Applicare il metodo di von Neumann per studiare le proprietà di due schemi alle differenze finite applicati all'equazione della diffusione del calore, con discretizzazione ai volumi finiti:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \alpha \dfrac{\partial^2 u}{\partial x^2} & x \in \mathbb{R} \\ u(x,0) = e^{ikx} \end{cases}$$

Analytical solution:

$$\text{Try } u(x,t) = \sigma(t)\, e^{ikx}$$

$$\dot{\sigma} = -\alpha k^2 \sigma \rightarrow \sigma = e^{-\alpha k^2 t}$$

$$u(x,t) = e^{-\alpha k^2 t} e^{ikx}$$

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Consider

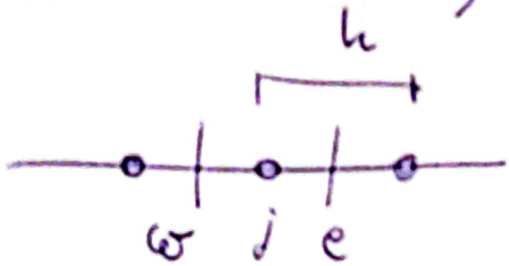
a) $\left.\frac{\delta u^{(1)}}{\delta x}\right|_{j+1/2} = \frac{u_j - u_{j-1}}{h}$ upwind, 1^{st}-order

and

b) $\left.\frac{\delta u^{(2)}}{\delta x}\right|_{j+1/2} = \frac{u_{j+1} - u_j}{h}$ central, 2^{nd}-order

Semi-analytical discretization:

$$h \frac{du_j}{dt} = \alpha \left(\left.\frac{\delta u}{\delta x}\right|_e - \left.\frac{\delta u}{\delta x}\right|_w \right)$$



Represent both schemes a) and b) as:

$$\left.\frac{\delta u}{\delta x}\right|_{j+1/2} = \beta \left.\frac{\delta u^{(1)}}{\delta x}\right|_{j+1/2} + (1-\beta) \left.\frac{\delta u^{(1)}}{\delta x}\right|_{j+1/2}$$

$$0 \leq \beta \leq 1$$

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Look for a numerical solution in the form

$$\hat{u}_j(t) = \hat{\sigma}(t)\, e^{ikx_j}$$

$$h\,\dot{\hat{\sigma}} = \alpha\,\hat{\sigma} \cdot \left[\frac{\delta}{\delta x} e^{ikx_j}\Big|_e - \frac{\delta}{\delta x} e^{ikx_j}\Big|_w \right]$$

$$\frac{\delta u^{(2)}}{\delta x}\Big|_{j+1/2} = \hat{\sigma}\, \frac{e^{ikx_{j+1}} - e^{ikx_j}}{h} =$$

$$= \hat{\sigma}\, e^{ikx_j}\, \frac{e^{ikh} - 1}{h} = \hat{\sigma}\, e^{ikx_j}\, \frac{e^{i\tilde{k}} - 1}{h}$$

$kh =: \tilde{k}$ modified wavenumber.

$$\frac{\delta u^{(1)}}{\delta x}\Big|_{j+1/2} = \hat{\sigma}\, \frac{e^{ikx_j} - e^{ikx_{j-1}}}{h} =$$

$$= \hat{\sigma}\, e^{ikx_j}\, \frac{1 - e^{-i\tilde{k}}}{h}$$

$$\frac{\delta u^{(2)}}{\delta x}\Big|_{j-1/2} = \hat{\sigma}\, e^{ikx_j}\, e^{-i\tilde{k}}\, \frac{e^{i\tilde{k}} - 1}{h} = \hat{\sigma}\, e^{ikx_j}\, \frac{1 - e^{-i\tilde{k}}}{h}$$

$$\frac{\delta u^{(1)}}{\delta x}\bigg|_{j-1/2} = \hat{\sigma}\, e^{ikx_j} e^{-i\tilde{k}}\, \frac{1-e^{-i\tilde{k}}}{h}$$

$$F(\tilde{k}) := \frac{1-e^{-i\tilde{k}}}{\tilde{k}}$$

complex conjugate

$$\frac{\delta u^{(2)}}{\delta x}\bigg|_{j+1/2} = \hat{\sigma}\, e^{ikx_j} \cdot k\left(-F^{*}\right)$$

