19 maggio

$f \in C^1(A, \mathbb{R}) \qquad A \subseteq \mathbb{R}^d$

cioè $\quad Df \in C^0(A, \underbrace{\mathcal{L}(\mathbb{R}^d, \mathbb{R})}_{\mathbb{R}^d})$

e tale che $D^2 f(x_0)$ esiste in $x_0 \in A$

$$D^2 f(x_0) = \left(\partial_i \partial_j f(x_0)\right)_{i,j=1}^{d} \quad \text{matrice Hessiana}$$

$d \times d$

$\exists \; A \in SO(\mathbb{R}^d)$ t.c.

$$A^{-1} D^2 f(x_0) A = {}^t A \, D^2 f(x_0) A$$

$$= \operatorname{diag}(\lambda_1, \dots, \lambda_d)$$

$$x = Ay + x_0$$

$$g(y) \doteq f(Ay + x_0), \qquad v \in \mathbb{R}^d$$

$$Dg(0)v = \frac{d}{dt} g(tv)\Big|_{t=0}$$

$$= \frac{d}{dt} f(Atv + x_0)\Big|_{t=0}$$

$$= Df(x_0)\,Av$$

$$\frac{d}{dt} Dg(y+tv)\Big|_{t=0} = Dg(y)\,v$$

$$= \frac{d}{dt} f(Ay + Atv + x_0)\Big|_{t=0}$$

$$= Df(Ay+x_0)\,Av$$

$$Dg(y)\,v = Df(Ay+x_0)\,Av$$

$$D^2g(0)(v,v) = D^2g(0)\,v^2$$

$$= \frac{d}{dt} Dg(tv)\,v\Big|_{t=0} = {}^t v\, D^2g(0)\,v$$

$$= \frac{d}{dt} Df(Atv+x_0)\,Av\Big|_{t=0} =$$

$$= {}^t(Av)\; D^2f(x_0)\,Av$$

$$= {}^t v \; \underbrace{{}^t A\, D^2f(x_0)\, A}\, v \qquad D^2g(0)$$

$$= {}^t v \; \mathrm{diag}(\lambda_1,\dots,\lambda_d)\; v$$

$$= \sum_{j=1}^{d} \lambda_j v_j^2$$

Notare che ~~da~~ se x_0 è un punto critico di f

$$Df(x_0) = 0$$

$$\Longleftrightarrow \quad Dg(0) = Df(x_0)\, A$$

$$\Longleftrightarrow \quad Dg(0) = 0.$$

Ovviamente x_0 è un punto max/min locale per f

$\Longleftrightarrow$ 0 è un punto max/min locale ~~per~~ g

$$g(y) = f(Ay + x_0)$$

Consideriamo l'espansione di Taylor

$$g(y) = \sum_{|\alpha| \leq 2} \frac{1}{\alpha!} \partial^\alpha g(0)\, y^\alpha + o(|y|^2)$$

$$\left\{\begin{aligned} &\alpha = (\alpha_1, \dots, \alpha_d) \\ &\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} \\ &y^\alpha = y_1^{\alpha} \cdots y_d^{\alpha_d} \end{aligned}\right.$$

$$\simeq g(y) = g(0) + \sum_{j=1}^{d} \frac{1}{2} \partial_j^2 \underline{g(0)} y^j + o(|y|^2)$$

$$\partial_1^2$$

$$\alpha = (2, 0, \dots, 0)$$

$$\alpha! = 2!\, \underbrace{0! \dots 0!}_{d-1}$$

$$g(y) = g(0) + \sum_{j=1}^{d} \frac{1}{2} \lambda_j y^j + o(|y|^2)$$

<u>Def</u> $D^2 f(x^0)$ è strett. (positiva) positivo se

$\lambda_j > 0 \quad \forall j = 1, \dots, d$

$(\lambda_j \geq 0)$

<u>Def</u> Diremo che un punto critico

x_0 di f è non degenere se

$D^2 f(x_0)$ è isomorfismo $\mathbb{R}^d$ ⟲.

$$\overset{\Uparrow}{\mathcal{L}^2}(\mathbb{R}^d, \mathbb{R}) \sim \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$$

