5

Waves and Vibrations

General Concepts – Transversal Waves in a String – The One-dimensional Wave Equation – The d'Alembert Formula – Second Order Linear Equations – Hyperbolic Systems With Constant Coefficients – The Multi-dimensional Wave Equation ($n > 1$) – Two Classical Models – The Cauchy Problem – Linear Water Waves

5.1 General Concepts

5.1.1 Types of waves

Our dayly experience deals with sound waves, electromagnetic waves (as radio or light waves), deep or surface water waves, elastic waves in solid materials. Oscillatory phenomena manifest themselves also in contexts and ways less macroscopic and known. This is the case, for instance, of rarefaction and shock waves in traffic dynamics or of electrochemical waves in human nervous system and in the regulation of the heart beat. In quantum physics, everything can be described in terms of wave functions, at a sufficiently small scale.

Although the above phenomena share many similarities, they show several differences as well. For example, progressive water waves propagate a disturbance, while standing waves do not. Sound waves need a supporting medium, while electromagnetic waves do not. Electrochemical waves interact with the supporting medium, in general modifying it, while water waves do not.

Thus, it seems too hard to give a general definition of *wave*, capable of covering all the above cases, so that we limit ourselves to introducing some terminology and general concepts, related to specific types of waves. We start with one-dimensional waves.

a. **Progressive** or **travelling** waves are disturbances described by a function of the following form:

$$u(x,t) = g(x - ct).$$

For $t = 0$, we have $u(x,0) = g(x)$, which is the "initial" profile of the perturbation. This profile propagates without change of shape with speed $|c|$, in the positive

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(negative) $x-$direction if $c > 0$ ($c < 0$). We have already met this kind of waves in Chapters 2 and 4.

b. **Harmonic** waves are particular progressive waves of the form

$$u(x,t) = A\exp\{i(kx-\omega t)\}, \qquad A,k,\omega \in \mathbb{R}. \tag{5.1}$$

It is understood that only the *real part* (or the imaginary part)

$$A\cos(kx-\omega t)$$

is of interest, but the complex notation may often simplify the computations. In (5.1) we distinguish, considering for simplicity ω and k positive:

- The wave *amplitude* $|A|$;
- The *wave number* k, which is the number of complete oscillations in the space interval $[0,2\pi]$, and the *wavelength*

  $$\lambda = \frac{2\pi}{k}$$

  which is the distance between successive maxima (*crest*) or minima (*troughs*) of the waveform;
- The *angular frequency* ω, and the *frequency*

  $$f = \frac{\omega}{2\pi}$$

  which is the number of complete oscillations in one second (Hertz) at a fixed space position;
- The *wave or phase speed*

  $$c_p = \frac{\omega}{k}$$

  which is the crests (or troughs) speed;

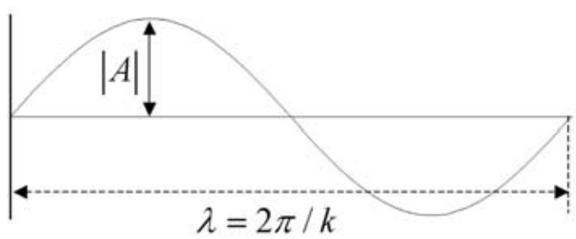

Fig. 5.1. Sinusoidal wave

c. Standing waves are of the form

$$u(x,t) = B \cos kx \cos \omega t.$$

In these disturbances, the basic sinusoidal wave, $\cos kx$, is modulated by the time dependent oscillation $B \cos \omega t$. A standing wave may be generated, for instance, by superposing two harmonic waves with the same amplitude, propagating in opposite directions:

$$A \cos(kx - \omega t) + A \cos(kx + \omega t) = 2A \cos kx \cos \omega t. \tag{5.2}$$

Consider now waves in dimension $n > 1$.

d. Plane waves. *Scalar* plane waves are of the form

$$u(\mathbf{x},t) = f(\mathbf{k} \cdot \mathbf{x} - \omega t).$$

The disturbance propagates in the direction of $\mathbf{k}$ with speed $c_p = \omega / |\mathbf{k}|$. The planes of equation

$$\theta(\mathbf{x},t) = \mathbf{k} \cdot \mathbf{x} - \omega t = \text{constant}$$

constitute the *wave-fronts.*

Harmonic or monochromatic plane waves have the form

$$u(\mathbf{x},t) = A \exp\{i(\mathbf{k} \cdot \mathbf{x} - \omega t)\}.$$

Here $\mathbf{k}$ is the *wave number* vector and ω is the *angular frequency.* The vector $\mathbf{k}$ is orthogonal to the wave front and $|\mathbf{k}|/2\pi$ gives the number of waves per unit length. The scalar $\omega/2\pi$ still gives the number of complete oscillations in one second (Hertz) at a fixed space position.

