12 maggio

$x_0 \in A$

<u>Lemma</u> $f : A \subseteq \mathbb{R}^d \to \mathbb{R}^N$,
$f \in C^{k-1}(A, \mathbb{R}^N)$, $\exists\ D^k f(x_0)$.

Allora vale

$$* D^k f(x_0)(v_1, .., v_k) = \sum_{i_1, .., i_k = 1}^{d} \partial_{i_1} .. \partial_{i_k} f(x_0)\, v_{1 i_1} \dots v_{k i_k}$$

$v_1, \dots, v_k \in \mathbb{R}^d$

$$v_1 = \begin{pmatrix} v_{11} \\ \vdots \\ v_{1d} \end{pmatrix}, \dots, v_k = \begin{pmatrix} v_{k1} \\ \vdots \\ v_{kd} \end{pmatrix}$$

$$\partial_{i_1} \dots \partial_{i_k} f(x_0) = D^k f(x_0)(e_{i_1}, \dots, e_{i_k})$$

<u>Dim</u> Per $k = 1$ è vero

$$Df(x_0)v = \partial_1 f(x_0) v_1 + \dots + \partial_d f(x_0) v_d$$

Sia il risultato vero per $k-1$.

$$\boxed{g(x) \doteq D^{k-1} f(x)(v_2, .., v_k)} =$$

$$= \sum_{i_2, .., i_k = 1}^{d} \partial_{i_2} .. \partial_{i_k} f(x)\, v_{2 i_2} \dots v_{k i_k}$$

$$D^{k-1}f(x)(e_{i_2},\dots,e_{i_k}) = \partial_{i_2}\dots\partial_{i_k} f(x)$$

Dimostriamo che g è diff in x_0 e

(1) $$Dg(x_0)v_1 = (D D^{k-1} f(x_0) v_1)(v_2,\dots,v_k) \in \mathbb{R}^N$$

dove $D^{k-1}f : A \longrightarrow \mathcal{L}^{k-1}(\mathbb{R}^d,\mathbb{R}^N)$

$$\underbrace{D D^{k-1} f(x_0)}_{D^k f(x_0)} \in \underbrace{\mathcal{L}(\mathbb{R}^d, \mathcal{L}^{k-1}(\mathbb{R}^d,\mathbb{R}^N))}_{\mathcal{L}^k(\mathbb{R}^d,\mathbb{R}^N)}$$

$$g(x) = D^{k-1}f(x)(v_2,\dots,v_k)$$

$$g = G \circ D^{k-1} f$$

$$G : \mathcal{L}^{k-1}(\mathbb{R}^d,\mathbb{R}^N) \longrightarrow \mathbb{R}^N$$

$$L \longmapsto L(v_2,\dots,v_k)$$

G è lineare $\qquad DG(L) \in \mathcal{L}(\mathcal{L}^{k-1}(\mathbb{R}^d,\mathbb{R}^N),\mathbb{R}^N)$

$$DG = G \qquad\qquad G$$

$$Dg(x_0)\underset{\mathbb{R}^d}{u} = D(G\circ D^{k-1}f)\big|_{x=x_0} u =$$

$$= G \; D D^{k-1} f(x_0) u = G \underbrace{D^k f(x_0) u}_{\mathcal{L}^{k-1}(\mathbb{R}^d, \mathbb{R}^N)} =$$

$$= \left(D^k f(x_0) u \right)(v_2, \dots, v_k)$$

$$= D^k f(x_0)(u, v_2, \dots, v_k)$$

$$g$$

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$$Dg(x_0) v_1 = \left(D D^{k-1} f(x_0) v_1 \right)(v_2, \dots, v_k) \in \mathbb{R}^N$$

$$= D^k f(x_0)(v_1, \dots, v_k)$$

Nel caso particolare $v_2 = e_{i_2}, \dots, v_k = e_{i_k}$

$$g(x) = D^{k-1} f(x)(e_{i_2}, \dots, e_{i_k})$$

$$= \partial_{i_2} \dots \partial_{i_k} f(x)$$

$$Dg(x_0) v_1 = D(\partial_{i_2} \dots \partial_{i_k} f)(x_0) v_1 =$$

$$= \sum_{i_1 = 1}^{d} \partial_{i_1} \partial_{i_2} \dots \partial_{i_k} f(x_0) \, v_{1 i_1}$$