<u>Lemma</u> $f \in C^1(A, \mathbb{R})$ $A \subseteq \mathbb{R}^d, x_0 \in A$

$D^2 f(x_0)$ esiste, $Df(x_0) = 0$, x_0

pt critico non degenere. Allora

1) $\lambda_j \neq 0 \quad \forall \; j = 1, \dots, d$

2) Se $D^2 f(x_0)$ è strett positivo
allora x_0 è un punto di min locale

3) Se $D^2 f(x_0)$ strett neg ...
... x_0 ... max locale

4) Se $\exists$ i e j con $\lambda_i < 0 < \lambda_j$
allora x_0 non è né un min
né un max locale.

Dim 1) $\underbrace{\det D^2 f(x_0)}_{\neq 0} = \lambda_1 \cdot \ldots \cdot \lambda_d$

$\Longleftrightarrow \quad \lambda_j \neq 0 \quad \forall j$

2) $D^2 f(x_0) > 0 \implies \lambda_j > 0 \quad \forall j$

$0 < c_0 = \min\{\lambda_1, \ldots, \lambda_d\}$

verifichiamo che 0 è un punto di min locale per $g(y)$

$$g(y) = g(0) + \sum_{j=1}^{d} \frac{1}{2} \lambda_j y_j^2 + o(|y|^2)$$

$o(|y|^2)$ significa che $\forall$

$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ t.c. $0 < |y| < \delta_\varepsilon$

$$\implies |o(|y|^2)| < \varepsilon |y|^2$$

$\varepsilon = \frac{c_0}{4}$ $\qquad 0 < |y| < \delta_\varepsilon$

$$g(y) = g(0) + \sum_{j=1}^{d} \frac{1}{2} \lambda_j y_j^2 + o(|y|^2)$$

$$\geq g(0) + \frac{c_0}{2} |y|^2 - |o(|y|^2)|$$

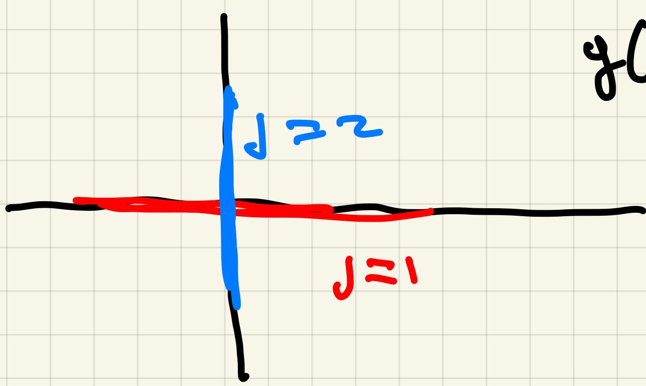
$$> g(0) + \frac{c_0}{2} |y|^2 - \frac{c_0}{4} |y|^2$$