e. Spherical waves are of the form

$$u(\mathbf{x},t) = v(r,t)$$

where $r = |\mathbf{x} - \mathbf{x}_0|$ and $\mathbf{x}_0 \in \mathbb{R}^n$ is a fixed point. In particular $u(\mathbf{x},t) = e^{i\omega t} v(r)$ represents a stationary spherical wave, while $u(\mathbf{x},t) = v(r - ct)$ is a progressive wave whose wavefronts are the spheres $r - ct = \text{constant}$, moving with speed $|c|$ (outgoing if $c > 0$, incoming if $c < 0$).

5.1.2 Group velocity and dispersion relation

Many oscillatory phenomena can be modelled by linear equations whose solutions are superpositions of harmonic waves with angular frequency depending on the wave number:

$$\omega = \omega(k). \tag{5.3}$$

A typical example is the wave system produced by dropping a stone in a pond.

If ω is linear, e.g. $\omega(k) = ck$, $c > 0$, the crests move with speed c, independent of the wave number. However, if $\omega(k)$ is not proportional to k, the crests move with speed $c_p = \omega(k)/k$, that *depends* on the wave number. In other words, the crests move at different speeds for different wavelengths. As a consequence, the various components in a wave packet given by the superposition of harmonic waves of different wavelengths will eventually separate or *disperse*. For this reason, (5.3) is called **dispersion relation**.

In the theory of dispersive waves, the **group velocity**, given by

$$c_g = \omega'(k)$$

is a central notion, mainly for the following three reasons.

1. *It is the speed at which an isolated wave packet moves as a whole.* A wave packet may be obtained by the superposition of dispersive harmonic waves, for instance through a Fourier integral of the form

$$u(x,t) = \int_{-\infty}^{+\infty} a(k)\, e^{i[kx - \omega(k)t]} dk \tag{5.4}$$

where the real part only has a physical meaning. Consider a localized wave packet, with wave number $k \approx k_0$, almost constant, and with amplitude slowly varying with x. Then, the packet contains a large number of crests and the amplitudes $|a(k)|$ of the various Fourier components are negligible except that in a small neighborhood of k_0, $(k_0 - \delta, k_0 + \delta)$, say.

Figure 5.2 shows the initial profile of a Gaussian packet,

$$\operatorname{Re} u(x,0) = \frac{3}{\sqrt{2}} \exp\left\{-\frac{x^2}{32}\right\} \cos 14x,$$

slowly varying with x, with $k_0 = 14$, and its Fourier transform:

$$a(k) = 6\exp\{-8(k-14)^2\}.$$

As we can see, the amplitudes $|a(k)|$ of the various Fourier components are negligible except when k is near k_0.

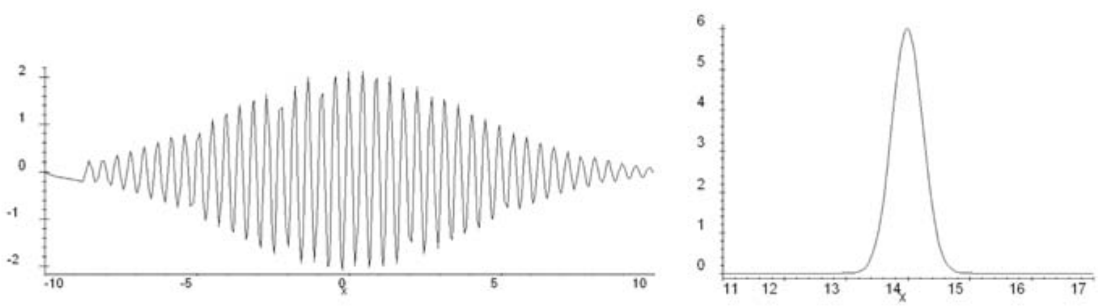

Fig. 5.2. Wave packet and its Fourier transform

Then we may write

$$\omega(k) \approx \omega(k_0) + \omega'(k_0)(k - k_0) = \omega(k_0) + c_g(k - k_0)$$

and

$$u(x,t) \approx e^{i\{k_0 x - \omega(k_0)t\}} \int_{k_0-\delta}^{k_0+\delta} a(k)\, e^{i(k-k_0)(x-c_g t)} dk. \tag{5.5}$$

Thus, u turns out to be well approximated by the product of two waves. The first one is a pure harmonic wave with relatively short wavelength $2\pi/k_0$ and phase speed $\omega(k_0)/k_0$. The second one depends on x, t through the combination $x - c_g t$, and is a superposition of waves of very small wavenumbers $k - k_0$, which correspond to very large wavelengths. We may interpret the second factor as a sort of envelope of the short waves of the packet, that is the packet as a whole, which therefore moves with the group speed.