Più in generale, se

$$g(x) = \underline{D^{k-1} f(x)(v_2, \dots, v_k)}$$

$$D^k f(x_0)(v_1,\dots,v_k) = D g(x_0)\, v_1 =$$

$$= D\left(\sum_{i_2,\dots,i_k=1}^{d} \partial_{i_2}\cdots\partial_{i_k} f(x)\, v_{2i_2}\cdots v_{ki_k}\right)\Bigg|_{x=x_0} v_1$$

$$= \sum_{i_2,\dots,i_k=1}^{d} \underbrace{D\partial_{i_2}\cdots\partial_{i_k} f(x_0)\, v_1}_{\sum_{i_1=1}^{d} \partial_{i_1}\cdots\partial_{i_k} f(x_0)\, v_{1i_1}}\; v_{2i_2}\cdots v_{ki_k}$$

$$= \sum_{i_1,\dots,i_k=1}^{d} \partial_{i_1}\cdots\partial_{i_k} f(x_0)\, v_{1i_1}\cdots v_{ki_k}$$

$$\partial_{i_1}\cdots\partial_{i_k} f(x_0) = D^k f(x_0)(e_{i_1},\dots,e_{i_k})$$

Osservazione $f : A \subseteq \mathbb{R}^d \to \mathbb{R}^N$

$D(x_0, r) \subseteq A$

$f \in C^{k-1}(A, \mathbb{R}^N)$

$\exists\ D^k f(x_0)$

Sia $h \in \mathbb{R}^d$ $\quad h \neq 0$

$t \to f(x_0 + th)$ è ben definito

nell' intervallo $\left(-\frac{r}{|h|}, \frac{r}{|h|}\right)$

Allora

$$\left(\frac{d}{dt}\right)^k f(x_0+th)\Big|_{t=0} = D^k f(x^0)\underbrace{(h,\dots,h)}_{k \text{ volte}}$$

$$\overbrace{\mathbb{R}^d\times\dots\times\mathbb{R}^d}^{k \text{ volte}}$$

$$= D^k f(x^0)\, h^k$$

Vera per $k=1$

$$\frac{d}{dt} f(x_0+th)\Big|_{t=0} = Df(x_0)\, h$$

sia vera per $k-1$

$$\left(\frac{d}{dt}\right)^{k-1} f(x_0+th) = D^{k-1} f(x_0+th)\, h^{k-1}$$

$$h^{k-1} = \underbrace{(h,\dots,h)}_{k-1 \text{ volte}}$$

$$\left(\frac{d}{dt}\right)^k f(x_0+th)\Big|_{t=0} =$$

$$= \frac{d}{dt}\left(\left(\frac{d}{dt}\right)^{k-1} f(x_0+th)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(D^{k-1} f(x_0+th)\, h^{k-1} \right)\Big|_{t=0}$$

$$= D\Big(\underbrace{D^{k-1} f(x_0+th)\, h^{k-1}}_{g(x_0+th)} \Big)\Big|_{t=0} h = Dg(x_0)h$$

$$\underset{①}{=} D^k f(x_0)\, h^k$$

①

$$Dg(x_0)v_1 = \left(D D^{k-1} f(x_0) v_1\right)(v_2, \dots, v_k) \in \mathbb{R}^N$$

