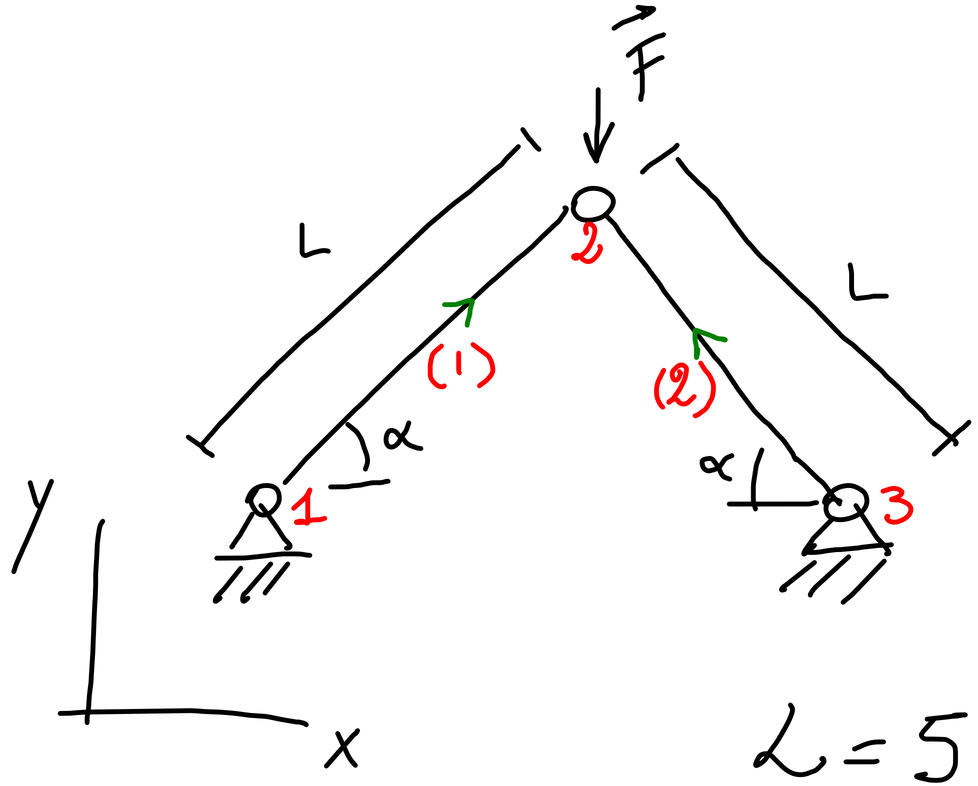
Esercizio: calcolare lo spostamento del nodo 2 della struttura di seguito riportata utilizzando il metodo FEM con assemblaggio diretto della matrice dei coefficienti.



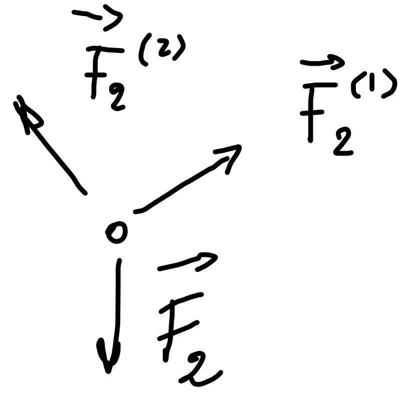
In verde è indicata l'orientazione da assumere per gli elementi.