$$\frac{\delta u^{(2)}}{\delta x}\bigg|_{j-1/2} = \hat{\sigma}\, e^{ikx_j} k F$$

$$\frac{\delta u^{(1)}}{\delta x}\bigg|_{j+1/2} = \hat{\sigma}\, e^{ikx_j} k F$$

$$\frac{\delta u^{(1)}}{\delta x}\bigg|_{j-1/2} = \hat{\sigma}\, e^{ikx_j} k F e^{-i\tilde{k}}$$

$$h\,\dot{\hat{\sigma}}\,\cancel{e^{ikx_j}} = K\alpha\hat{\sigma}\Big[\beta(-F^* - F) +$$

$$+ (1-\beta)\left(F - F e^{-i\tilde{k}}\right)\Big]\cancel{e^{ikx_j}}$$

$$\dot{\hat{\sigma}} = \frac{\alpha K}{h}\Big[F\left(-\beta + 1 - \beta - e^{-i\tilde{k}}(1-\beta)\right)$$

$$- F^*\beta\Big]\hat{\sigma}$$

$$= \frac{\alpha K}{h}\underbrace{\Big[F\left(1 - 2\beta - (1-\beta)e^{-i\tilde{k}}\right) - F^*\beta\Big]}_{-G(\tilde{k},\beta)\tilde{k}}\hat{\sigma}$$

$$\hat{\sigma}(t) = e^{-\frac{\alpha K}{h}G\tilde{k}t}$$

$$= e^{-\alpha k^2 t \cdot \frac{G\tilde{k}}{\tilde{k}}} = e^{-\alpha k^2 t G}$$

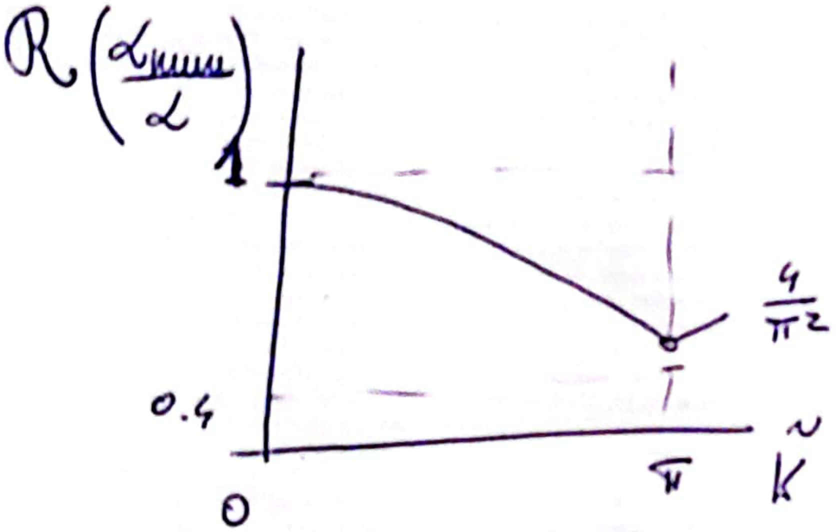
$$\hat{u}(x_j,t) = e^{-\alpha k^2 t G}\, e^{ikx_j}$$

$$\alpha_{num} \equiv \alpha G(\tilde{k},\beta)$$

$$\frac{\alpha_{num}}{\alpha} = G(\tilde{k}, \beta) \text{ error function.}$$

CD2

$$\beta = 1 : \quad \frac{\alpha_{num}}{\alpha} = \left[\frac{2 \cdot (1 - \cos \tilde{k})}{\tilde{k}^2} \right] + i \left[0 \right]$$