$$= g(0) + \frac{c_0}{4} |y|^2 > g(0)$$

$\Rightarrow$ e' un punto di minimo

4) Possiamo supporre $\lambda_1 > 0$ e $\lambda_2 < 0$

$$h_j(t) = g(t e_j) \qquad j = 1, 2$$



$$g(y) = g(0) + \sum \frac{1}{2} \lambda_j y_j^2 + o(|y|^2)$$

$$h_1(t) = g(t e_1) =$$

$$= g(0) + \frac{\lambda_1}{2} t^2 + o(t^2)$$

$$h_2(t) = g(t e_2) = g(0) + \frac{\lambda_2}{2} t^2 + o(t^2)$$

$c_0 = \min\{\lambda_1, -\lambda_2\}$

$h_1(t) \geq g(0) + \frac{c_0}{2} t^2 - |o(t^2)|$

$h_2(t) \leq g(0) - \frac{c_0}{2} t^2 + |o(t^2)|$

$\varepsilon = \frac{c_0}{4}$ $\qquad 0 < |t| < \delta_\varepsilon$

$$|o(t^2)| < \frac{c_0}{4} t^2$$

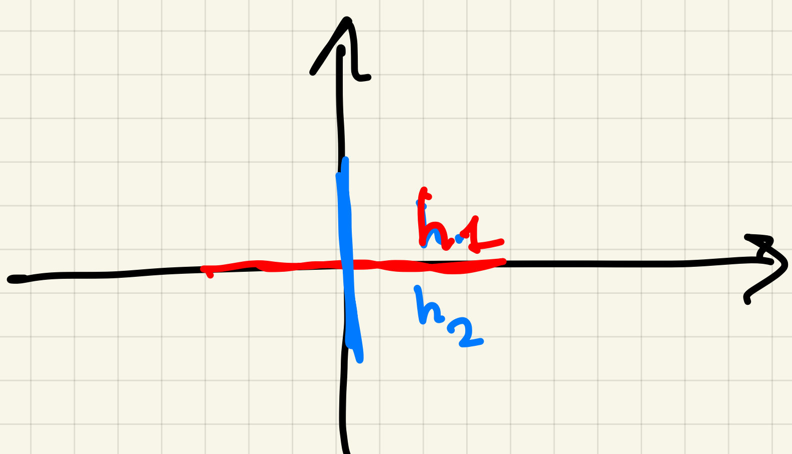
$0 < |t| < \delta_\varepsilon$

$h_1(t) \geq g(0) + \frac{c_0}{2} t^2 - \frac{c_0}{4} t^2 =$

$= g(0) + \frac{c_0}{4} t^2 \Rightarrow t = 0$ è punt di min

$h_2(t) < g(0) - \frac{c_0}{2} t^2 + \frac{c_0}{4} t^2 = g(0) - \frac{c_0}{4} t^2$

$\Rightarrow t = 0$ ho punt di max



$$f(x,y) = \frac{x^4}{4} - \frac{x^3}{3} - x^2 + y^4 + y^2$$

$$\lim_{(x,y)\to\infty} f(x,y) = \lim_{(x,y)\to\infty} \left(\frac{x^4}{4} + y^4\right) = +\infty$$

$$\begin{cases} \partial_x f = x^3 - x^2 - 2x = x(x^2 - x - 2) = 0 \\ \partial_y f = 4y^3 + 2y = 2(2y^2+1)y = 0 \end{cases}$$

$$x = 0 \qquad x_{\pm} = \frac{1}{2} \pm \frac{\sqrt{9}}{2} = \begin{cases} 2 \\ -1 \end{cases}$$

$$y = 0$$

$$(0,0), \qquad (2,0) \qquad (-1,0)$$

$$H_f(x,y) = \begin{pmatrix} \partial_x^2 f & 0 \\ 0 & \partial_y^2 f \end{pmatrix} =$$