$$= D^k f(x_0)(v_1, \dots, v_k)$$

Espansioni di Taylor

$\mathbb{N} = \{1, 2, \dots\}$

$\mathbb{N}_0 = \{0, 1, \dots\}$

$\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$

$|\alpha| = \alpha_1 + \dots + \alpha_d$

$\alpha! = \alpha_1! \dots \alpha_d!$

$h \in \mathbb{R}^d$ $\qquad h^\alpha = h_1^{\alpha_1} \cdots h_d^{\alpha_d}$

$h = (h_1, \ldots, h_d)$

Osservazione Sia $A \in \mathcal{L}^k(\mathbb{R}^d, \mathbb{R}^N)$ simmetrico e definiamo $f : \mathbb{R}^d \to \mathbb{R}^N$

$$f(x) = \frac{1}{k!} A x^k = \frac{1}{k!} A(\underbrace{x, \ldots, x}_{k})$$

Risulta $D^k f(x) \equiv A$ $\quad$ (2)

$f \in C^\infty(\mathbb{R}^d, \mathbb{R}^N)$

Per $k = 1$ $\quad$ (2) è vera

$$f(x) = Ax \qquad Df(x) \equiv A$$

Supponiamo (2) sia vera per $k-1$.

$$f(x+th) = \frac{1}{k!} A(x+th)^k =$$

$$= \frac{1}{k!} A(x+th, \ldots, x+th)$$

$$\frac{d}{dt} f(x+th) =$$

$$= \frac{1}{k!} \frac{d}{dt} A(x+th, \dots, x+th)$$

$$= \frac{1}{k!} \sum_{i=1}^{k} A(x+th, \dots, x+th, \underset{i}{h}, x+th, \dots, x+th)$$

$$= \frac{1}{k!} \sum_{i=1}^{k} A(h, x+th, \dots, x+th)$$

$$= \frac{1}{(k-1)!} A(h, \underbrace{x+th, \dots, x+th}_{k-1}) = \frac{d}{dt} f(x+th)$$

$$\left(\frac{d}{dt}\right)^k f(x+th) = \left(\frac{d}{dt}\right)^k \left(\frac{1}{k!} A(x+th)^k \right) =$$

$$= \left(\frac{d}{dt}\right)^{k-1} \frac{d}{dt} \left(\frac{1}{k!} A(x+th)^k \right)$$

$$= \left(\frac{d}{dt}\right)^{k-1} \frac{1}{(k-1)!} A(h, x+th, \dots, x+th)$$

$$= A(h, \underbrace{h, \dots h}_{k-1})$$

$$\left(\frac{d}{dt}\right)^k f(x+th)\Big|_{t=0} = A\, h^k$$
$$= D^k f(x)\, h^k$$

$$B = A - D^k f(x) \in \mathcal{L}^k(\mathbb{R}^d, \mathbb{R}^N)$$

simmetrica

$$B\, h^k = 0 \overset{?}{\Longrightarrow} B \equiv 0$$

Formula di polarizzazione

$$B(x_1, \dots, x_k) =$$
$$= \frac{1}{2^k\, k!} \sum_{s_1, \dots, s_k \in \{-1, 1\}} s_1 \cdots s_k\, B(s_1 x_1 + \dots + s_k x_k)^k$$

$k=1$

$$B x = \frac{Bx - B(-x)}{2} = \frac{1}{2}\left(Bx - B(-x)\right)$$

$$(a_1 + \cdots + a_d)^k =$$
$$= \underbrace{(a_1 + \cdots + a_d) \cdots (a_1 + \cdots + a_d)}_{k \text{ volte}}$$

$$= \sum_{i_1,\dots,i_d=1}^{d} a_{i_1} \cdots a_{i_d}$$

$$= \sum_{|\beta|=k} \frac{k!}{\beta!} a_1^{\beta_1} \cdots a_d^{\beta_d}$$

$$\frac{1}{2^k k!} \sum_{s_1,\dots,s_k \in \{-1,1\}} s_1 \cdots s_k B(s_1 x_1 + \dots + s_k x_k)^k$$

