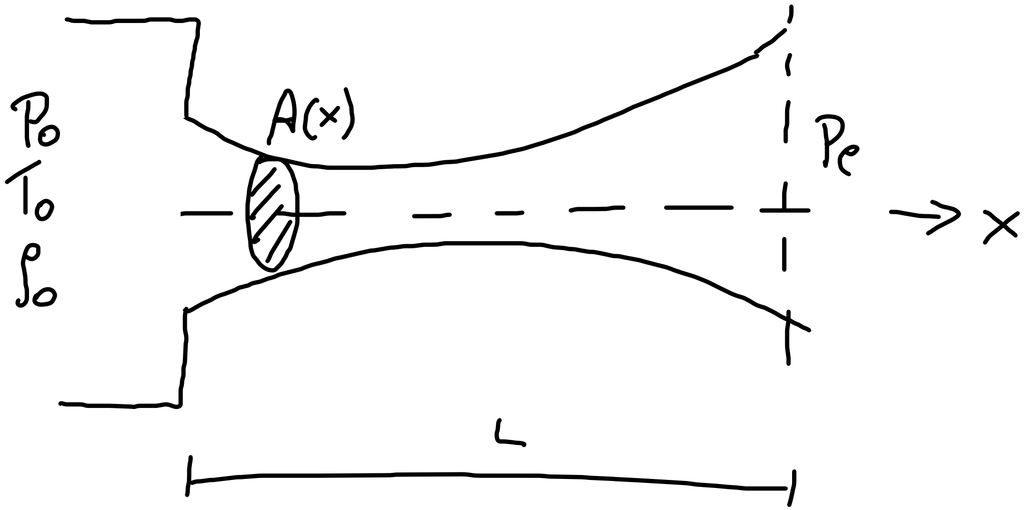
Quasi-1D nozzle flow. Finite-difference solution with the MacCormack's method

- Compressible, isentropic flow through a quasi-1D nozzle. Ideal gas:



Mass conservation:

$$\frac{\partial \rho}{\partial t} = -\frac{1}{A}\frac{\partial}{\partial x}\left(\rho u A\right)$$

Momentum equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{R}{\rho}\left[T\frac{\partial}{\partial x}\rho + \rho\frac{\partial}{\partial x}T\right]$$

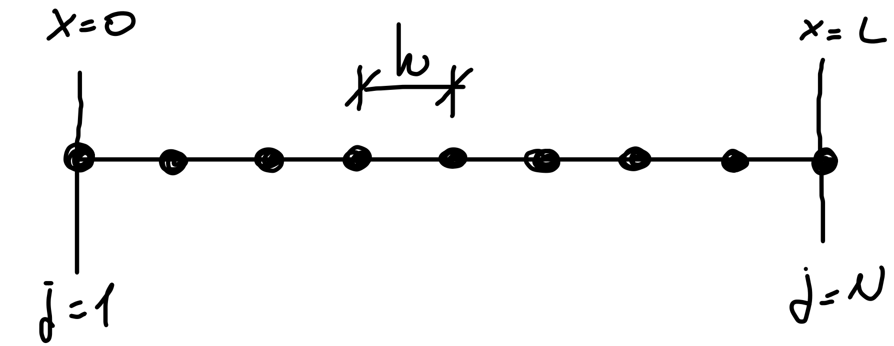
(Internal) energy equation

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - (\gamma-1)\,T\left[\frac{\partial}{\partial x}u + u\frac{\partial}{\partial x}\ln A\right]$$

Equations of state:

$$P = \rho R T\ ; \qquad e = c_v T = \frac{R}{\gamma-1}T$$

- Uniform mesh.



- Predictor step of Mac-Cormack's method:

Forward finite-differences used to approximate first derivatives:

$$\left.\frac{\delta u}{\delta x}\right|_j^t = \frac{u_{j+1}^t - u_j^t}{h}$$

$$\left.\frac{\delta \rho}{\delta x}\right|_j^t = \frac{\rho_{j+1}^t - \rho_j^t}{h}$$

$$\left.\frac{\delta T}{\delta x}\right|_j^t = \frac{T_{j+1}^t - T_j^t}{h}$$

$$\left.\frac{\delta}{\delta x}\ln A\right|_j = \frac{\ln A_{j+1} - \ln A_j}{h}$$

$$\left.\frac{\delta}{\delta x}(\rho u A)\right|_j^t = \frac{(\rho u A)_{j+1}^t - (\rho u A)_j^t}{h}$$

$$\left.\frac{\partial \rho}{\partial t}\right|_j^t = -\frac{1}{A_j}\left.\frac{\delta}{\delta x}(\rho u A)\right|_j^t \quad ; \quad j = 2 \dots N-1$$

$$\left.\frac{\partial u}{\partial t}\right|_j^t = -u_j^t \left.\frac{\delta u}{\delta x}\right|_j^t - \frac{R}{\rho_j^t}\left[T_j^t \left.\frac{\delta \rho}{\delta x}\right|_j^t + \rho_j^t \left.\frac{\delta T}{\delta x}\right|_j^t\right] \quad ; \quad j = 2 \dots N-1$$

$$\left.\frac{\partial T}{\partial t}\right|_j^t = -u_j^t \left.\frac{\delta T}{\delta x}\right|_j^t - (\gamma-1)\, T_j^t \left[\left.\frac{\delta u}{\delta x}\right|_j^t + u_j^t \left.\frac{\delta}{\delta x}\ln A\right|_j\right] \quad ; \quad j = 2 \dots N-1$$