2. *An observer that travels at the group velocity sees constantly waves of the same wavelength* $2\pi/k$, after the transitory effects due to a localized initial perturbation (e.g. a stone thrown into a pond). In other words, c_g is the propagation speed of the wave numbers.

Imagine dropping a stone into a pond. At the beginning, the water perturbation looks complicated, but after a sufficiently long time, the various Fourier components will be quite dispersed and the perturbation will appear as a slowly modulated wave train, almost sinusoidal near every point, with a *local wave number* $k(x,t)$ and a *local frequency* $\omega(x,t)$. If the water is deep enough, we expect that, at each fixed time t, the wavelength increases with the distance from the stone (longer waves move faster, see subsection 5.10.4) and that, at each fixed point x, the wavelength tends to decrease with time.

Thus, the essential features of the wave system can be observed at a relatively long distance from the location of the initial disturbance and after some time has elapsed.

Let us assume that the free surface displacement u is given by a Fourier integral of the form (5.4). We are interested on the behavior of u for $t \gg 1$. An important tool comes from the method of stationary phase[1] which gives an asymptotic formula for integrals of the form

$$I(t) = \int_{-\infty}^{+\infty} f(k)\, e^{it\varphi(k)} dk \tag{5.6}$$

as $t \to +\infty$. We can put u into the form (5.6) by writing

$$u(x,t) = \int_{-\infty}^{+\infty} a(k)\, e^{it[k\frac{x}{t} - \omega(k)]} dk,$$

then by moving from the origin at a fixed speed V (thus $x = Vt$) and defining

$$\varphi(k) = kV - \omega(k).$$

Assume for simplicity that φ has only one stationary point k_0, that is

$$\omega'(k_0) = V,$$

[1] See subsection 5.10.6

and that $\omega''(k_0) \neq 0$. Then, according to the *method of stationary phase*, we can write

$$u(Vt,t) = \sqrt{\frac{\pi}{|\omega''(k_0)|}}\frac{a(k_0)}{\sqrt{t}}\exp\{it[k_0V - \omega(k_0)]\} + O\left(t^{-1}\right). \tag{5.7}$$

Thus, if we allow errors of order t^{-1}, moving with speed $V = \omega'(k_0) = c_g$, *the same wave number* k_0 *always appears* at the position $x = c_g t$. Note that the amplitude decreases like $t^{-1/2}$ as $t \to +\infty$. This is an important attenuation effect of dispersion.

3. *Energy is transported at the group velocity by waves of wavelength* $2\pi/k$. In a wave packet like (5.5), the energy is proportional to[2]

$$\int_{k_0-\delta}^{k_0+\delta} |a(k)|^2\, dk \simeq 2\delta\,|a(k_0)|^2$$

so that it moves at the same speed of k_0, that is c_g.

Since the energy travels at the group velocity, there are significant differences in the wave system according to the sign of $c_g - c_p$, as we will see in Section 10.

5.2 Transversal Waves in a String

5.2.1 The model

We derive a classical model for the small transversal vibrations of a tightly stretched horizontal string (e.g. a string of a guitar). We assume the following hypotheses:

1. *Vibrations of the string have small amplitude.* This entails that the changes in the slope of the string from the horizontal equilibrium position are very small.
2. *Each point of the string undergoes vertical displacements only.* Horizontal displacements can be neglected, according to 1.
3. *The vertical displacement of a point depends on time and on its position on the string.* If we denote by u the vertical displacement of a point located at x when the string is at rest, then we have $u = u(x,t)$ and, according to 1, $|u_x(x,t)| \ll 1$.
4. *The string is perfectly flexible.* This means that it offers no resistance to bending. In particular, the stress at any point on the string can be modelled by a tangential[3] force $\mathbf{T}$ of magnitude τ, called *tension*. Figure 5.3 shows how the forces due to the tension acts at the end points of a small segment of the string.
5. *Friction is negligible.*

Under the above assumptions, the equation of motion of the string can be derived from *conservation of mass* and *Newton law.*

[2] See *A. Segel*, 1987.