$$B(y_1,\dots,y_k)$$

$$= \frac{1}{2^k k!} \sum_{s_1,\dots,s_k \in \{-1,1\}} s_1 \cdots s_k \sum_{|\alpha|=k} \frac{k!}{\alpha!} B (s_1 x_1)^{\alpha_1} \cdots (s_k x_k)^{\alpha_k}$$

$$= \frac{1}{2^k \cancel{k!}} \sum_{s_1,\dots,s_k \in \{-1,1\}} s_1^{\alpha_1+1} \cdots s_k^{\alpha_k+1} \sum_{|\alpha|=k} \frac{\cancel{k!}}{\alpha!} B x_1^{\alpha_1} \cdots x_k^{\alpha_k}$$

Tutti i Termini dove anche un solo esponente $\alpha_j + 1$ è dispari non contribuiscono alla somma

Gli unici α che contribuiscono sono quelli $\alpha = (\underset{\neq 0}{\alpha_1}, \dots, \underset{\neq 0}{\alpha_k})$

$$|\alpha| = \alpha_1 + \dots + \alpha_k = k \Rightarrow \alpha = (1, \dots, 1)$$

$$\left| \frac{1}{2^k} \sum_{s_1,\dots,s_k \in \{-1,1\}} B(x_1, x_2, \dots, x_k) \right.$$

$$= \cancel{\frac{1}{2^k}}\, \cancel{2^k}\, B(x_1, x_2, \dots, x_k)$$

<u>Teor</u> (Peano) $f : A \subseteq \mathbb{R}^d \to \mathbb{R}^N$ $x^0 \in A$

$D(x^0, r) \subseteq A$, $p \in \mathbb{N}$, $f \in C^{p-1}(A, \mathbb{R}^N)$

$\exists\ D^p f(x_0)$. Allora per $|h| < r$, $h \in \mathbb{R}^d$

ho

$$f(x_0 + h) = P_p(h) + o(|h|^p)$$

$$P_p(h) := \sum_{k=0}^{p} \frac{1}{k!} D^k f(x_0)\, h^k$$

$$\lim_{h \to 0} \frac{o(|h|^p)}{|h|^p} = 0$$

<u>Dim</u> Basta $N = 1$

$$g(h) := f(x_0 + h) - P_p(h)$$

Ho $D^{k_0} g(0) = 0 = D^{k_0} f(x_0) - \underbrace{D^{k_0} P_p(h)\Big|_{h=0}}_{D^{k_0} f(x_0)}$

$k_0 \leq p$

$$D^{k_0} P_p(h)\Big|_{h=0} = D^{k_0} f(x_0)$$

$$D^{k_0} \sum_{k=0}^{p} \frac{1}{k!} D^k f(x_0)\, h^k \Big|_{h=0} =$$

$$= D^{k_0} \frac{1}{k_0!} D^{k_0} f(x_0)\, h^{k_0} = D^{k_0} f(x_0)$$

Si tratta di dimostrare che se

$D^k g(0) = 0$ per $k \leq p$

$D^{k-1} g \in C^0\left(D_{\mathbb{R}^d}(0,r), \mathcal{L}^{k-1}(\mathbb{R}^d, \mathbb{R})\right)$

allora $\boxed{g(h) = o(|h|^p) \quad *}$

Per $p = 1$ è vero da

$$g(h) = Dg(0)\,h + o(|h|)$$

$$\lim_{h\to 0} \frac{g(h) - g(0) - Dg(0)\,h}{|h|} = 0$$

$$\frac{g(h) - Dg(0)\,h}{|h|} = o(1)$$

$$g(h) = Dg(0)\,h + |h|\,o(1)$$
$$= Dg(0)\,h + o(|h|)$$

Supponiamo che per induzione il caso $p-1$ sia vero. Allora

$$Dg(h) = o(|h|^{p-1})$$

$$g(h) = g(h) - g(0) = g(th)\Big]_{t=0}^{1}$$

$$= \frac{d}{dt}\, g(th)\Big|_{t=t_*} \qquad t_* \in (0,1)$$

$$= Dg(t_* h)\, h = o(|t_* h|^{p-1})\, h$$
$$= o(|h|^{p-1})\, h$$
$$= o(|h|^{p})$$

