Advanced Math. Phys and A –

Notes on fluid dynamics.

GF, May 2023

AMPA Fluid dynamics 1. - Introduction and basic definitions.

The purpose of this last set of lectures is to provide some notion of the dynamics of fluids, as well as address and "solve" a few interesting "exercises".

Fluid dynamics is a branch of the so called "continuum" mechanics, that is, the study of the evolution of systems with an infinite # of degrees of freedom, whose parameter space is a "continuum", that is, a non-zero measure set in $\mathbb{R}^3$.(*)

The so-called continuum hypothesis for deformable bodies asserts that:

- 1) $\exists$ a reference state Ω_0 for the body B, Ω_0 open in $\mathbb{R}^3$
- 2) The motion of the body is described by a C^∞ (**) map

$$\phi_t : \underset{\substack{\Downarrow \\ t}}{[0, \infty)} \times \Omega_0 \longrightarrow \Omega_t$$

invertible (with regular inverse) $\forall$ fixed t.

- 3) ϕ_t is differentiable (twice, at least) in t.

Let us examine 3+2 The motion of the body (or of any subportion $\Omega' \subset \Omega$) is thus described, in cartesian coordinates, by $(t, a) \in (0, \infty) \times \Omega_0 \longrightarrow x(t, \underline{a}) \in \Omega_t$

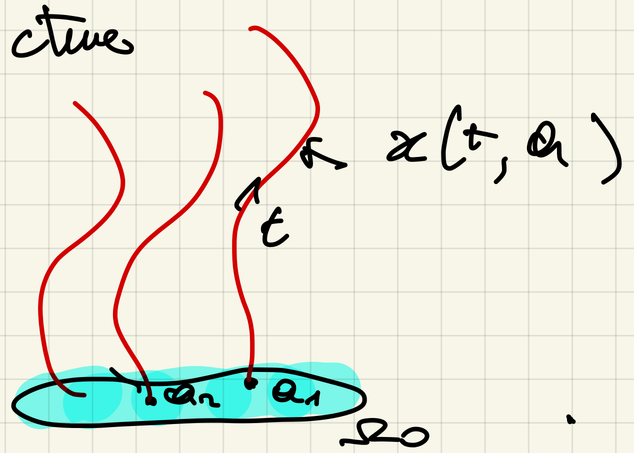
(*) Analogously, we can consider - as we shall do in the applications - to 2d bodies.

(**) Actually, we can relax to C^k, $k \geq 2$

This is akin to the usual mechanical representation, where, for a system of N particles located at $\underline{x}^0_i$ @ $t = t_0$, we have

$$x_i(t) = x_i(t; x^0_i).$$

In a picture



This is called "Lagrangian description". The Lagrangian velocity is

$$\underline{v}(t, \underline{a}) = \frac{\partial}{\partial t} \underline{x}(t, a).$$

In words, this is the velocity @ time t of the fluid parcel that, @ $t = 0$, was labeled by a.

The Eulerian velocity is, instead, a vector in $T_x \Omega_t$ and is defined as

$$\underline{u}(\underline{x}, t) = \underline{v}(t, \phi_t^{-1}(\underline{x})). \qquad (1.1)$$

Here we make use of the invertibility ($\forall t$) of the body motion, that is, given $(t, \underline{x}) \in (0, \infty) \times \Omega_t$ $\exists!$ $a(\underline{x}, t)$ s.t. $\phi_t(\underline{a}(\underline{x}, t)) = \underline{x}$.

[The Lagrangian description of any quantity representing a physical property will be given by

$$q(t, a) .$$

Its Eulerian counterpart will be

$$f(t, \underline{x}) = f(x(t; \underline{a}), t) = q(a, t)$$

seen as a scalar field, defined on $(0,\infty) \times \mathbb{R}_t$ — this is defined on $(0,\infty) \times \Omega_0$.]

To define the equation of motion, characterizing φ, we notice that, once $u(t, \underline{x})$ is known, then one can reconstruct the motion $(t, a) \mapsto x(t; a)$ by solving the ODE's with initial value

$$(1.2) \quad \begin{cases} \dfrac{d\underline{x}}{dt} = \underline{u}(t, \underline{x}) \\ \underline{x}(t_0) = a \end{cases}$$

So the basic kinematical quantity to be determined is the Euler velocity field $\underline{u}(t, \underline{x})$.