$$= \begin{pmatrix} 3x^2 - 2x - 2 & 0 \\ 0 & 12y^2 + 2 \end{pmatrix}$$

$$H_f(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{sella}$$

$$H_f(-1,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{pt min loc}$$

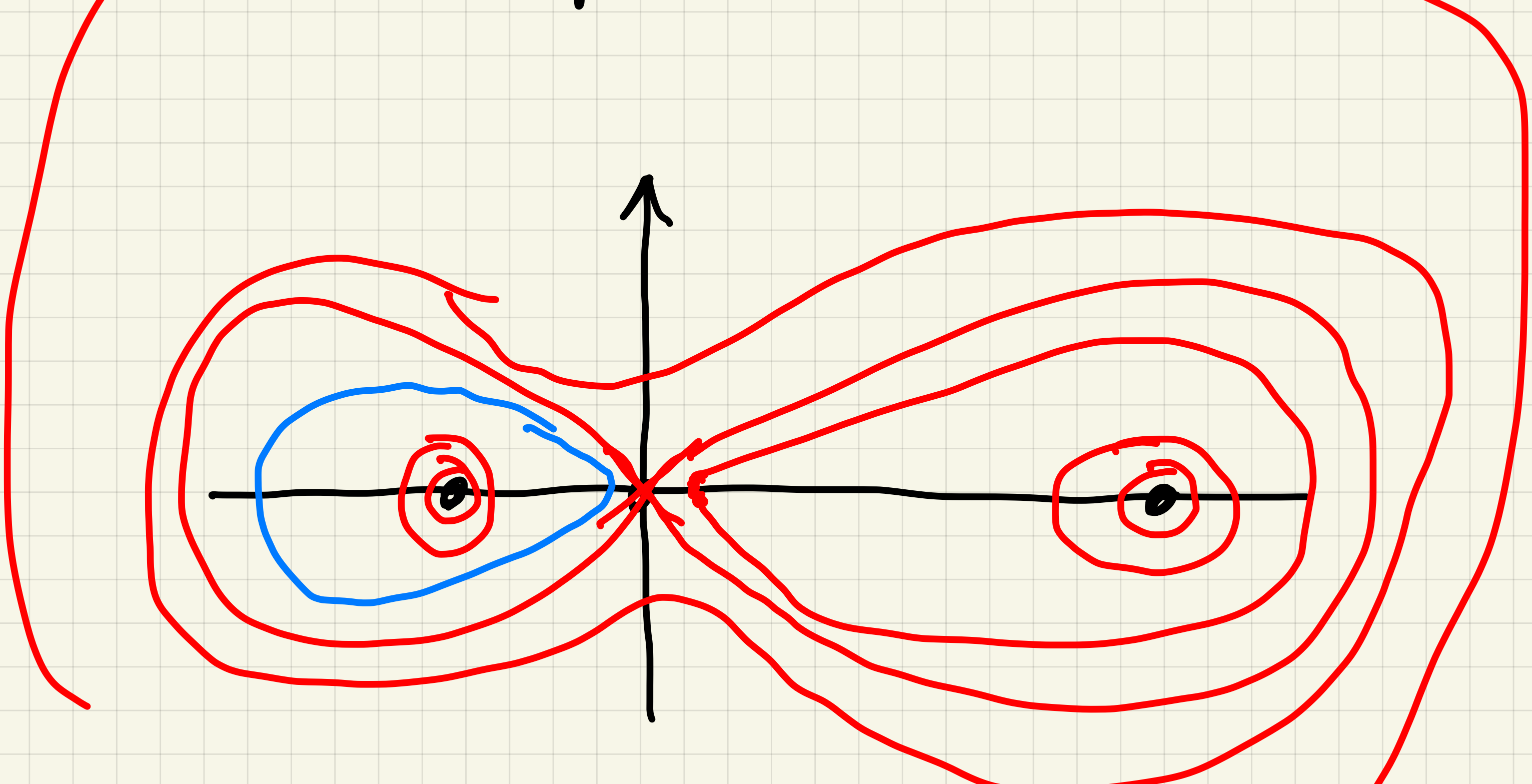
$$H_f(2,0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{pt min loc}$$

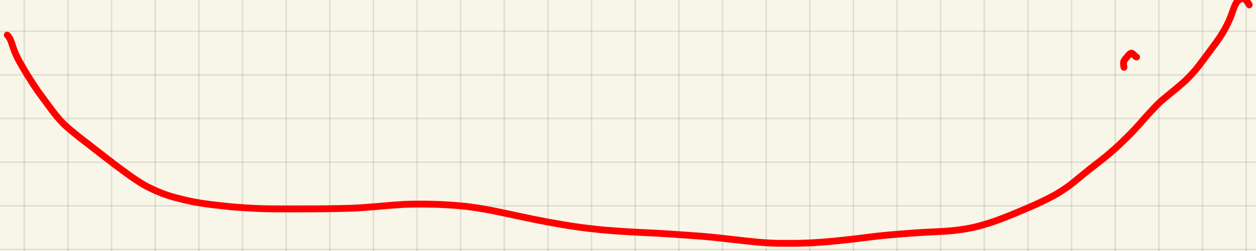
$$f(x,y) = \frac{x^4}{4} - \frac{x^3}{3} - x^2 + y^4 + y^2$$