$L = 5\,m$ $\alpha = 45^\circ$

$F_x = 0$ $F_y = 1000\,kgf$ $E = 11\,GPa$ $A = 280\,cm^2$

Soluzione

- Equilibrio nodo 2:

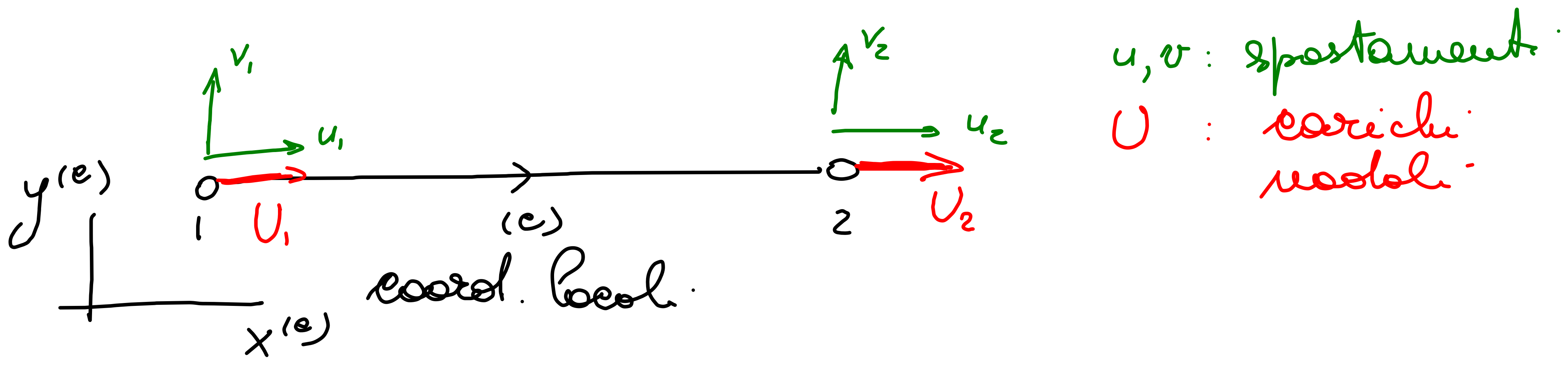


$$\sum_e \vec{F}_2^{(e)} + \vec{F}_2 = 0$$

$\vec{F}_2^{(e)}$: forza esercitata dall'asta (e) sul nodo 2

$\vec{F}_2$: forza esterna applicata sul nodo 2.

- Forza esercitata dalle aste ai nodi:
  - È conveniente usare un sistema di coordinate locali in quanto i carichi nodali sono puramente assiali e possono essere espressi in modo semplice in funzione degli spostamenti assiali dei nodi.

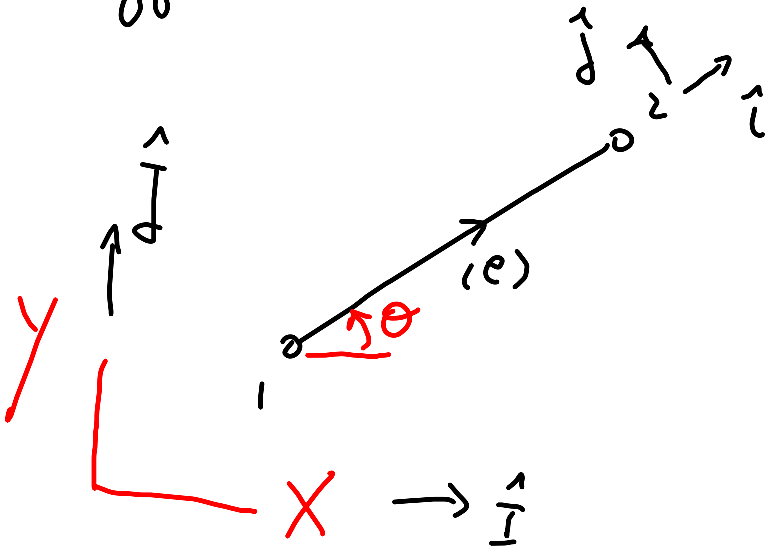


$$U_1 = -EA\,\varepsilon = -EA\,\frac{u_2 - u_1}{L}$$

$$U_2 = EA\,\varepsilon = EA\,\frac{u_2 - u_1}{L}$$

$$\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} U_1 \\ 0 \\ U_2 \\ 0 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = [K]\{u\}$$

