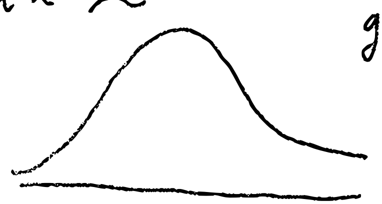
Laplace expansion

$$\int_a^b g(x)\,dx = \int e^{h(x)}\,dx \approx$$

$$h(x) = \log(g(x))$$



$$= \int e^{h(\tilde{x}) + \underbrace{h'(\tilde{x})}_{=0}(x-\tilde{x}) + \frac{1}{2}h''(\tilde{x})(x-\tilde{x})^2}\,dx$$

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$$= e^{h(\tilde{x})} \int \underbrace{e^{\frac{1}{2}h''(\tilde{x})(x-\tilde{x})^2}}_{\propto \mathcal{N}(\tilde{x},\, -h''(\tilde{x})^{-1})}\,dx$$

$$= e^{h(\tilde{x})}(2\pi)^{1/2}\left(-h''(\tilde{x})\right)^{-1/2}$$

$$= \boxed{e^{h(\tilde{x})}\sqrt{\frac{2\pi}{-h''(\tilde{x})}}}$$

$$\int_a^b g(x)\, dx_1 \ldots dx_p \simeq e^{h(\tilde{x})} (2\pi)^{p/2} \left|-h''(\tilde{x})\right|^{-1/2}$$

1) $\boxed{p(y)} = \int p(y|\theta)\pi(\theta)\, d\theta = \int e^{\log(p(y|\theta)\pi(\theta))} d\theta$

$h(\theta) \equiv \log(p(y|\theta)\pi(\theta))$

$$\simeq e^{h(\hat{\theta})} (2\pi)^{p/2} \left|-h''(\hat{\theta})\right|^{-\frac{1}{2}}$$

$$= \boxed{p(y|\hat{\theta})\pi(\hat{\theta})(2\pi)^{p/2} \left|-h''(\hat{\theta})\right|^{-1/2}}$$

2) $\boxed{p(y)} = \int p(y|\theta)\pi(\theta)\, d\theta = \int e^{n\left(\frac{1}{n}\log p(y|\theta) + \frac{1}{n}\log \pi\right)}$

$h(\theta) = \frac{1}{n} p(y|\theta) + \frac{1}{n}\pi(\theta)$

$$\simeq \boxed{n^{-\frac{p}{2}} p(y|\hat{\theta})\pi(\hat{\theta})(2\pi)^{p/2} \left|-h''(\hat{\theta})\right|^{-1/2}}$$

$\rightarrow \hat{\theta} = \underset{\theta \in \Theta}{\operatorname{argmax}}\, h(\theta)$

$\rightarrow h''(\hat{\theta})$

$$\boxed{E(\theta|y)} = \int \theta\, \pi(\theta|y)\, d\theta = \frac{\int \theta\, p(y|\theta)\pi(\theta)\, d\theta}{\int p(y|\theta)\pi(\theta)\, d\theta}$$

$$= \frac{\int \theta\, e^{h(\hat\theta) + h'(\hat\theta) + \cdots}}{\int e^{h(\hat\theta) + h'(\hat\theta) + \cdots}} = \frac{\int \theta \sqrt{\frac{2\pi}{-h''(\hat\theta)}}\, \varphi(\hat\theta, -h''^{-1})}{\boxed{\int \sqrt{\frac{2\pi}{-h''(\hat\theta)}}\, \varphi(\hat\theta, -h''}}$$

$$= \int \theta\, \varphi(\hat\theta, -h''(\hat\theta)^{-1})\, d\theta = \boxed{\hat\theta}$$

$$\text{BIC} = -2 \log p(y|\hat\theta_{MLE}) + \underline{p \log n}$$

From 2) $\underline{\log p(y)} = \log p(y|\hat\theta) + \log \pi(\hat\theta) + \frac{p}{2}\log(2\pi) + \frac{1}{2}\log|-h''(\hat\theta)|^{-1/2} - \frac{p}{2}\log n$

$$\approx \log p(y|\hat\theta) - \frac{p}{2}\log n$$

$$= \widehat{elpd}_{BIC}$$

$\boxed{\text{Variational Bayes}}$ $\boxed{KL(Q\|\pi(\theta|y)) = -VLB + \log p(y)}$

Find $Q_\phi(\theta) \approx \pi(\theta|y) \rightarrow$

$$KL(Q\|\pi(\theta|y)) = -\int Q(\theta) \log\left(\frac{\pi(\theta|y)}{Q(\theta)}\right) d\theta$$