29 Maggio

<u>Teor</u> $x_0 \in \mathbb{R}^d$ $y_0 \in \mathbb{R}^N$

$A \subseteq \mathbb{R}^d \times \mathbb{R}^N$ $(x_0, y_0) \in A$

$F : A \to \mathbb{R}^N$ $(x, y) \to F(x, y) \in \mathbb{R}^N$

$F(x_0, y_0) = 0 \in \mathbb{R}^N$

$F \in C^1(A, \mathbb{R}^N)$

$$D_y F(x_0, y_0) \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$$

$$\left(\begin{pmatrix} \partial_1 F_1(x_0,y_0) & \partial_2 F_1(x_0,y_0) & \dots & \partial_N F_1(x_0,y_0) \\ \vdots & & & \\ \partial_1 F_N(x_0,y_0) & \dots & \dots & \partial_N F_N(x_0,y_0) \end{pmatrix} \right.$$

è invertibile.

Allora vale quanto segue

(a) Esistono un intorno $U (= D_{\mathbb{R}^d}(x_0, r_1))$ di x_0 e un intorno $V (= D_{\mathbb{R}^N}(y_0, r_2))$

di y_0 ed una funzione

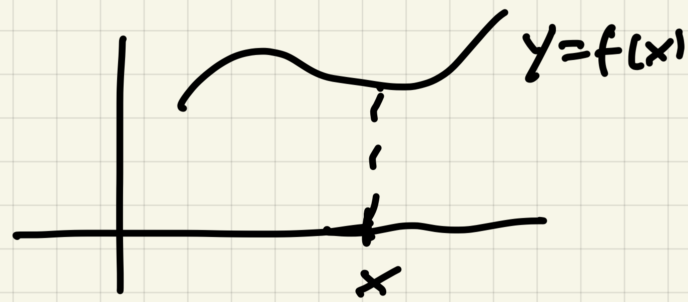
$$f : U \longrightarrow V \qquad U \ni x \longrightarrow \underset{\substack{\parallel \\ y}}{f(x)} \in V$$

tali che

$$F(x,y)=0 \quad \text{per } (x,y) \in U \times V \ (\subseteq A)$$

$$\Updownarrow$$

$$y = f(x) \quad \text{con } x \in U.$$



(b) Si ha $f \in C^1(U, \mathbb{R}^N)$ con

$$Df(x) = -\left(D_y F(x,f(x))\right)^{-1} D_x F(x,f(x))$$

<u>Teor</u> Sia $F \in C^1(A, \mathbb{R}^N)$

$\underset{\text{aperto}}{A \subseteq \mathbb{R}^{d+N}}$, $G \in C^1(A, \mathbb{R})$

$$F(z) = \begin{pmatrix} F_1(z) \\ \vdots \\ F_N(z) \end{pmatrix}$$

Sia $z_0 \in A$ t.c. $F(z_0) = 0$

e sia $DF(z_0) : \mathbb{R}^{d+N} \to \mathbb{R}^N$ sia suriettivo.

Sia $\mathcal{C} = \{ z \in A : F(z) = 0 \}$

Allora, se z_0 è un punto ~~di~~ max locale per $G|_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}$

$\exists!$ una scelta $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$

t.c.
$$\nabla G(z_0) = \sum_{j=1}^{N} \lambda_j \nabla F_j(z_0)$$

$z_0 = (x_0, y_0)$

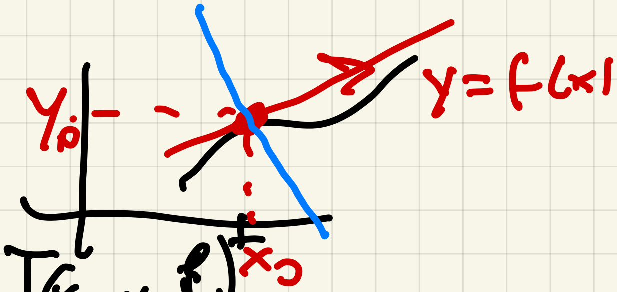
Dim (Idea) Non è restrittivo assumere che $z = (x, y) \in \mathbb{R}^d \times \mathbb{R}^N$ con

$D_y F(x_0, y_0) : \mathbb{R}^N \to \mathbb{R}^N$ isomorfismo.