- Passaggio da coordinate locali a coordinate globali:



$$\vec{u} = u\,\hat{i} + v\,\hat{j} = \tilde{u}\,\hat{I} + \tilde{v}\,\hat{J}$$

$$\tilde{u} = u\,\hat{i}\cdot\hat{I} + v\,\hat{j}\cdot\hat{I} = u\cos\theta - v\,\mathrm{sen}\,\theta$$

$$\tilde{v} = u\,\hat{i}\cdot\hat{J} + v\,\hat{j}\cdot\hat{J} = u\,\mathrm{sen}\,\theta + v\cos\theta$$

$\tilde{u}, \tilde{v}$: componenti globali spostamento

$$\begin{Bmatrix} \tilde{u} \\ \tilde{v} \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \quad e \quad \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \tilde{u} \\ \tilde{v} \end{Bmatrix}$$

Analogamente per i carichi nodali:

$$\begin{Bmatrix} \tilde{U} \\ \tilde{V} \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} U \\ 0 \end{Bmatrix}$$

Considerando ambo i nodi di un'asta:

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{bmatrix} [R] & [0] \\ [0] & [R] \end{bmatrix} \begin{Bmatrix} \tilde{u}_1 \\ \tilde{v}_1 \\ \tilde{u}_2 \\ \tilde{v}_2 \end{Bmatrix} \quad e \quad \begin{Bmatrix} \tilde{U}_1 \\ \tilde{V}_1 \\ \tilde{U}_2 \\ \tilde{V}_2 \end{Bmatrix} = \begin{bmatrix} [R]^T & [0] \\ [0] & [R]^T \end{bmatrix} \begin{Bmatrix} U_1 \\ 0 \\ U_2 \\ 0 \end{Bmatrix}$$

dove

$$[R] \equiv \begin{bmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{bmatrix}$$

Indichiamo con $[L]$ la matrice di rotazione dell'asta:

$$[L] \equiv \begin{bmatrix} [R] & [0] \\ [0] & [R] \end{bmatrix}$$

Si ha:

$$\{u\} = [L]\{\tilde{u}\} \qquad \{\tilde{U}\} = [L]^T\{U\} \quad \text{dove } \{U\} = \begin{Bmatrix} U_1 \\ 0 \\ U_2 \\ 0 \end{Bmatrix}$$

- A livello di elemento il legame tra carichi nodali e spostamenti nodali in coordinate globali è quindi:

$$\begin{Bmatrix} U_1 \\ 0 \\ U_2 \\ 0 \end{Bmatrix} = [K] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \tilde{U}_1 \\ \tilde{V}_1 \\ \tilde{U}_2 \\ \tilde{V}_2 \end{Bmatrix} = [L]^T \begin{Bmatrix} U_1 \\ 0 \\ U_2 \\ 0 \end{Bmatrix} = [L]^T[K][L] \begin{Bmatrix} \tilde{u}_1 \\ \tilde{v}_1 \\ \tilde{u}_2 \\ \tilde{v}_2 \end{Bmatrix}$$

$$\equiv [\tilde{K}] \begin{Bmatrix} \tilde{u}_1 \\ \tilde{v}_1 \\ \tilde{u}_2 \\ \tilde{v}_2 \end{Bmatrix}$$

- Calcoliamo $[\tilde{K}]$:

$$[L] = \begin{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \end{bmatrix} \qquad [K] = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[L]^T[K] = \frac{EA}{L} \begin{bmatrix} c & 0 & -c & 0 \\ s & 0 & -s & 0 \\ -c & 0 & c & 0 \\ -s & 0 & s & 0 \end{bmatrix}; \quad [L]^T[K][L] = \frac{EA}{L} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ cs & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

- Assemblaggio matrice globale:

Siccome dobbiamo fare i calcoli "a mano", consideriamo direttamente il sistema ridotto, con incognite $\tilde{u}_2$ e $\tilde{v}_2$ ("2" nodo globale):

- Equilibrio nodo 2 lungo X:

$$-\tilde{U}_2^{(1)} - \tilde{U}_2^{(2)} + F_X = 0$$

forza agente sul nodo locale 2 dell'asta 1

forza agente sul nodo locale 2 dell'asta 2.

$$-\tilde{K}_{31}^{(1)}\cancel{\tilde{u}_1^{(1)}}-\tilde{K}_{32}^{(1)}\cancel{\tilde{v}_1^{(1)}}-\tilde{K}_{33}^{(1)}\tilde{u}_2^{(1)}-\tilde{K}_{34}^{(1)}\tilde{v}_2^{(1)}$$

$$-\tilde{K}_{31}^{(2)}\cancel{\tilde{u}_1^{(2)}}-\tilde{K}_{32}^{(2)}\cancel{\tilde{v}_1^{(2)}}-\tilde{K}_{33}^{(2)}\tilde{u}_2^{(2)}-\tilde{K}_{34}^{(2)}\tilde{v}_2^{(2)}+F_x=0$$

$$\tilde{u}_2^{(1)}=\tilde{u}_2^{(2)}\equiv\tilde{u} \qquad \tilde{u}_1^{(1)}=\tilde{v}_1^{(1)}=\tilde{u}_1^{(2)}=\tilde{v}_1^{(2)}=0$$

$$\tilde{v}_2^{(1)}=\tilde{v}_2^{(2)}\equiv\tilde{v}$$

$$\Rightarrow\ -\tilde{K}_{33}^{(1)}\tilde{u}-\tilde{K}_{34}^{(1)}\tilde{v}-\tilde{K}_{33}^{(2)}\tilde{u}-\tilde{K}_{34}^{(2)}\tilde{v}+F_x=0$$

$$\Rightarrow\ \boxed{\left(\tilde{K}_{33}^{(1)}+\tilde{K}_{33}^{(2)}\right)\tilde{u}+\left(\tilde{K}_{34}^{(1)}+\tilde{K}_{34}^{(2)}\right)\tilde{v}=F_x}$$

• Equilibrio nodo 2 lungo y:

$$-\tilde{V}_2^{(1)} - \tilde{V}_2^{(2)} + F_y = 0$$

forza agente sul nodo locale 2 dell'asta 1

forza agente sul nodo locale 2 dell'asta 2.

$$\boxed{\left(\tilde{K}_{43}^{(1)} + \tilde{K}_{43}^{(2)}\right)\tilde{u} + \left(K_{44}^{(1)} + K_{44}^{(2)}\right)\tilde{v} = F_y}$$

$$[\hat{k}] = \frac{EA}{L}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ cs & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

<u>Per elemento 1:</u>

$c \equiv \cos\alpha$

$s \equiv \operatorname{sen}\alpha$

<u>Per elemento 2:</u>

$c \equiv \cos(\pi-\alpha) = -\cos\alpha$

$s \equiv \operatorname{sen}(\pi-\alpha) = \operatorname{sen}\alpha$

Eq. lungo x:

$$\frac{EA}{L}\left\{\left(\tilde{K}_{33}^{(1)} + \tilde{K}_{33}^{(2)}\right)\tilde{u} + \left(K_{34}^{(1)} + K_{34}^{(2)}\right)\tilde{V}\right\} = F_x$$

$$\left(C_\alpha^2 + (-C_\alpha)^2\right)\tilde{u} + \left(s_\alpha C_\alpha + s_\alpha(-C_\alpha)\right)\tilde{V} = \frac{F_x L}{EA}$$

$$\boxed{2(\cos\alpha)^2\,\tilde{u} = F_x L/EA} \Rightarrow \tilde{u} = 0$$

$$[\tilde{k}] = \frac{EA}{L}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ cs & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