Further axioms:

4) Mass density and its conservation

$\exists$ a non-negative function $\rho_L = \rho_L(t, a)$ giving rise to its Eulerian counterpart $\rho(\underline{x}, t)$.

As we have seen in one of the lectures on the D'Alembert equation mass conservation translates into the PDE

$$\rho_t + \nabla\cdot(\rho \underline{u}) = 0 \qquad (1.3)$$

II : The translation of the fundamental physical laws

[classical physics!]

The question is how to implement in this setting the fundamental laws of physics, that is:

•) The momentum balance law,

$$\frac{d\vec{p}}{dt} = \vec{F}^{ext}, \quad \text{"} p = \sum m_i \cdot \underline{u}_i \text{"}$$

($\vec{F}^{ext}$ is the "resultant", i.e. the sum, of external forces, internal ones not contributing to the balance law by the 3rd Newton principle)

••) The angular momentum balance,

$$\frac{d\vec{L}}{dt} = \vec{M}^{ext} \quad \text{"} \vec{L} = \underline{x} \wedge \underline{p} \text{"}$$

$$\text{"} \vec{M}^{ext} = \sum_i \underline{x} \wedge F^{ext} \text{"}$$

•••) The first and second principle of thermodynamics

2 : in this set of lectures this last part will be only loosely addressed

The idea is to consider a generic subset $B_0 \subset \Omega_0$ and "follow it" along the – still unknown – motion, i.e consider $B_t = \phi_t(B_0)$. Then, if f is the volume density of some physical quantity (e.g. density, velocity, ...), the amount of such quantity transported by B_t will be

$$F_{B_t}(t) = \int_{B_t} f(t, x)\, d^3x \quad .$$

It is thus natural to assume that the time-variation of $F_{B_t}(t)$ will be due to two types of terms:

A) a volume source $\varphi_b(x;t)\, d^3x_t$

B) a contribution from the boundary, with surface density σ, so that general evolution equations will be written in the form

$$\frac{d}{dt}\int_{B_t} f(t;x)\, d^3x = \int_{B_t} \varphi(t;x)\, d^3x + \int_{\partial B_t} \sigma \, dA .$$

Let us first deal with the left hand side of this general equation.

Remark that we are defining a quantity (i.e $\int_{B_t} f(x,t)\, d^3x$) following the motion B_t in its time-evolution. Hence we are adopting a Lagrangian point of view. But we want, in the end, to have an expression involving Eulerian quantities. The key to arrive at such a result is the so-called Reynold's Transport Theorem (RTT)

Theorem (RTT)

Let f be a C^2 function of its argument, and let B_t the evolution of B_0 along the displacement $x = x_t(a)$ then

$$(1.4) \quad \frac{d}{dt}\int_{B_t} f(t,\underline{x})\, d^3x = \int_{B_t} \left(\partial_t f + (u\cdot\nabla)f + f\,\underline{\nabla}\cdot u\right) d^3x,$$

where $\underline{u}$ is the Euler velocity vector field.

Proof:

$$\frac{d}{dt}\int_{B_t} f(t,\underline{x})\, d^3x =$$

$$(1.5) \quad = \lim_{h\to 0}\frac{1}{h}\left[\int_{B_{t+h}} f(t,x_{t+h})\, d^3x_{t+h} - \int_{B_t} f(t,x_t)\, d^3x_t\right],$$

where subscripts $t+h$, t stand to emphasize where the integration is to be taken.

Since we are "moving with the body", we have

$$(1.6) \quad \underline{x}_{t+h} = \underline{x}_t + h\,\underline{u}(t,x_t) + o(h)$$

and, in the limit $h \to 0$, we can use this as a coordinate change.