$\mathcal{C}$ $\qquad y = f(x) \qquad f : D(x_0, r_1) \to D(y_0, r_2)$

$$T_{(x_0,y_0)}\mathcal{C} = \{ (u, Df(x_0)u) : u \in \mathbb{R}^d \} \quad d$$

$$\left(T_{(x_0,y_0)}\mathcal{C}\right)^{\perp} \quad N$$



$$\mathbb{R}^{d+N} = T_{(x_0,y_0)}\mathcal{C} \oplus \left(T_{(x_0,y_0)}\mathcal{C}\right)^{\perp}$$

$$\nabla G(x_0,y_0),\ \nabla F_1(x_0,y_0), \ldots, \nabla F_N(x_0,y_0) \in \left(T_{(x_0,y_0)}\mathcal{C}\right)^{\perp}$$

$$\forall b \quad \begin{cases} \partial_1 G(x) = \sum_{j=1}^{N} \lambda_j(b)\, \partial_1 F_j(x) \\ \vdots \\ \partial_d G(x) = \sum_{j=1}^{N} \lambda_j(b)\, \partial_d F_j(x) \\ F_1(x) = b_1 \\ F_N(x) = b_N \end{cases}$$

$x = (x_1, \ldots, x_d)$

$\lambda_1, \ldots, \lambda_N$

$b = (b_1, \ldots, b_N)$

Supponiamo che $x = x(b)$ sia il punto di Massimo e sia $\underset{\mathbb{R}^N}{b} \to \underset{\mathbb{R}^d}{x(b)}$ differenziabile.

$G(x(b))$

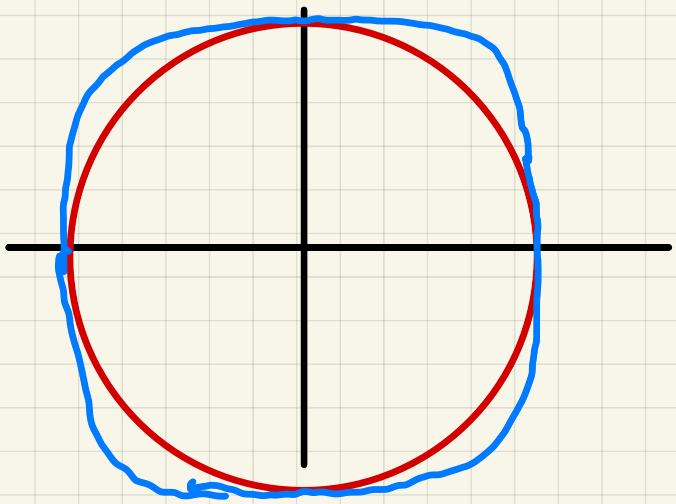
$$\frac{\partial}{\partial b_k} G(x(b)) = \lambda_k(b)$$

$$\begin{cases} f(x,y) = x^4 + y^4 \\ \text{con vincolo} \\ \quad x^6 + y^6 = 1 \end{cases}$$

trovare massimi e minimi

$y = \pm \sqrt[6]{1-x^6}$

$$\begin{cases} \cancel{4}^{\,2} x^3 = \cancel{6}^{\,3} \lambda x^5 \\ \cancel{4}^{\,2} y^3 = \cancel{6}^{\,3} \lambda y^5 \\ x^6 + y^6 = 1 \end{cases}$$



Per $x = 0$

$y = \pm 1$

$(0, \pm 1)$ $\quad \left(\lambda = \frac{2}{3}\right)$

$(\pm 1, 0)$

$$f(\pm 1, 0) = f(0, \pm 1) = 1$$

Devono esistere altri punti critici

con $x \neq 0$ e con $y \neq 0$

$$\begin{cases} 2x^3 = 3\lambda x^5 \\ 2y^3 = 3\lambda y^5 \end{cases} \qquad \begin{cases} 2 = 3\lambda x^2 \\ 2 = 3\lambda y^2 \end{cases}$$