Por elemento 1:

$c \equiv \cos\alpha$

$s \equiv \operatorname{sen}\alpha$

Por elemento 2:

$c \equiv \cos(\pi-\alpha) = -\cos\alpha$

$s \equiv \operatorname{sen}(\pi-\alpha) = \operatorname{sen}\alpha$

Eq. luego y:

$$\left(\tilde{K}^{(1)}_{43} + \tilde{K}^{(2)}_{43}\right)\tilde{u} + \left(K^{(1)}_{44} + K^{(2)}_{44}\right)\tilde{V} = F_y L/EA$$

$$\left(\cos\alpha \operatorname{sen}\alpha + (-\cos\alpha)\operatorname{sen}\alpha\right)\tilde{u} + \left((\operatorname{sen}\alpha)^2 + (\operatorname{sen}\alpha)^2\right)\tilde{V} = F_y L/EA$$

$$2(\operatorname{sen}\alpha)^2\,\tilde{V} = F_y L/EA$$

$$2 \cdot (\sin 45^\circ)^2\, \tilde{V} = -\frac{1000 \times 9.81\, N \times 5\, m}{11\, GPa \times 280\, cm^2}$$

$$2 \cdot \frac{1}{2}\, \tilde{V} = \tilde{v} = -\frac{9.81 \times 1000 \times 5}{11 \times 10^9 \times 280 \times 10^{-4}} =$$

$$= -\frac{9.81 \times 5}{11 \times 10^2 \times 280} = -\frac{9.81}{11 \cdot 5600} \cong 0,16\ mm$$

Sforzo nelle travi :

$$\tilde{U}_2^{(1)} = -\tilde{K}_{33}^{(1)}\tilde{u} - \tilde{K}_{35}\tilde{v} = \left(-\frac{1}{2}\cancel{\tilde{u}} - \frac{1}{2}\tilde{v}\right)\frac{EA}{L} = -4905\ N$$

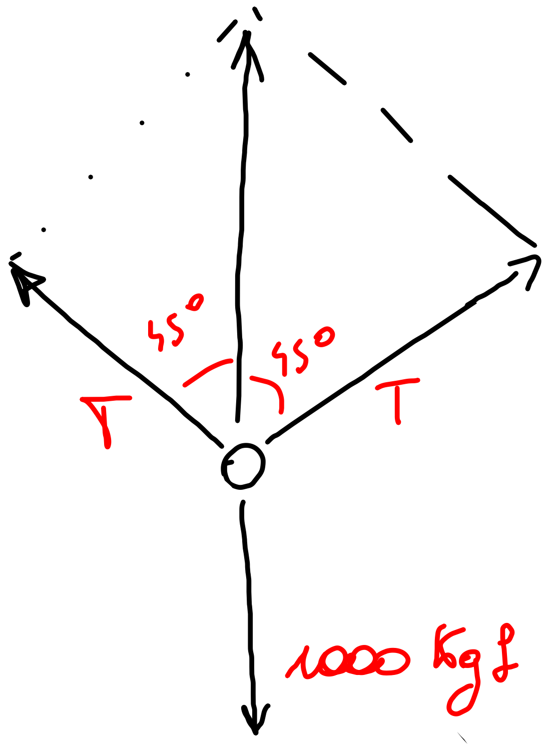
$$= -500\ Kgf$$

$$\tilde{V}_2^{(1)} = \ldots \ldots \ldots = -500\ Kgf$$

$$\{U\} = [R]^T \{\tilde{U}\} \Rightarrow \quad U_2^{(1)} = -707{,}107\ Kgf$$

$$V_2^{(1)} = 0\ Kgf$$

- Verifico resultado:



$$2T\frac{\sqrt{2}}{2} = 1000\ Kgf$$

$$T = \frac{1000}{\sqrt{2}}\ Kgf = 707{,}107\ Kgf$$

compressive