Problema agli autovalori: soluzione con il metodo degli elementi finiti

$$\begin{cases} y'' = \lambda y & x \in (0,1) \\ y(0) = 0 \\ y'(1) = 0 \end{cases}$$

$$\int_0^1 \left[\hat{y}'' - \lambda \hat{y}\right] v \, dx = \int_0^1 (\hat{y}' v)' - \hat{y}' v' - \lambda \hat{y} v \, dx$$

$$= \hat{y}' v \Big|_0^1 - \int_0^1 \hat{y}' v' \, dx - \lambda \int_0^1 \hat{y} v \, dx = 0$$

Galerkin: $\left.\begin{array}{l} \hat{y} = \sum_j N_j(x) \, y_j \\ v = N_i \end{array}\right\} \; i = 1 \ldots N$

Boundary term disappears (why??)

$$\Rightarrow \quad \sum_{1}^{n} {}_{j} \int_{0}^{1} N_i' N_j' \, dx \; y_j = \lambda \sum_{1}^{n} {}_{j} \int_{0}^{1} N_i N_j \, dx \; y_j$$

$$[K] \equiv [K_{ij}] \qquad K_{ij} \equiv \int_0^1 N_i' N_j' \, dx$$

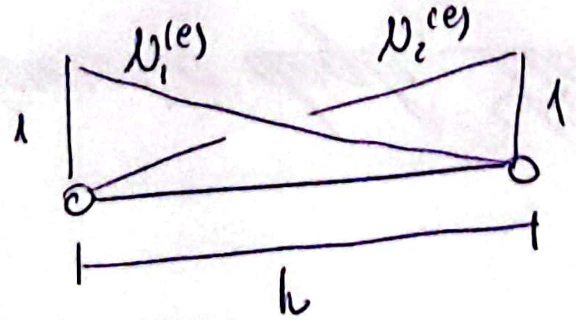
$$[M] \equiv [M_{ij}] \qquad M_{ij} \equiv \int_0^1 N_i N_j \, dx$$

$$\{y\} := \begin{Bmatrix} y_1 \\ | \\ y_n \end{Bmatrix}$$

$$[K]\{y\} = \lambda [M] \{y\}$$

"generalized" eigenvalue problem.

$$K_{ij} = \int_0^1 N_i' N_j' \, dx = \sum_e \int_{\Omega_e} N_i^{e\,\prime}(x) \, N_j^{e\,\prime}(x) \, dx$$



Local stiffness matrix:

$$[K^e] = \begin{bmatrix} 1/h & -1/h \\ -1/h & 1/h \end{bmatrix}$$

Similarly, the local mass matrix:

$$[M^e] = \begin{bmatrix} h/3 & h/6 \\ h/6 & h/3 \end{bmatrix}$$

Global matrices:

$$[K] = \begin{bmatrix} \ddots & & & \\ 1/h+1/h & -1/h & & \\ -1/h & 1/h+1/h & -1/h & \\ & -1/h & 1/h+1/h & \\ & & & \ddots \end{bmatrix} = \frac{1}{h}\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}$$

$$[M] = h\begin{bmatrix} 1/3 & 1/6 & & & \\ & \ddots & & & \\ & 1/6 & 2/3 & 1/6 & \\ & & & \ddots & \\ & & & 1/6 & 1/3 \end{bmatrix}$$

- Boundary condition at left endpoint:

$$[\tilde{K}] = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & \\ -1 & 2 & -1 & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

NB

$$[\tilde{M}] = h\begin{bmatrix} 2/3 & 1/6 & 0 & \cdots \\ 1/6 & 2/3 & 1/6 & \cdots \\ & \ddots & \ddots & \\ & & 1/6 & 1/3 \end{bmatrix}$$

The resulting eigenvalue problem (could be solved analytically ...):

$$\boxed{[\tilde{M}]^{-1}\{y\} = \lambda\{y\}}$$

STOP