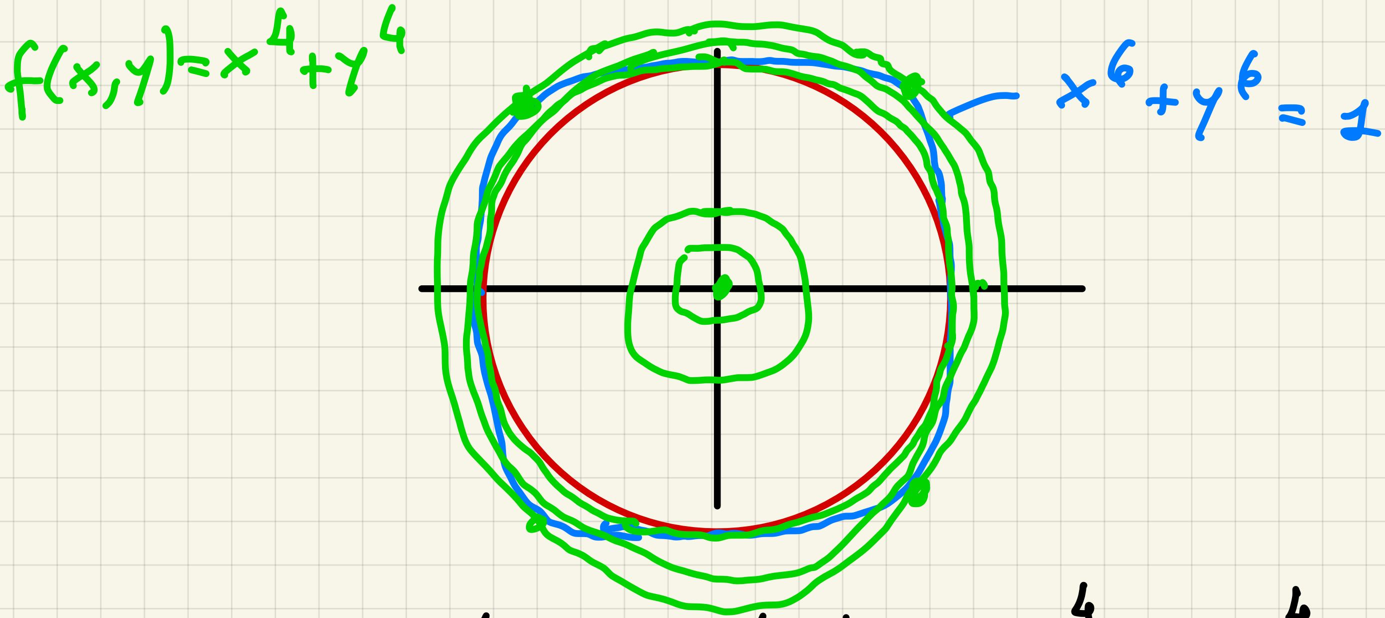
$\lambda x^6 + y^6 = 1$ $\qquad\qquad$ $\lambda x^6 + y^6 = 1$