[3] Consequence of absence of distributed moments along the string.

5.10.5 Asymptotic behavior

As we have already observed, the behavior of a wave packet is dominated for short times by the initial conditions and only after a relatively long time it is possible to observe the intrinsic features of the perturbation. For this reason, information about the asymptotic behavior of the packet as $t \to +\infty$ are important. Thus, we need a good asymptotic formula for the integral in (5.149) when $t \gg 1$.

For simplicity, consider gravity waves only, for which

$$\omega(k) = \sqrt{g|k|}.$$

Let us follow a particle $x = x(t)$ moving along the positive $x-$direction with constant speed $v > 0$, so that $x = vt$. Inserting $x = vt$ into (5.149) we find

$$\begin{aligned} h(vt,t) &= \frac{1}{4\pi}\int_{\mathbb{R}} e^{it(kv-\omega(k))}\widehat{h}_0(k)\ dk + \frac{1}{4\pi}\int_{\mathbb{R}} e^{it(kv+\omega(k))}\widehat{h}_0(k)\ dk \\ &\equiv h_1(vt,t) + h_2(vt,t). \end{aligned}$$

According to Theorem 5.6 in the next subsection (see also Remark 5.10), with

$$\varphi(k) = kv - \omega(k),$$

if there exists exactly one stationary point for φ, i.e. only one point k_0 such that

$$\omega'(k_0) = v \quad \text{and} \quad \varphi''(k_0) = -\omega''(k_0) \neq 0,$$

we may estimate h_1 for $t \gg 1$ by the following formula:

$$h_1(vt,t) = \frac{A(k_0)}{t}\exp\{it[k_0 v - \omega(k_0)]\} + O\left(t^{-1}\right) \tag{5.151}$$

where

$$A(k_0) = \widehat{h}_0(k_0)\sqrt{\frac{1}{8\pi|\omega''(k_0)|}}\exp i\left\{-\frac{\pi}{4}\text{sign } \omega''(k_0)\right\}.$$

We have

$$\omega'(k) = \frac{1}{2}\sqrt{g}\,|k|^{-1/2}\,\text{sign}(k)$$

and

$$\omega''(k) = -\frac{\sqrt{g}}{4}|k|^{-3/2}.$$

Since $v > 0$, equation $\omega'(k_0) = v$ gives the unique *point of stationary phase*

$$k_0 = \frac{g}{4v^2} = \frac{gt^2}{4x^2}.$$

Moreover,

$$k_0 v - \omega(k_0) = -\frac{g}{4v} = -\frac{gt}{4x}$$

and

$$\omega''(k_0) = -\frac{2v^3}{g} = -\frac{2x^3}{gt^3} < 0$$

so that from (5.151) we find

$$h_1(vt,t) = \frac{1}{4}\widehat{h}_0\left(\frac{g}{4v^2}\right)\sqrt{\frac{g}{\pi t v^3}}\exp i\left\{-\frac{gt}{4v}+\frac{\pi}{4}\right\} + O\left(t^{-1}\right)$$

Similarly, since

$$\widehat{h}_0(k_0) = \widehat{h}_0(-k_0),$$

we find

$$h_2(vt,t) = \frac{1}{4}\widehat{h}_0\left(\frac{g}{4v^2}\right)\sqrt{\frac{g}{\pi t v^3}}\exp i\left\{\frac{gt}{4v}-\frac{\pi}{4}\right\} + O\left(t^{-1}\right).$$

Finally,

$$\begin{aligned} h(vt,t) &= h_1(vt,t) + h_2(vt,t) \\ &= \widehat{h}_0\left(\frac{g}{4v^2}\right)\sqrt{\frac{g}{4\pi v^3 t}}\cos\left\{\frac{gt}{4v}-\frac{\pi}{4}\right\} + O\left(t^{-1}\right). \end{aligned}$$

This formula shows that, for large x and t, with $x/t = v$, constant, the wave packet is locally sinusoidal with wave number

$$k(x,t) = \frac{gt}{4vx} = \frac{gt^2}{4x^2}.$$

In other words, an observer moving at the constant speed $v = x/t$ sees a dominant wavelength $2\pi/k_0$, where k_0 is the solution of $\omega'(k_0) = x/t$. The amplitude decreases as $t^{-1/2}$. This is due to the dispersion of the various Fourier components of the initial configuration, after a sufficiently long time.

