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# The minority game: A statistical physics perspective

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**Abstract**

A brief review is given of the minority game, an idealized model stimulated by a market of speculative agents, and its complex many-body behaviour. Particular consideration is given to analytic results for the model rather than discussions of its relevance in real-world situations.


*Keywords:* Minority game; Statistical physics; Econophysics

There is currently much interest in the statistical physics community in the emergence of complex co-operative behaviour as a consequence of frustration and disorder in systems of simple microscopic constituents and simple rules of interaction [1]. Appropriate minimalist models, designed to capture the essence of real-world problems without peripheral complications, have played crucial roles in the understanding of such systems. The minority game is such a minimalist model, introduced in econophysics to mimic a market of speculators trying to profit by buying low and selling high. In this paper, we review it from the perspective of statistical physics, with a view to exposing relevant cooperative and complex features, to demonstrate the significant, but incomplete, degree of analytic solubility currently achievable, and to illustrate the possibilities for potentially soluble extensions.

The model describes a system of a large number $N$ of agents each of whom at each step of a discrete dynamics makes a bid that can be either positive or negative (buy or sell). The objective of each agent is to make a bid of opposite sign from that of the sum of all the bids (i.e., a minority choice). No agent has any direct knowledge of the actions or propensities of the others but is aware of the cumulative action (total bid) made at each step. Each agent decides his/her bid through the application of a personal strategy operator to some common information, available identically to all. In the simplest versions of the model, to which we restrict here, the strategy operators are allocated randomly and independently for each agent before play commences and are not modified during play. Each agent has a finite set of strategies, one of which is chosen and used at each step. The choice is determined by 'points' allocated to the strategies and augmented regularly via a comparison between the bid associated with playing the strategy and the actual total bid, being increased for minority prediction. This is the only mechanism for co-operation but is sufficient to yield complex macroscopic behaviour.

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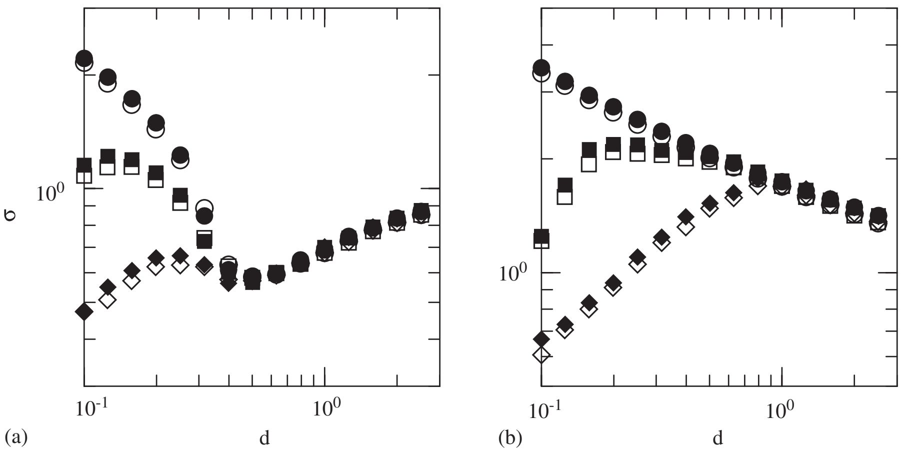


Fig. 1. Volatilities in batch minority games with 2 strategies per agent; (a) with completely uncorrelated strategies, (b) with each agent's 2 strategies mutually anti-correlated but with no correlation between agents. Shown are three different bias asymmetries between the points allocated initially to each agent's 2 strategies: $p_i(0) = 0.0$ (circles), 0.5 (squares) and 1.0 (diamonds). Also exhibited is a comparison between the results of simulation of the deterministic many-agent dynamics (open symbols) and the numerical evaluation of the analytically derived stochastic single-agent ensemble dynamics (closed symbols). From Ref. [5].

In the original version of the model [2] the information on which decisions were made was the history of the actual play over a finite window (the last $m$ time-steps). However, simulations demonstrated that utilising instead a random fictitious 'history' (information) at each time-step produces essentially identical behaviour, suggesting that its relevance is just to provide a mechanism for an effective interaction between agents. A natural non-trivial measure of the macroscopic behaviour is the volatility, the standard deviation of the total bid. It demonstrates statistical physics interest in several ways: (i) in exhibiting non-trivial scaling behaviour as a function of $d = D/N$, where $D$ is the information dimension [3], (ii) in exhibiting a cusp at a critical $d_c$ following a *tabula rasa* start, and especially (iii) in that the system is ergodic with volatility independent of starting point allocations for $d > d_c$ but non-ergodic and preparation-dependent for $d < d_c$; see Fig. 1. This is reminiscent of the susceptibility of an infinite-range spin glass where a critical temperature $T_c$ separates a preparation-dependent regime from an equilibrating one.

