$ is the displacement of the roof relative to the ground.

Substituting Equation (8.48) into the first expression of Equation (8.47), pre-multiplying it by $\{A^{(1)}\}^T$, and using the orthogonality conditions of the mode shapes yields:

$$M_1\ddot{x}_N(t) + K_1x_N(t) = c\dot{z}(t) + kz(t) - \alpha_1M_1\ddot{x}_g \tag{8.49}$$

where:

$$\begin{aligned} M_1 &= \{A^{(1)}\}^T[m]\{A^{(1)}\} = \text{generalized mass in first mode} \\ K_1 &= \{A^{(1)}\}^T[k]\{A^{(1)}\} = \text{generalized stiffness coefficient in first mode} \\ \alpha_1 &= \frac{\{A^{(1)}\}^T[m]\{r\}}{M_1} = \text{modal participation factor in first mode} \end{aligned} \tag{8.50}$$

6. Analysis of Structures with Tuned-Mass Dampers

By comparing the second expression of Equation (8.47) and Equation (8.49) with Equations (8.24) and (8.25), one can conclude that the first modal representation of a multi-degree-of-freedom structure is exactly the same as that of a single-degree-of-freedom structure, except that the modal mass and the modal stiffness are employed instead of the physical parameters in the SDOF case. Therefore, the tuning of a TMD for the fundamental mode of a multi-degree-of-freedom structural system can be performed by using Equations (8.43) and (8.46) or Figures 8.9 and 8.10 or Table 8-1 with:

$$\begin{aligned} \mu &= \frac{m}{M_1} = \text{mass ratio} \\ \Omega_N &= \omega_1 = \text{fundamental frequency of main structure} \end{aligned} \tag{8.51}$$

$$\begin{aligned} M_1\ddot{x}_N(t) + K_1x_N(t) &= c\dot{z}(t) + kz(t) - \alpha_1 M_1\ddot{x}_g \\ m\ddot{z} + c\dot{z} + kz &= -m\ddot{x}_N - m\ddot{x}_g \end{aligned}$$

$$\begin{aligned} M\ddot{x}_1 + Kx_1 + k(x_1 - x_2) + c(\dot{x}_1 - \dot{x}_2) &= P_0 \sin\bar{\omega}t \\ m\ddot{x}_2 + k(x_2 - x_1) + c(\dot{x}_2 - \dot{x}_1) &= 0 \end{aligned}$$

$x_1 = x_N$

$x_2 = x_N + z$

6. Analysis of Structures with Tuned-Mass Dampers

- Same approach can mitigate vibrations of any mode.
- A structure with a TMD may experience inelastic deformations during a strong earthquake.
- When inelastic deformations occur, fundamental frequency decreases.
- TMD may lose effectiveness due to de-tuning effect.
- Detuning phenomenon discussed in next section.

7. Seismic Response of Inelastic Buildings with Tuned-Mass Dampers

- Carr (2005) investigated seismic response of shear wall reinforced concrete buildings equipped with TMDs
- Main objective to investigate seismic fragility of elastic and inelastic reinforced concrete buildings with TMDs of various sizes
- Ensembles of ground motions representing various seismic hazard levels

7. Seismic Response of Inelastic Buildings with Tuned-Mass Dampers

- Building Models
  - Shear wall-type buildings, 3 and 10, and 25 stories.