5.10.6 The method of stationary phase

The *method of stationary phase*, essentially due to Laplace, gives an asymptotic formula for integrals of the form

$$I(t) = \int_a^b f(k)\,e^{it\varphi(k)}dk \qquad (-\infty \le a < b \le \infty)$$

as $t \to +\infty$. Actually, only the real part of $I(t)$, in which the factor $\cos[t\varphi(k)]$ appears, is of interest. Now, as t increases and $\varphi(k)$ varies, $\cos[t\varphi(k)]$ oscillates more and more and eventually much more than f. For this reason, the contributions of the intervals where $\cos[t\varphi(k)] > 0$ will balance those in which $\cos[t\varphi(k)] < 0$, so that we expect that $I(t) \to 0$ as $t \to +\infty$, just as the Fourier coefficients of an integrable function tend to zero as the frequency goes to infinity.

To obtain information on the vanishing speed, assume φ is constant on a certain interval J. On this interval $\cos[t\varphi(k)]$ is constant as well and hence there are neither oscillations nor cancellations. Thus, it is reasonable that, for $t \gg 1$, the relevant contributions to $I(t)$ come from intervals where φ is constant or at least almost constant. The same argument suggests that eventually, a however small interval, containing a stationary point k_0 for φ, will contribute to the integral much more than any other interval without stationary points.

The method of stationary phase makes the above argument precise through the following theorem.

Theorem 5.6. *Let f and φ belong to $C^2([a,b])$. Assume that*

$$\varphi'(k_0) = 0, \varphi''(k_0) \neq 0 \qquad \text{and} \qquad \varphi'(k) \neq 0 \text{ for } k \neq k_0.$$

Then, as $t \to +\infty$

$$\int_a^b f(k)\, e^{it\varphi(k)} dk = \sqrt{\frac{2\pi}{|\varphi''(k_0)|}} \frac{f(k_0)}{\sqrt{t}} \exp\left\{i\left[t\varphi(k_0) + \frac{\pi}{4}\operatorname{sign}\varphi''(k_0)\right]\right\} + O\left(t^{-1}\right)$$

First a lemma.

Lemma 5.3. *Let f, φ as in Theorem 5.6. Let $[c,d] \subseteq [a,b]$ and assume that $|\varphi'(k)| \geq C > 0$ in (c,d). Then*

$$\int_c^d f(k)\, e^{it\varphi(k)} dk = O\left(t^{-1}\right) \qquad t \to +\infty. \tag{5.152}$$

Proof. Integrating by parts we get (multiplying and dividing by φ'):

$$\int_c^d \frac{f}{\varphi'}\varphi' e^{it\varphi} dk = \frac{1}{it}\left\{\frac{f(d)\, e^{it\varphi(d)}}{\varphi'(d)} - \frac{f(c)\, e^{it\varphi(c)}}{\varphi'(c)} - \int_c^d \frac{f'\varphi' - f\varphi''}{(\varphi')^2} e^{it\varphi} dk\right\}.$$

Thus, from $\left|e^{it\varphi(k)}\right| \leq 1$ and our hypotheses, we have

$$\begin{aligned}\left|\int_c^d f\ e^{it\varphi} dk\right| &\leq \frac{1}{Ct}\left\{|f(d)| + |f(c)| + \frac{1}{C}\int_c^d |f'\varphi' - f\varphi''|\, dk\right\} \\ &\leq \frac{K}{t}\end{aligned}$$

which gives (5.152). □

Proof of Theorem 5.6. Without loss of generality, we may assume $k_0 = 0$, so that $\varphi'(0) = 0$, $\varphi''(0) \neq 0$. From Lemma 5.3, it is enough to consider the integral

$$\int_{-\varepsilon}^{\varepsilon} f(k)\, e^{it\varphi(k)} dk$$

where $\varepsilon > 0$ is as small as we wish. We distinguish two cases.

Case 1: φ is a quadratic polynomial, that is

$$\varphi(k) = \varphi(0) + Ak^2, \qquad A = \frac{1}{2}\varphi''(0).$$

Then, write

$$f(k) = f(0) + \frac{f(k) - f(0)}{k}k \equiv f(0) + q(k)\,k,$$

and observe that, since $f \in C^2([-\varepsilon, \varepsilon])$, $q'(k)$ is bounded in $[-\varepsilon, \varepsilon]$. Then, we have:

$$\int_{-\varepsilon}^{\varepsilon} f(k)\,e^{it\varphi(k)}dk = 2f(0)e^{it\varphi(0)} \int_0^{\varepsilon} e^{itAk^2}dk + e^{it\varphi(0)} \int_{-\varepsilon}^{\varepsilon} q(k)\,ke^{itAk^2}dk.$$

Now, an integration by parts shows that the second integral is $O(1/t)$ as $t \to \infty$ (the reader should check the details).