$$f(-1,0) = \frac{1}{4} + \frac{1}{3} - 1 = \frac{3+4}{12} - 1 = -\frac{5}{12}$$

$$f(2,0) = -\frac{8}{3}$$

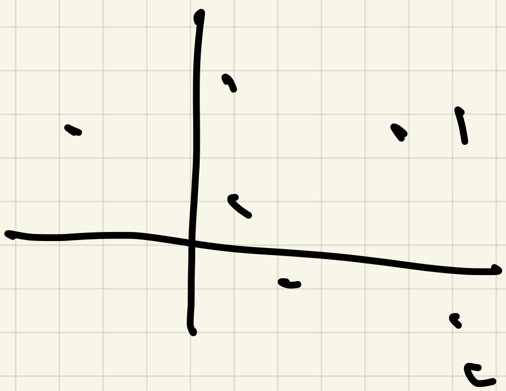
(2,0) pt min assoluto





Esempio

$(x_1, y_1) \dots (x_n, y_n)$ non allineati



$$\left| \frac{a x_j - y_j + b}{\sqrt{a^2+1}} \right|$$

$y = ax + b \qquad ax - y + b = 0$

$$f(a,b) = \sum_{j=1}^{n} (a x_j + b - y_j)^2$$

$$= a^2 \overbrace{\sum_{j=1}^{n} x_j^2}^{P} + n b^2 + \overbrace{\sum_{j=1}^{n} y_j^2}^{Q} + 2ab \overbrace{\sum_{j=1}^{n} x_j}^{X} - 2a \overbrace{\sum_{j=1}^{n} x_j y_j}^{S} - 2b \overbrace{\sum_{j=1}^{n} y_j}^{Y}$$

$$f(a,b) = a^2 P + b^2 n + Q + 2Xab - 2Sa - 2bY$$

$$f_a = 2(aP + bX - S) = 0$$

$$f_b = 2(aX + nb - Y) = 0$$

$$\begin{cases} aP + bX = S \\ aX + nb = Y \end{cases}$$

$$\begin{pmatrix} P & X \\ X & n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} S \\ Y \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{Pn - X^2} \begin{pmatrix} n & -X \\ -X & P \end{pmatrix} \begin{pmatrix} S \\ Y \end{pmatrix}$$

$$nP - X^2 > 0$$

$$nP - X^2 = n \sum_{j=1}^{n} x_j^2 - \left(\sum_{j=1}^{n} x_j \right)^2 =$$

$$= n \sum_{j=1}^{n} x_j^2 - \sum_{j,k=1}^{n} x_j x_k$$

$$> n \sum_{j=1}^{n} x_j^2 - \sum_{j,k=1}^{n} \frac{x_j^2 + x_k^2}{2}$$

$$x_j x_k \le \frac{x_j^2 + x_k^2}{2} \iff (x_j - x_k)^2 \ge 0$$

$$= nP - \frac{1}{2} \sum_{k=1}^{n} \underbrace{\sum_{j=1}^{n} x_j^2}_{P} - \frac{1}{2} \sum_{j=1}^{n} \underbrace{\sum_{k=1}^{n} x_k^2}_{P}$$