$\bar{\rho}_j = \rho_j^t + \left.\frac{\partial \rho}{\partial t}\right|_j^t \Delta t \quad j = 2 \dots N-1$; $\bar{\rho}_1$ and $\bar{\rho}_N$ from b.c.'s

$\bar{u}_j = u_j^t + \left.\frac{\partial u}{\partial t}\right|_j^t \Delta t \quad j = 2 \dots N-1$; $\bar{u}_1$ and $\bar{u}_N$ " "

$\bar{T}_j = T_j^t + \left.\frac{\partial T}{\partial t}\right|_j^t \Delta t \quad j = 2 \dots N-1$; $\bar{T}_1$ and $\bar{T}_N$ " "

- Corrector step of MacCormack's method:

Backward finite-differences used to approximate first derivatives:

$$\left.\frac{\delta \bar{u}}{\delta x}\right|_j = \frac{\bar{u}_j - \bar{u}_{j-1}}{h}$$

$$\left.\frac{\delta \bar{\rho}}{\delta x}\right|_j = \frac{\bar{\rho}_j - \bar{\rho}_{j-1}}{h}$$

$$\left.\frac{\delta \bar{T}}{\delta x}\right|_j = \frac{\bar{T}_j - \bar{T}_{j-1}}{h}$$

$$\left.\frac{\delta}{\delta x}\ln A\right|_j = \frac{\ln A_j - \ln A_{j-1}}{h}$$

$$\left.\frac{\delta(\overline{\rho u A})}{\delta x}\right|_j = \frac{(\bar{\rho}\bar{u}A)_j - (\bar{\rho}\bar{u}A)_{j-1}}{h}$$

$$\left.\frac{\partial \bar{\rho}}{\partial t}\right|_j = -\frac{1}{A_j}\left.\frac{\delta(\overline{\rho u A})}{\delta x}\right|_j$$

$$\left.\frac{\partial \bar{u}}{\partial t}\right|_j = -\bar{u}_j\left.\frac{\delta \bar{u}}{\delta x}\right|_j - \frac{R}{\bar{\rho}_j}\left[\bar{T}_j\left.\frac{\delta \bar{\rho}}{\delta x}\right|_j + \bar{\rho}_j\left.\frac{\delta \bar{T}}{\delta x}\right|_j\right]; \quad j = 2 \dots N-1$$

$$\left.\frac{\partial \bar{T}}{\partial t}\right|_j = -\bar{u}_j\left.\frac{\delta \bar{T}}{\delta x}\right|_j - (\gamma-1)\bar{T}_j\left[\left.\frac{\delta \bar{u}}{\delta x}\right|_j + \bar{u}_j\left.\frac{\delta}{\delta x}\ln A\right|_j\right]; \quad j = 2 \dots N-1$$

$$\rho_j^{t+\Delta t} = \rho_j^t + \frac{\Delta t}{2}\left(\left.\frac{\partial \rho}{\partial t}\right|_j^t + \left.\frac{\partial \bar{\rho}}{\partial t}\right|_j\right), \quad j = 2 \dots N-1; \quad \rho_1^{t+\Delta t} \text{ and } \rho_N^{t+\Delta t} \text{ from b.c.'s}$$

$$u_j^{t+\Delta t} = u_j^t + \frac{\Delta t}{2}\left(\left.\frac{\partial u}{\partial t}\right|_j^t + \left.\frac{\partial \bar{u}}{\partial t}\right|_j\right), \quad j = 2 \dots N-1; \quad u_1^{t+\Delta t} \text{ and } u_N^{t+\Delta t} \text{ "}$$

$$T_j^{t+\Delta t} = T_j^t + \frac{\Delta t}{2}\left(\left.\frac{\partial T}{\partial t}\right|_j^t + \left.\frac{\partial \bar{T}}{\partial t}\right|_j\right), \quad j = 2 \dots N-1; \quad T_1^{t+\Delta t} \text{ and } T_N^{t+\Delta t} \text{ "}$$

Boundary conditions: (-0 denotes reservoir conditions)

- Subsonic inflow:

$\rho_1 = \rho_0$

$T_1 = T_0$

$u_1 = 2u_2 - u_3$

Subsonic outflow:

$T_N = 2T_{N-1} - T_{N-2}$

$\rho_N = \dfrac{p_e}{R\,T_N}$

$u_N = 2u_{N-1} - u_{N-2}$

Supersonic outflow:

$\rho_N = 2\rho_{N-1} - \rho_{N-2}$

$u_N = 2u_{N-1} - u_{N-2}$

$T_N = 2T_{N-1} - T_{N-2}$