In the first integral, if $A > 0$, let

$$tAk^2 = y^2.$$

Then

$$\int_0^{\varepsilon} e^{itAk^2}dk = \frac{1}{\sqrt{tA}} \int_0^{\varepsilon\sqrt{tA}} e^{iy^2}dy.$$

Since[38]

$$\int_0^{\varepsilon\sqrt{tA}} e^{iy^2}dy = \frac{\sqrt{\pi}}{2}e^{i\frac{\pi}{4}} + O\left(\frac{1}{\varepsilon\sqrt{tA}}\right),$$

we get

$$\int_0^{\varepsilon} f(k)\,e^{it\varphi(k)}dk = \sqrt{\frac{2\pi}{|\varphi''(0)|}}\frac{f(0)}{\sqrt{t}} \exp\left\{i\left[\varphi(0)t + \frac{\pi}{4}\right]\right\} + O\left(\frac{1}{t}\right),$$

which proves the theorem when $A > 0$. The proof is similar if $A < 0$.

Case 2. General φ. By a suitable change of variable we reduce case 2 to case 1. First we write

$$\varphi(k) = \varphi(0) + \frac{1}{2}a(k)\,k^2 \tag{5.153}$$

where

$$a(k) = 2\int_0^1 (1 - r)\,\varphi''(rk)\,dr.$$

[38] Recall that $e^{i\pi/4} = \left(\sqrt{2} + i\sqrt{2}\right)/2$. Moreover, the following formulas hold:

$$\left|\frac{\sqrt{\pi}}{2\sqrt{2}} - \int_0^{\lambda} \cos(y^2)dy\right| \le \frac{\sqrt{\pi}}{\lambda}$$
$$\left|\frac{\sqrt{\pi}}{2\sqrt{2}} - \int_0^{\lambda} \sin(y^2)dy\right| \le \frac{\sqrt{\pi}}{\lambda}.$$

Equation (5.153) follows by applying to $\psi(s) = \varphi(sk)$ the following Taylor formula:

$$\psi(1) = \psi(0) + \psi'(0)\, s + \frac{1}{2}\int_0^1 (1-r)\,\psi''(r)\, dr.$$

Note that $a(0) = \varphi''(0)$. Consider the function

$$p(k) = k\sqrt{a(k)/\varphi''(0)}.$$

We have $p(0) = 0$ and $p'(0) = 1$. Therefore, p is invertible near zero. Let

$$k = p^{-1}(y).$$

Then, since

$$\varphi(k) = \varphi(0) + \frac{\varphi''(0)}{2}\,[p(k)]^2,$$

we have,

$$\begin{aligned}\tilde{\varphi}(y) &\equiv \varphi\left(p^{-1}(y)\right)\\ &= \varphi(0) + \frac{\varphi''(0)}{2}\left[p\left(p^{-1}(y)\right)\right]^2\\ &= \varphi(0) + \frac{\varphi''(0)}{2}y^2\end{aligned}$$

and

$$\int_{-\varepsilon}^{\varepsilon} f(k)\, e^{it\varphi(k)} dk = \int_{p^{-1}(-\varepsilon)}^{p^{-1}(\varepsilon)} F(y)\, e^{it\tilde{\varphi}(y)} dy$$

where

$$F(y) = \frac{f\left(p^{-1}(y)\right)}{p'\left(p^{-1}(y)\right)}.$$

Since $F(0) = f(0)$ and $\tilde{\varphi}$ is a quadratic polynomial with $\tilde{\varphi}(0) = \varphi(0)$, $\tilde{\varphi}''(0) = \varphi''(0)$, case 2 follows from case 1. □

Remark 5.7. Theorem 5.6 holds for integrals extended over the whole real axis as well (actually this is the most interesting case) as long as, in addition, f is bounded, $|\varphi'(\pm\infty)| \geq C > 0$, and $\int_{\mathbb{R}} |f'\varphi' - f\varphi''|\,(\varphi')^{-2}\, dk < \infty$. Indeed, it is easy to check that Lemma 5.3 is true under these hypotheses and then the proof of Theorem 5.6 is exactly the same.

Problems

5.1. The chord of a guitar of length L is plucked at its middle point and then released. Write the mathematical model which governs the vibrations and solve it. Compute the energy $E(t)$.