$$= nP - \frac{1}{2} nP - \frac{1}{2} nP = 0$$

$$nP - X^2 \ge 0$$

$$f(a,b) = a^2 P + b^2 n + Q + 2Xab - 2Sa - 2by$$

$$H_f(a,b) = 2 \begin{pmatrix} P & X \\ X & n \end{pmatrix}$$

$$\begin{pmatrix} P-\lambda & X \\ X & n-\lambda \end{pmatrix} = 0$$

$$= (P-\lambda)(n-\lambda) - X^2 =$$

$$= \lambda^2 - (P+n)\lambda + Pn - X^2 = 0$$

$$\lambda_{\pm} = \frac{P+m}{2} \pm \frac{\sqrt{(P+m)^2 - 4Pm + 4X^2}}{2}$$

$$X^2 < Pm$$

$$f(1,1) = 8$$

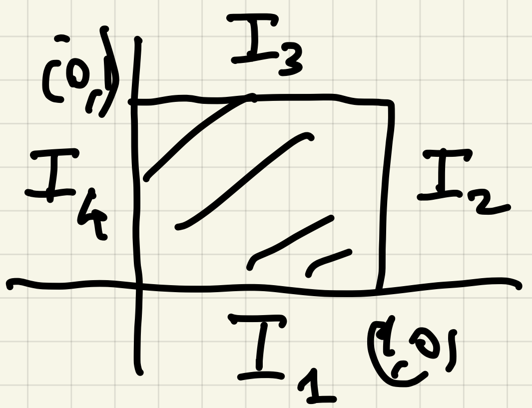
Domini con bordo

$$f(0,0) = 2$$

$$f(1,0) = -1$$

$$f(x,y) = x^2 + y^2 + 8y - 4x + 2 \qquad f(0,1) = 11$$

$$Q = [0,1] \times [0,1]$$



$$\mathring{Q}$$

$$f_x = 2x - 4 = 2(x-2) = 0 \qquad (2,-4)$$

$$f_y = 2y + 8 = 2(y+4) = 0$$

I_1

$$f(x,0) = x^2 - 4x + 2$$

$$f_x(x,0) = 2(x-2) < 0$$

I_2

$$f_y(1,y) = 2y + 8 > 0$$

	min	max
I_1	$(1,0)$	$(0,0)$
I_2	$(1,0)$	$(1,1)$
I_3	$(1,1)$	$(0,1)$
I_4	$(0,0)$	$(0,1)$