Since the information on which the agents act is the same for all, this problem is manifestly mean-field. It therefore offers the potential for exact solution for its macro-behaviour in the sense of the elimination of the microscopic variables in favour of self-consistently determined macro-parameters in the limit of large $N$ [4]. The physics seems robust to variations of detail, but for completeness we indicate the version discussed explicitly.

Each agent $i$; $i = 1, \ldots, N$, is taken to have two $D = \mathrm{d}N$-dimensional strategies $\mathbf{R}_{ia} = (R^1_{ia}, \ldots, R^{\mathrm{d}N}_{ia})$; $a = \pm 1$, with each component $R^\mu_{ia}$ chosen independently randomly $\pm 1$ at the outset and thereafter fixed. The common random information enters in that $\mu(t)$ is chosen stochastically randomly at each time-step $t$ from the set $\mu(t) \in \{1, \ldots, D\}$ and each agent plays one of his/her two strategies $R^{\mu(t)}_{ia_i}$; $a_i = \pm 1$. The actual choices of $a_i$ used, $b_i(t)$, are determined by the current values of point differences $p_i(t)$. Let us restrict initially to deterministic choices, $b_i(t) = \mathrm{sgn}(p_i(t))$. The $p_i(t)$ are updated every $M$ time-steps according to

$$p_i(t+M) = p_i(t) - M^{-1} \sum_{\ell=t}^{t+M-1} \xi_i^{\mu(\ell)} \left\{ N^{-1/2} \sum_j (\omega_j^{\mu(\ell)} + \xi_j^{\mu(\ell)} \,\mathrm{sgn}(p_j(t))) \right\}, \tag{1}$$

where $\boldsymbol{\omega}_i = (\mathbf{R}_{i1} + \mathbf{R}_{i2})/2$, $\boldsymbol{\xi}_i = (\mathbf{R}_{i1} - \mathbf{R}_{i2})/2$. In the so-called 'online' game $M = 1$ but here we consider the 'batch' game where $M \geqslant O(N)$ so that the sum on the actual $\mu(\ell)$ in (1) may be replaced by an average [6] so that [7,8]

$$p_i(t+1) = p_i(t) - \sum_j J_{ij}\,\mathrm{sgn}(p_j(t)) - h_i \equiv p_i(t) - \partial H/\partial s_i|_{s_i = \mathrm{sgn}(p_i(t))}, \tag{2}$$

where $J_{ij} = N^{-1} \sum_{\mu=1}^{D} \xi_i^\mu \xi_j^\mu$, $h_i = N^{-1/2} \sum_{\mu=1}^{D} \omega_i^\mu \xi_i^\mu$, $H = \sum_{(ij)} J_{ij} s_i s_j + h_i s_i$.

To proceed we use the dynamical generating functional method [9] with

$$Z = \int \prod_t \mathrm{d}\mathbf{p}(t) W(\mathbf{p}(t+1) \mid \mathbf{p}(t)) P(\mathbf{p}(0)), \tag{3}$$

where $\mathbf{p}(t) = (p_1(t), \ldots, p_N(t))$, $W(\mathbf{p}(t+1) \mid \mathbf{p}(\mathbf{t}))$ denotes the transformation operation of Eq. (2) and $P(\mathbf{p}(0))$ denotes the probability distribution of the initial score differences from which the dynamics is started. We consider the typical case by averaging over the specific choices of quenched strategies. The averaged generating functional may then be transformed exactly into a form involving only macroscopic but temporally non-local variables ($\tilde{\mathbf{C}}$, $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{K}}$), relatable to the correlation and response functions of the original many-agent problem:

$$Z = \int D\tilde{C}(t,t') D\tilde{G}(t,t') D\tilde{K}(t,t') \exp\big(N\Phi(\tilde{\mathbf{C}}, \tilde{\mathbf{G}}, \tilde{\mathbf{K}})\big), \tag{4}$$

where $\Phi$ is independent of $N$ and the bold-face notation here denotes matrices in time. This expression is extremally dominated in the large $N$ limit and steepest descents yields an effective stochastic single-agent dynamics

$$p(t+1) = p(t) - d \sum_{t' \leqslant t} (\mathbf{1} + \mathbf{G})^{-1}_{tt'} \operatorname{sgn} p(t') + \sqrt{d}\eta(t), \tag{5}$$

where

$$\langle \eta(t)\eta(t') \rangle = [(\mathbf{1} + \mathbf{G})^{-1}(\mathbf{1} + \mathbf{C})(\mathbf{1} + \mathbf{G}^{\mathbf{T}})^{-1}]_{tt'} \tag{6}$$

and the $\mathbf{G}$ and $\mathbf{C}$ are two-time response and correlation functions determined self-consistently as averages over an ensemble of such single agents [10]; see Ref. [9] for details. In the limit of large $N$ this analysis is believed to be exact, but it is highly non-trivial. Empirical evidence is shown in Fig. 1 where comparison is made between the results of simulations over many instances of the many-agent Eq. (2) and numerical evaluations of the analytically derived single-agent dynamics of Eq. (5), including extension to anti-correlated strategies [6].

