Finite-volume approximation of

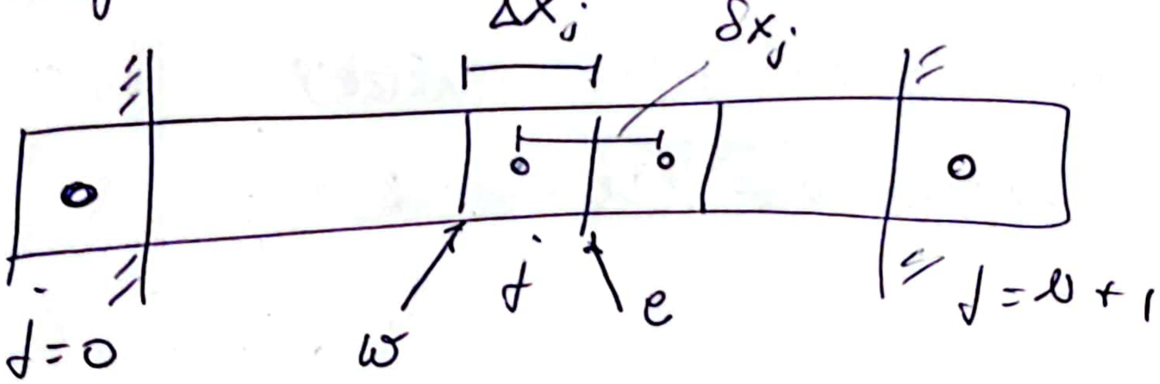
$$\begin{cases} u_t + u u_x = a u_{xx} \\ u(0,t) = 0 \\ u(L,t) = 1 \end{cases}$$

Solution

Recast in flux-form:

$$u_t + \left(\frac{u^2}{2}\right)_x = a u_{xx}$$

On arbitrary mesh:



$$(u_t)_j \Delta x_j + \underbrace{\left(\frac{u^2}{2}\right)_e - \left(\frac{u^2}{2}\right)_w}_{\equiv A_j} = a\left[(u_x)_e - (u_x)_w\right]$$

Scansionato con CamScanner

$$\underbrace{u_{j-1}^{n+1}\left[-\frac{a\Delta t}{2\delta x_{j-1}}\right]}_{a_W} + \underbrace{u_j^{n+1}\left[\Delta x_j + \frac{a\Delta t}{2}\left(\frac{1}{\delta x_j} + \frac{1}{\delta x_{j-1}}\right)\right]}_{a_P}$$

$$+ \underbrace{u_{j+1}^{n+1}\left[-\frac{a\Delta t}{2\delta x_j}\right]}_{a_E} =$$

$$\underbrace{= u_j^n \Delta x_j + \Delta t\left[\frac{3}{2}A_j^n - \frac{1}{2}A_j^{n-1}\right] + \frac{a\Delta t}{2}\left[\frac{u_{j+1}^n - u_j^n}{\delta x_j} - \frac{u_j^n - u_{j-1}^n}{\delta x_{j-1}}\right]}_{b}$$

for $j = 1 \ldots N$

Boundary conditions (using ghost cells):

$\boxed{j=1}$ $\quad a_W \underline{\underline{u_0^{n+1}}} + a_P u_1^{n+1} + a_E u_2^{n+1} = b$

?

$$\frac{u_0^{n+1} + u_1^{n+1}}{2} = 0 \Rightarrow u_0^{n+1} = -u_1^{n+1}$$

$$u_e = \frac{u_{j+1}\Delta x_j + u_j \Delta x_{j+1}}{\Delta x_j + \Delta x_{j+1}}$$ linear interpolation

$$u_w = \frac{u_j \Delta x_{j-1} + u_{j-1}\Delta x_j}{\Delta x_j + \Delta x_{j-1}}$$

$$(u_x)_e \simeq \frac{u_{j+1} - u_j}{\delta x_j}$$

$$(u_x)_w \simeq \frac{u_j - u_{j-1}}{\delta x_{j-1}}$$

Adams-Bashfort for advective term and Crank-Nicolson for diffusive term:

$$(u_j^{n+1} - u_j^n)\Delta x_j + \Delta t\left[\frac{3}{2}A_j^n - \frac{1}{2}A_j^{n-1}\right]$$

$$= a\frac{\Delta t}{2}\left[\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\delta x_j} - \frac{u_j^{n+1} - u_{j-1}^{n+1}}{\delta x_{j-1}}\right]$$

$$+ a\frac{\Delta t}{2}\left[\frac{u_{j+1}^n - u_j^n}{\delta x_j} - \frac{u_j^n - u_{j-1}^n}{\delta x_{j-1}}\right]$$

$$\Rightarrow \left(a_P - a_W\right) u_1^{n+1} + a_E u_2^{n+1} = b$$

$$\boxed{j = N}$$

$$a_W u_{N-1}^{n+1} + a_P u_N^{n+1} + a_E \underbrace{u_{N+1}^{n+1}}_{2} = b$$

$$\frac{u_N^{n+1} + u_{N+1}^{n+1}}{2} = 1 \Rightarrow u_{N+1}^{n+1} = 2 - u_N^{n+1}$$

$$\Rightarrow \left(a_P - a_E\right) u_N^{n+1} + a_W u_{N-1}^{n+1} = b - 2 a_E$$