Prop Nelle ipotesi del teorema

$$\sum_{k=0}^{p} \frac{1}{k!} D^k f(x_0) h^k =$$

$$= \sum_{|\alpha| \leq p} \frac{1}{\alpha!} \partial^\alpha f(x_0)\, h^\alpha$$

$\alpha \in \mathbb{N}_0^d$

$h \in \mathbb{R}^d$

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$$

$$h^\alpha = h_1^{\alpha_1} \cdots h_d^{\alpha_d}$$

$h = h_1 e_1 + \dots + h_d e_d$

Dim

$$\frac{1}{k!} D^k f(x_0) (\underbrace{h, \dots, h}_{k \text{ volte}}) =$$

$$= \frac{1}{k!} D^k f(x_0) \left(h_1 e_1 + \dots + h_d e_d, \dots, h_1 e_1 + \dots + h_d e_d\right)$$

$$= \frac{1}{\cancel{k!}} \sum_{|\alpha| = k} \frac{\cancel{k!}}{\alpha!} D^k f(x_0) (h_1 e_1)^{\alpha_1} \cdots (h_d e_d)^{\alpha_d}$$

$$= \sum_{|\alpha| = k} \frac{1}{\alpha!} \underbrace{h_1^{\alpha_1} \cdots h_d^{\alpha_d}}_{h^\alpha} \underbrace{D^k f(x_0) e_1^{\alpha_1} \cdots e_d^{\alpha_d}}_{\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f(x_0)}$$

$$= \sum_{|\alpha| = k} \frac{1}{\alpha!} \partial^\alpha f(x_0)\, h^\alpha$$

Test derivate seconde $\qquad f : \mathbb{R}^d \to \mathbb{R}^N$

$$D^2 f = D\,Df = D\left(\partial_1 f, \dots, \partial_d f\right) =$$

$$= \left(D\partial_1 f, \dots, D\partial_d f\right)$$

$$= \begin{pmatrix} \partial_1\partial_1 f & \cdots & \partial_1\partial_d f \\ \vdots & & \vdots \\ \partial_d\partial_1 f & \cdots & \partial_d\partial_d f \end{pmatrix} = Hf$$

$N = 1$ $\qquad$ matrice Hessiana

$$\left(\partial_i \partial_j f\right)_{i,j=1}^{d}$$

Osservazione $\qquad N = 1$

$f : A \subseteq \mathbb{R}^d \to \mathbb{R} \qquad x_0 \in A$

$f \in C^2(A, \mathbb{R})$

$D^2 f(x_0)$, $\quad$ sia $Df(x_0) = 0$

Siccome $D^2 f(x_0)$ è simmetrica

esiste $\qquad A \in SO(\mathbb{R}^d)$

$$ {}^tA\, D^2 f(x_0)\, A = \mathrm{diag}(\lambda_1, \dots, \lambda_d) $$

$$ g(y) \doteq f(Ay + x_0) \qquad x = Ay + x_0 $$

$$ Dg(y) = Df(x)\, A $$

$$ Df(x_0) = 0 \iff Dg(0) = 0 $$

$$ Dg(y) = Df(Ay + x_0)\, A $$

$$ Dg(y)h = Df(Ay + x_0)\, Ah $$

$$ D^2 g(0)(h, k) = D^2 f(x_0)(Ah, Ak) $$

$$ = {}^tk\, \underbrace{{}^tA\, D^2 f(x_0)\, A}\, h $$

$$ D^2 g(0) = \mathrm{diag}(\lambda_1, \dots, \lambda_d) $$