Hence, naive characterization in terms of a unique deterministic ‘representative agent’, in the sense of conventional economics theory, is not possible. However, a single effective-agent description is available in a much more subtle sense. This is that one can consider the system to behave as though one has a ‘*representative stochastic ensemble*’ of non-interacting agents experiencing memory-weighting and coloured noise, both determined self-consistently over the ensemble. Note that Eq. (5) is stochastic even though Eq. (2) is deterministic.

To go further one would need to solve the effective single-agent ensemble dynamics in a closed form. A complete solution is not currently possible. However, one can solve for certain quantities in the ergodic equilibrating region. This concerns the aymptotic long-time behaviour, which is stationary so that the two-time correlation and response functions become functions only of the relative times (i.e., $G(t,t')$ and $C(t,t')$ become functions only of $(t'-t)$). Assuming also finite integrated response and weak long-term memory leads to a formulation determining self-consistently the asymptotic order parameters $Q = \lim_{\tau\to\infty} C(\tau)$ and the integrated response $\chi = \sum_\tau G(\tau)$. Breakdown of the ergodic regime is signalled by diverging integrated response. Again the analytic theory works well within this ergodic regime, as is demonstrated in Fig. 2. The volatility, however, requires also the non-stationary parts of $C$ and $G$ and remains incompletely solved in general, even in the ergodic regime [6].

The origin of the large volatilities found for *tabula rasa* starts in the non-ergodic regime can be ascribed to oscillatory behaviour, clearly visible empirically for such starts in quantities like the temporal correlation function

$$C(\tau) = \lim_{t\to\infty} N^{-1} \sum_i \operatorname{sgn}(p_i(t+\tau)) \operatorname{sgn}(p_i(t)), \tag{7}$$

which exhibits persistent oscillations (with period 2 in the rescaled time units of Eq. (2)) for $d < d_c$ [6]. *Tabula rasa* starts in this region exhibit essentially no frozen agents, whereas highly biased starts result in mostly

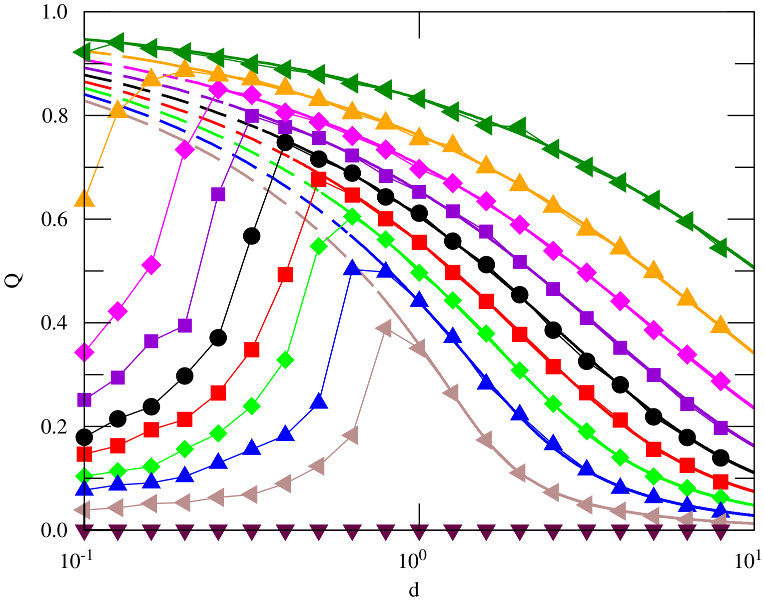


Fig. 2. Persistent part $Q$ of the correlation function for the batch MG with *tabula rasa* initial conditions. Symbols are simulation data. Solid lines are the theoretical predictions for the ergodic regime, extrapolated as dashed lines into the non-ergodic phase below $d_c$ (where they are no longer valid), the changeover signalling the predicted breakdown of the ergodic assumption. The different curves are for different degrees of mutual correlation between agents' two strategies; from anticorrelated at the bottom to highly correlated at the top. From Ref. [6].

frozen agents and hence reduce the oscillations and with them the excess time-averaged volatility. The oscillations and the excess volatility are also reduced by random asynchronous point updating [6] and by adding appropriate stochasticity to the original MG dynamics [11,12].

Thus, as well as its possible relevance as an idealized economics model, the minority game is of interest as a novel complex many-body system with both similarities and differences compared with other problems previously studied in statistical physics. Techniques developed within the spin glass community have proven useful in its analysis and suggest extensions to other dynamical many-body systems characterised by a combination of local/personal and global/range-free parameters, such as typified by stockmarkets (and in contrast to those of most conventional condensed matter systems), without the need for Markovian or detailed balance dynamics. A complete solution to the effective single-agent stochastic ensemble remains still a challenge.

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