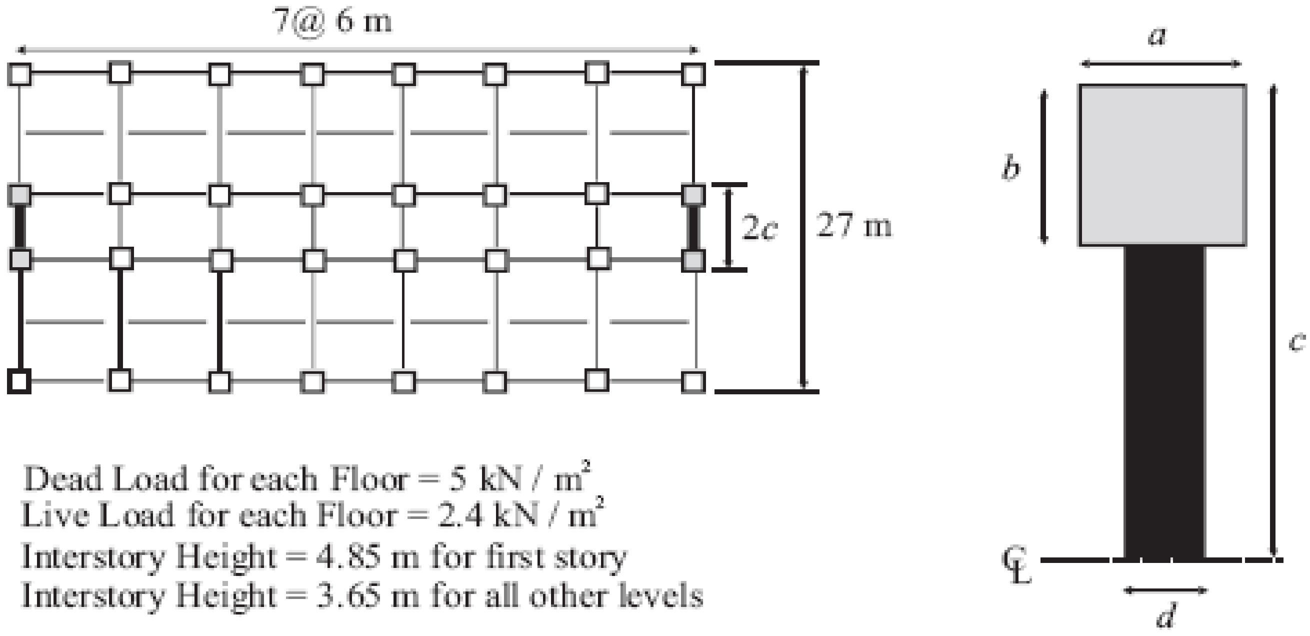


Figure 8.12 Layout and Design Loads for 10-Storey Building Model

7. Seismic Response of Inelastic Buildings with Tuned-Mass Dampers

- Analysis Procedure
  - Two-dimensional lumped-mass model of each building model with and without TMDs.
  - Torsional effects neglected.
  - Each model included only one wall with one gravity column.
  - Total dead loads acting on interior columns applied to gravity column.
  - TMD modeled as a SDOF system on roof of building.
  - TMD tuned to Equations (8.43) and (8.46).

Table 8-3: Properties of TMDs for 10-Storey Building

Mass Ratio	TMD Natural Period (s)	TMD Damping Ratio
0.05	1.72	0.13
0.10	1.80	0.17
0.20	1.97	0.21

7. Seismic Response of Inelastic Buildings with Tuned-Mass Dampers

- Ground Motions
  - Ensembles of synthetic strong ground motions generated for Southern California site.
  - Ground motions for 2%, 5%, 10%, and 20% probabilities of exceedence in 50 years (return periods of 2475, 975, 475, and 224 years).
  - Each ensemble comprised of 25 earthquake records.
  - Total of 100 strong ground motions considered.
  - Strong motions simulated using the Specific Barrier Model (Papageorgiou and Aki 1983a, 1983b).