$$x^2 = \frac{2}{3\lambda} = y^2$$

$y = \pm x$ $\qquad$ $2x^6 = 1$ $\qquad$ $x = \pm 2^{-\frac{1}{6}}$

$$\pm\left(\pm 2^{-\frac{1}{6}}, 2^{-\frac{1}{6}}\right)$$

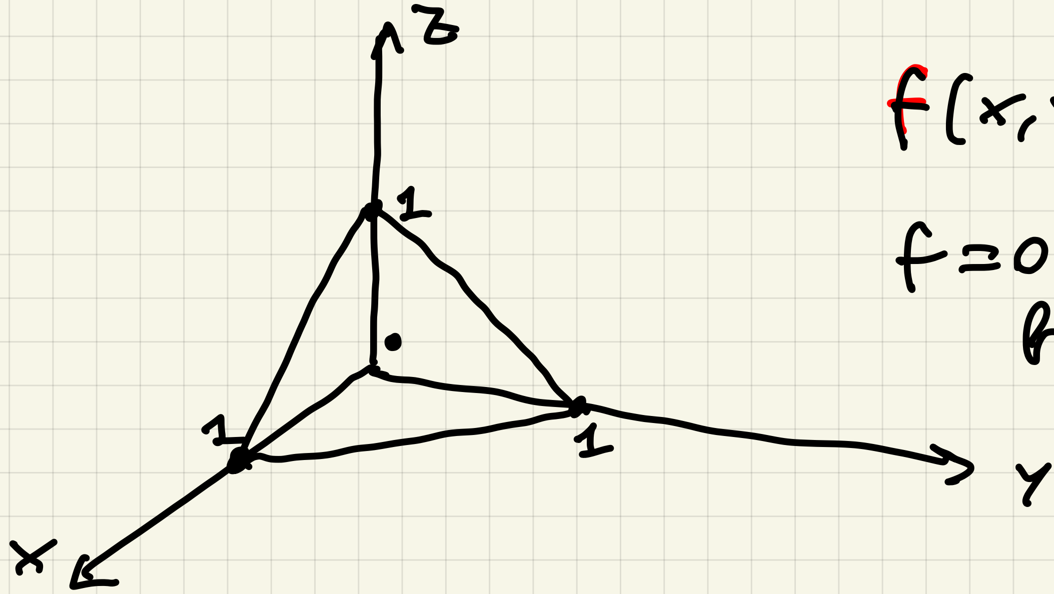
$2 = 3\lambda\, 2^{-\frac{1}{3}}$ $\qquad$ $\lambda = \frac{2^{\frac{2}{3}}}{3}$



$$f\left(\pm 2^{-\frac{1}{6}}, \pm 2^{-\frac{1}{6}}\right) = 2^{-\frac{4}{6}} + 2^{-\frac{4}{6}}$$

$$= 2\, 2^{-\frac{2}{3}} = 2^{\frac{1}{3}} > 1$$

$$\begin{cases} f(x,y,z) = xyz \\ x+y+z=1 \quad x \geq 0,\ y \geq 0,\ z \geq 0 \end{cases}$$



$f(x,y,z) \geq 0$

$f = 0$ sul bordo del triangolo

$$\begin{cases} x+y+z=1 \\ yz = \lambda \\ xz = \lambda \\ xy = \lambda \end{cases}$$

$$xz = \lambda = xy$$

$$\cancel{x} z = \cancel{x} y$$

$$x = y = z$$

$$3x = 1 \qquad x = \frac{1}{3}$$

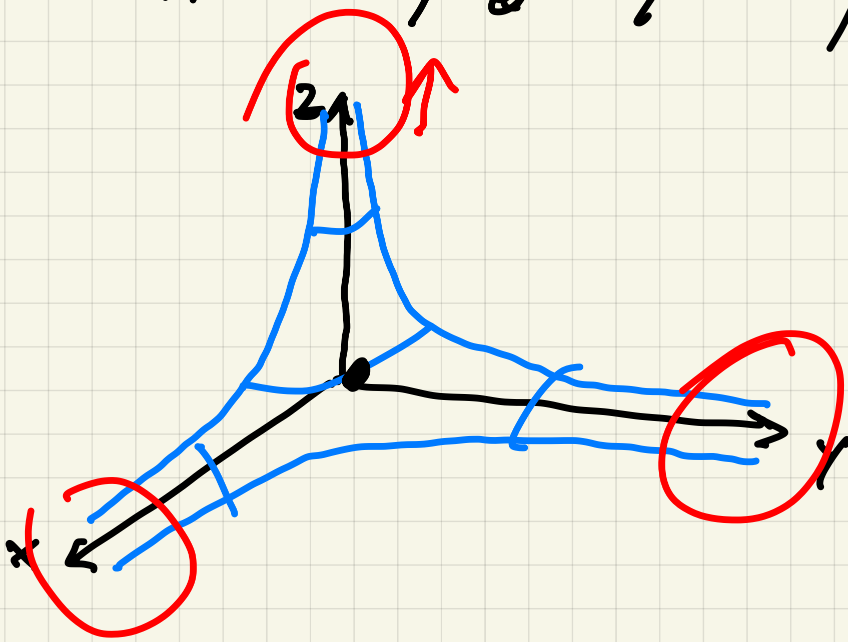
$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ è il punto di max assoluto