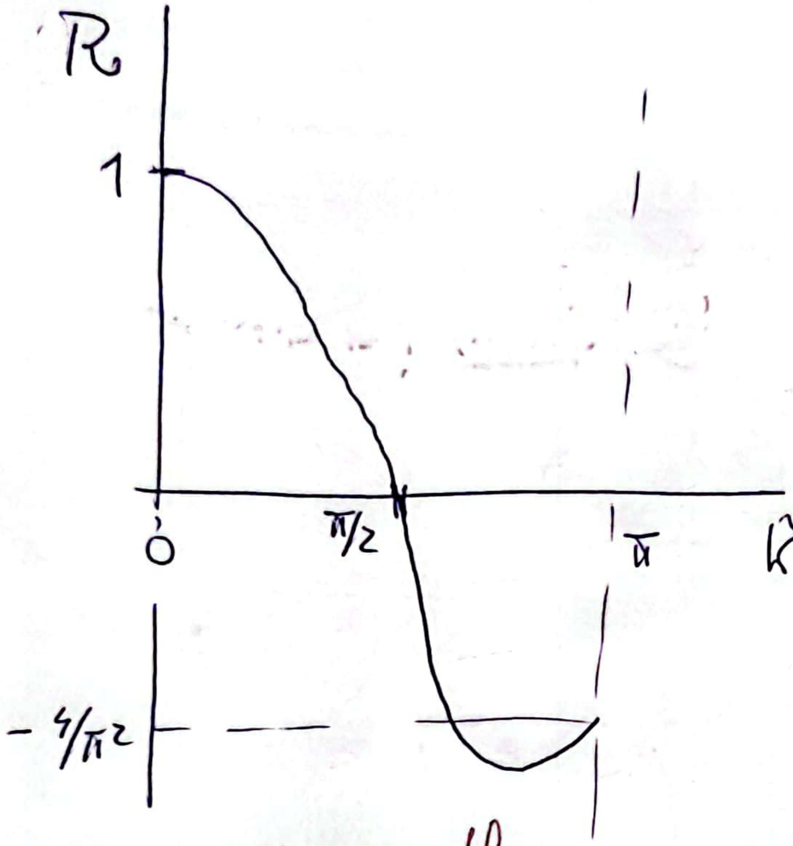
CD2 is less diffusive at higher wavenumbers but there is no dispersion error.

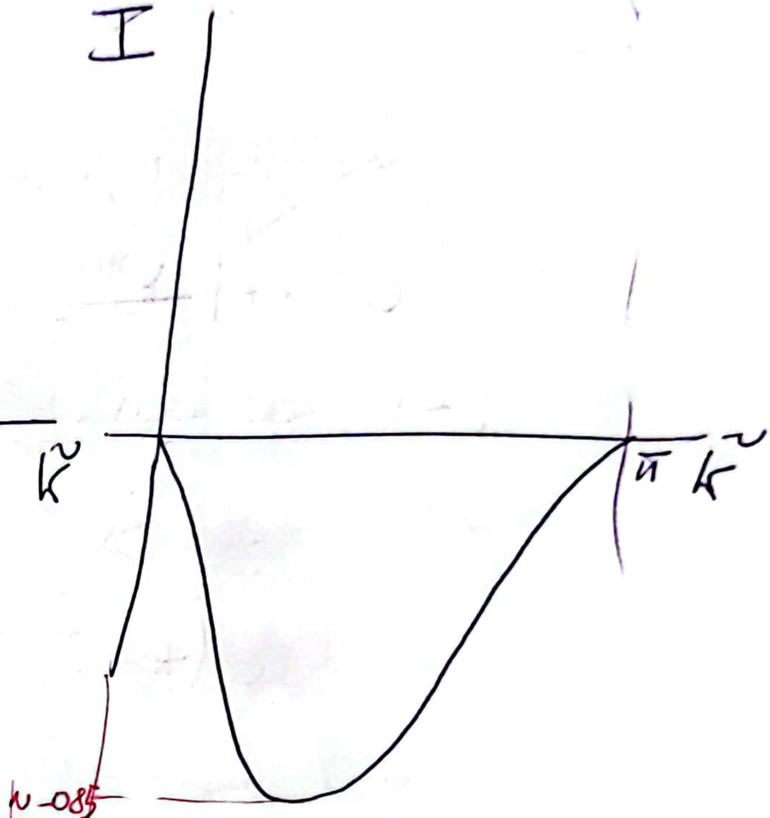
UPWIND

$$\beta = 0$$

$$\frac{\alpha_{num}}{\alpha} = \left[\frac{2 \cos \tilde{k} (1 - \cos \tilde{k})}{\tilde{k}^2} \right] + i \left[\frac{\sin(2\tilde{k}) - 2 \sin \tilde{k}}{\tilde{k}^2} \right]$$

$$= R + i I$$





Less diffusive than expected for $\tilde{k} < \pi/2$ and counter-diffusive for $\tilde{k} > \pi/2$!