7. Seismic Response of Inelastic Buildings with Tuned-Mass Dampers

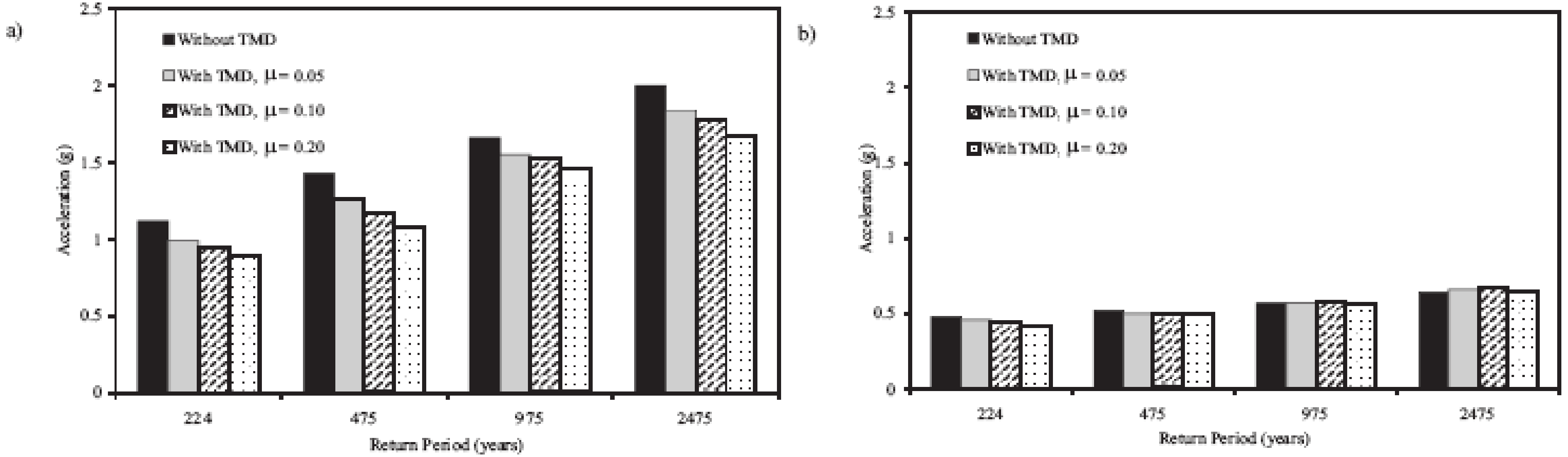


Figure 8.13 Mean Peak Absolute Floor Acceleration, 10-Storey Building with and without TMD: a) Elastic Response, b) Inelastic Response

8. Design Considerations

The design procedure of a TMD for a building structure oscillating mainly in its fundamental frequency and mode can be carried out with the following steps.

a) **Step 1: Evaluation of Mass Ratio**: First, the equivalent viscous damping ratio of the structure-TMD assembly needs to be identified. A procedure proposed by Luft (1979) can be used for this purpose. This involves looking at design acceleration and displacement response spectra, S_D and S_A, and selecting an appropriate damping value ξ_{eq} that satisfies:

$$\begin{cases} \alpha_1 S_D(\omega_1, \xi_{eq}) \le x_{N(max)} \\ \alpha_1 S_A(\omega_1, \xi_{eq}) \le \ddot{x}_{n(max)} \end{cases} \tag{8.52}$$

where $x_{N(max)}$ and $\ddot{x}_{N(max)}$ are the target maximum relative displacement and maximum absolute acceleration at the roof level of the building.

8. Design Considerations

The required mass ratio μ can then be estimated by the following equation (Luft, 1979):

$$\mu = 16(\xi_{eq} - 0.8\xi_1)^2 = \frac{m}{M_1} \tag{8.53}$$

e.g.:
$\zeta_{eq} = 0.20$
$\zeta_1 = 0.05$
$\mu = 0.41$ → very difficult

where ξ_1 is the first modal damping ratio of the main structure. Note that, in most practical applications, the selection of μ is limited by physical considerations.

b) **Step 2: Tuning of TMD properties**: Figures 8.9 and 8.10 or Table 8-1 can be used to estimate the optimum frequency ratio and damping of the TMD.

c) **Step 3: Structural Dynamic Analysis Check**: The final step in designing the TMD is to check that the selected TMD parameters give the building response that is in the range of predetermined response threshold. Otherwise, the preliminary design has to be refined by trial-and-error.

Thank you!