I_3 $\quad f_x(x,1) = 2(x-2) < 0$

I_4 $\quad f_y(0,y) = 2y+8 > 0$

$(0,1)$ max assoluto

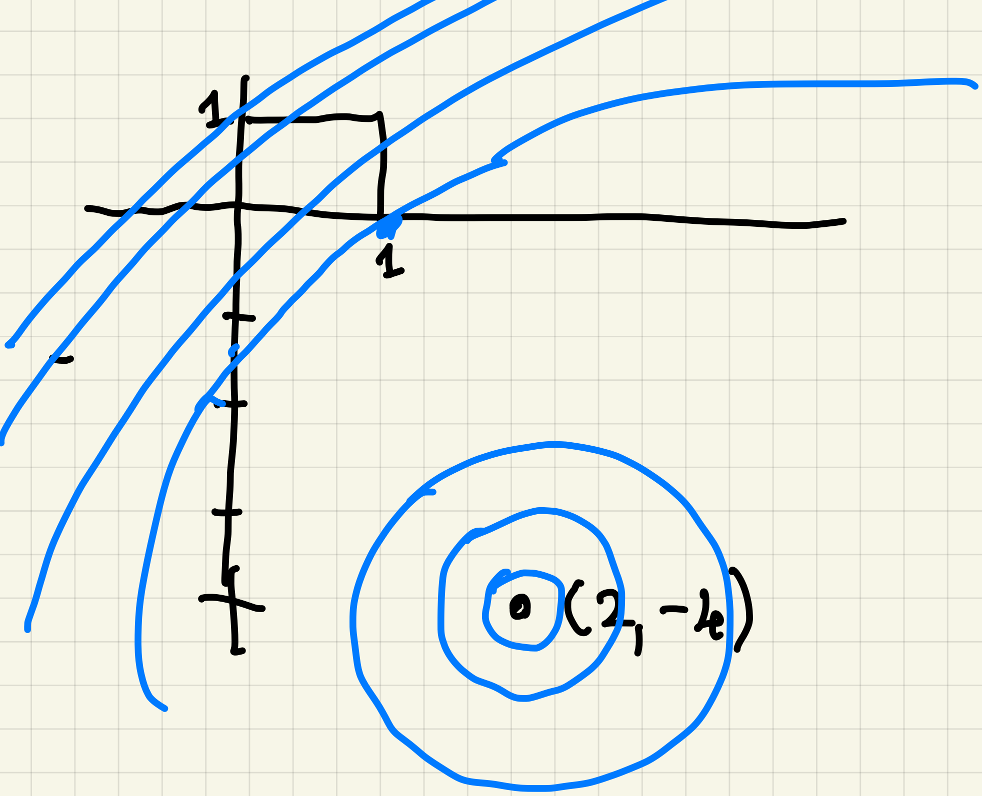
$(1,0)$ min assoluto

$f($

$$f(x,y) = x^2+y^2+8y-4x+2 =$$

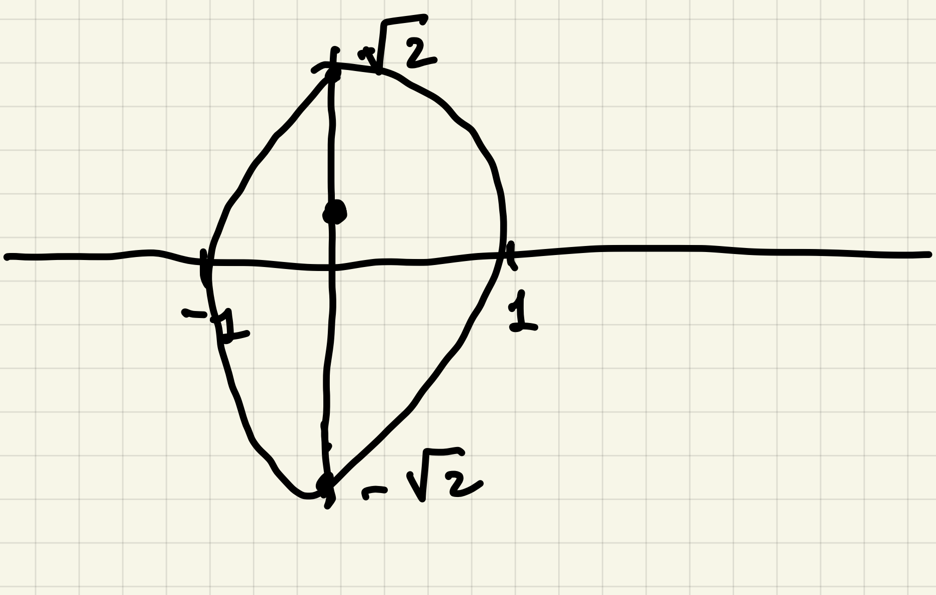
$$= x^2-4x+y^2+8y+2 =$$

$$= (x-2)^2+(y+4)^2-4-16+2 = c$$



$$f(x,y) = x^2 - y^2 + y$$

$$x^2 + \frac{y^2}{2} \leq 1 \qquad \Omega$$



$$f_x = 2x = 0 \qquad x = 0 \qquad \left(0, \tfrac{1}{2}\right)$$

$$f_y = -2y + 1 = 0 \qquad y = \frac{1}{2}$$

$$f_{xx} = 2 \qquad f_{yy} = -2$$

$$H_f(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$f \quad x^2 - y^2 + y$$

$$x^2 + \frac{y^2}{2} = 1$$

$$\begin{cases} x = \cos\vartheta \\ y = \sqrt{2}\sin\vartheta \end{cases}$$

$$f(\cos\vartheta, \sqrt{2}\sin\vartheta) =$$

$$h(\vartheta) = \cos^2\vartheta - 2\sin^2\vartheta + \sqrt{2}\sin\vartheta =$$

$$= 1 - 3\sin^2\vartheta + \sqrt{2}\sin\vartheta$$

$$h'(\vartheta) = \left(-6\sin\vartheta + \sqrt{2}\right)\cos\vartheta = 0$$

$\cos\vartheta = 0 \qquad \vartheta = \frac{\pi}{2}, \frac{3}{2}\pi$

$\sin\vartheta = \frac{\sqrt{2}}{6}$

$\vartheta = \frac{\pi}{2} \qquad (0, \sqrt{2})$

$\vartheta = \frac{3}{2}\pi \qquad (0, -\sqrt{2})$

per $\sin\vartheta = \frac{\sqrt{2}}{6}$

$y = \sqrt{2}\,\sin\vartheta = \sqrt{2}\;\frac{\sqrt{2}}{6} = \frac{2}{6} = \frac{1}{3}$

$x = \pm\sqrt{\frac{17}{18}}$

$x^2 = 1 - \frac{y^2}{2} = 1 - \frac{1}{18} = \frac{17}{18}$

$f(x,y) = x^2 - y^2 + y$

$f(0,\sqrt{2}) = -2 + \sqrt{2} < 0$

$f(0,-\sqrt{2}) = -2 - \sqrt{2} < 0$ pt min ass.

$$f\left(\pm\sqrt{\frac{17}{18}}, \frac{1}{3}\right) = \frac{17}{18} - \frac{1}{9} + \frac{1}{3}$$

punti max assoluti

$$z = f(x, y)$$