$$\lambda = \frac{1}{9}$$

$$\begin{cases} f(x,y,z) = xyz \\ xy + xz + yz = 1 \end{cases} \qquad x \geq 0,\ y \geq 0,\ z \geq 0$$

Punti di minimo vincolato si hanno per

$x = 0$, $y = 0$, $z = 0$.



ma devo considerare

$$\lim_{(x,y,z)\to\infty} xyz =$$

se ad esempio $z \to +\infty$

$$xy + (x+y)z = 1$$

devo avere $x + y \to 0 \Rightarrow x, y \to 0$

$\Rightarrow \quad x \to 0,\ y \to 0,\ xy \to 0 \Rightarrow (x+y)z \to 1$

$$0 \leq xyz \leq x\,\underbrace{(x+y)z}$$

0 1

$$\Longrightarrow \lim_{(x,y,z)\to\infty} f(x,y,z) = 0$$

$$\begin{cases} yz = \lambda (y+z) \\ xz = \lambda (x+z) \\ xy = \lambda (x+y) \\ xy + xz + yz = 1 \end{cases}$$

$$\frac{yz}{y+z} = \lambda = \frac{xz}{x+z}$$

$$\frac{y\cancel{z}}{y+z} = \frac{x\cancel{z}}{x+z}$$

$$\frac{y}{y+z} = \frac{x}{x+z}$$

$t \longrightarrow \dfrac{t}{t+z}$ è strettamente crescente in $[0,+\infty)$

$$\left(\frac{t}{t+z}\right)' = \frac{(t+z)-t}{(t+z)^2} = \frac{z}{(t+z)^2} > 0$$

$$\Longrightarrow x = y = z$$

$$3x^2 = 1 \qquad x = 3^{-\frac{1}{2}}$$

Lemma Sia $T \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$ invertibile

e poniamo $C_1 = |T^{-1}|$

Sia $B \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$

Allora

se $C_1 |B - T| \leq \frac{1}{2} \Rightarrow B^{-1}$ esiste

e $|B^{-1}| \leq 2C_1$

Dim

$$B = T + B - T = T\left(1 + T^{-1}(B-T)\right)$$

$$|T^{-1}(B-T)| \leq |T^{-1}|\,|B-T| =$$

$$= C_1 |B-T| \leq \frac{1}{2}$$

Allora

$$\left(1 + T^{-1}(B-T)\right)^{-1} = \sum_{j=0}^{\infty} (-1)^j \left(T^{-1}(B-T)\right)^j$$

$$\left| \left(1+T^{-1}(B-T)\right)^{-1} \right| \le \sum_{j=0}^{\infty} \left| \left(T^{-1}(B-T)\right)^{j} \right|$$

$$\le \sum_{j=0}^{\infty} \left| T^{-1}(B-T) \right|^{j}$$

$$\le \sum_{j=0}^{\infty} \Big(\underbrace{|T^{-1}|}_{C_1} \; |B-T| \Big)^{j}$$

$$\le \sum_{j=0}^{\infty} 2^{-j} = 2 = \frac{1}{1-\frac{1}{2}} = 2$$

$$|B^{-1}| \le \underbrace{\left| \left(1+T^{-1}(B-T)\right)^{-1} \right|}_{\le 2} \; \underbrace{|T^{-1}|}_{=\, C_1}$$