Dispersive error

Dispersive error :

$$\alpha_{num} = \alpha \left(R + i I \right)$$

$$\hat{u}(x_j, t) = e^{-\alpha k^2 t R} \, e^{-i \alpha k^2 t I} \, e^{i k x_j}$$

$$= e^{-(\alpha R) k^2 t} \, e^{i k (x_j - c_{num} t)}$$

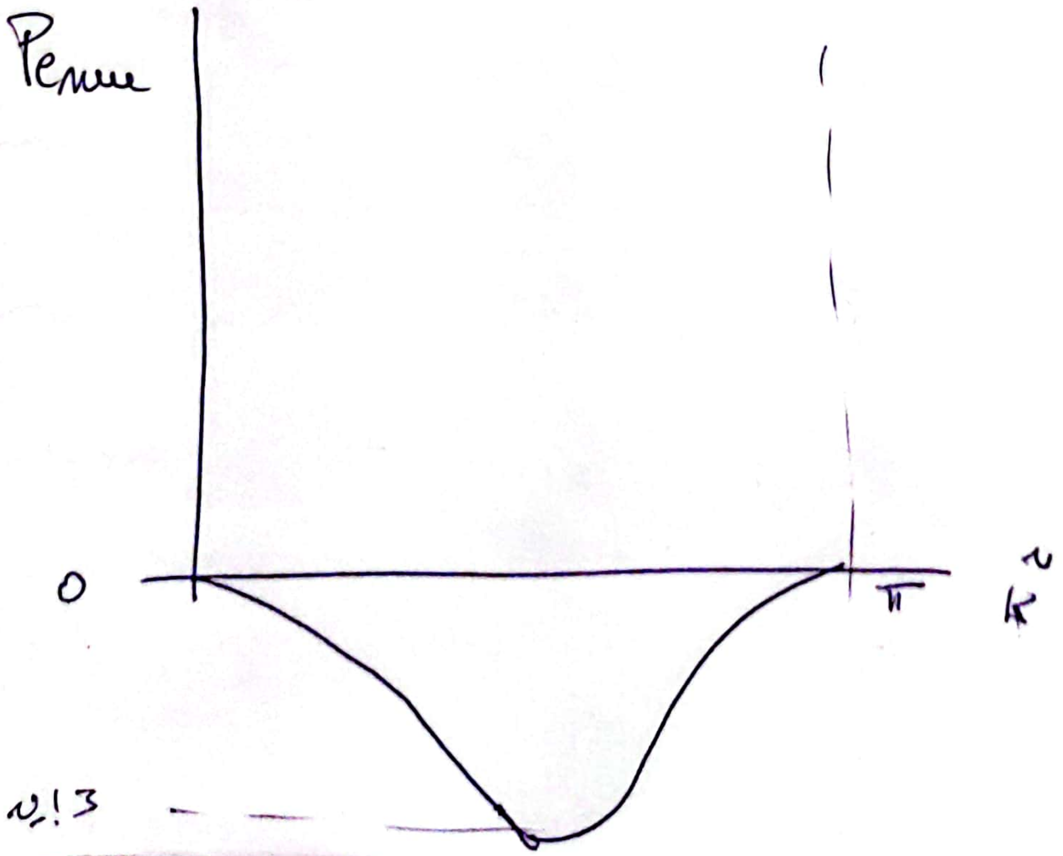
Dispersion !

$$C_{num} = \alpha k I$$

Pèclet number: $Pe_{num} = \dfrac{C_{num} h}{\alpha} =$

$$= \tilde{k} I$$

"Dispersion" means that the initial waveform is not just progressively damped: it also moves with a velocity C_{num} that depends on $\tilde{k}$.



$P_{emme} < 0 \Rightarrow$ waveform moves towards negative "x".

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