The Jacobian matrix of the transformation (1.6) is

$$(1.7) \quad J_{ke} = \frac{\partial x^k_{t+h}}{\partial x^e_t} = \delta^{ke} + h\left.\frac{\partial u^k}{\partial x^e}\right|_{x^e_t} + o(h),$$

so that its determinant is

$$\det \delta^k{}_e = 1 + h \sum_k \frac{\partial u^k}{\partial x^k} + o(h).$$

$$\left(= 1 + h \operatorname{Tr} J + o(h) \right)$$

So the LHS of (1.5) becomes

$$(1.8)\quad \lim_{h\to 0} \frac{1}{h} \int_{B_t} \left[f(t+h, x_{t+h})(1 + h\, \nabla\cdot \underline{u} + o(h)) - f(t, \underline{x}_t) \right] d^3x_t.$$

Taylor-expanding $f(t+h, x_{t+h})$ yields, still using (1.6)

$$(1.8) = \lim_{h\to 0} \frac{1}{h} \left(\int_{B_t} \left[\left(f(t, x_t) + h \frac{\partial f}{\partial t} + h\, \underline{u}\cdot\nabla f + o(h) \right)\left(1 + h \nabla\cdot\underline{u} + o(h) \right) + \right.\right.$$
$$\left.\left. - f(t, x_t) \right] d^3x_t \right)$$

Thus we get

$$\frac{d}{dt} \int_{B_t} f(t, x_t)\, d^3x_3 = \int_{B_t} \left(\frac{\partial}{\partial t} f + \underline{u}\cdot\nabla f + f \cdot \nabla\cdot u \right) d^3x_g$$

QED.

<u>Remarks:</u> a) let $f \equiv 1$. Then $\int_{B_t} d^3x_t = \operatorname{Vol}(B_t)$.

Then, using RTT we obtain

$$\frac{d}{dt} \int_{B_t} d^3x_t = \int_{B_t} \nabla\cdot\underline{u}\; d^3x_t$$, i.e the known

formula that the variation of volumes along a vector field $\underline{u}$ are given by the divergence of u.

b) let $f = \rho$ be the density. By definition Cor, rather, by our procedure of "following a particle of the body in its motion",

$$\frac{d}{dt}\int_{B_t} \rho(t,x_t)\, d^3x_t = 0$$ (This is sometimes and somewhere called Lavoisier principle).

Using RTT we have

$$0 = \frac{d}{dt}\int_{B_t} \rho(t,x_t)\, d^3x_t = \int_{B_t} (\rho_t + \underline{u}\cdot\nabla\rho + \rho\,\nabla\cdot\underline{u})\, d^3x_t =$$

$$= \text{by the usual formulas} = \int_{B_t} \rho_t + \nabla\cdot(\rho\underline{u})\, d^3x_t .$$

Since this must hold $\forall\, B_t$ we "localize" this integral formula and recover the mass conservation law (pls see the lectures on the derivation of the D'Alembert equation) as

$$\boxed{\rho_t + \nabla\cdot(\rho\underline{u}) = 0} \tag{1.9}$$

(to be referred to as MCE).

<u>Notational remark</u>:

The quantity $f_t + \underline{u}\cdot\nabla f$ is called Material or convective derivative. It is often denoted by $\frac{Df}{Dt}$. Anyhow, in Cartesian coordinates, it is given by $\partial_t f + \sum_J u^J \partial_J f$.

Notice that $\frac{D}{Dt}$ satisfies the Leibnitz rule, $\frac{D}{Dt} fg = \left(\frac{Df}{Dt}\right) g + f\frac{Dg}{Dt}$.

LEMMA ("per-unit mass lemma").

Let Φ be the "per-unit mass" density of a scalar quantity f, i.e. set

$$F_t(B) := \int_{B_t} f \, d^3x_t = \int \rho \Phi \, d^3x_t .$$

Then:

$$\frac{d}{dt} F_t(B) = \frac{d}{dt} \int_{B_t} \rho \Phi \, d^3x_t = \int_{B_t} \rho \left(\Phi_t + u \cdot \nabla \Phi \right) d^3x_t .$$

<u>Proof:</u> It consists in a suitable regrouping of the terms appearing in the RTT formula (1.4).

We have

$$\frac{d}{dt} \int_{B_t} \rho \Phi \, d^3x_t \overset{RTT}{=} \int_{B_t} \left(\partial_t(\rho\Phi) + u \cdot \nabla(\rho\Phi) + \rho\Phi \, \nabla \cdot u \right) d^3x_t$$

$$= \int_{B_t} \left(\rho_t \Phi + \rho \Phi_t + \rho \, \underline{u} \cdot \nabla \Phi + \Phi \, \underline{u} \cdot \underline{\nabla} \rho + \Phi \rho \, \underline{\nabla} \cdot u \right) d^3x_t$$

$$= \int_{B_t} \Big[\Phi \cdot \underbrace{\left(\rho_t + \underline{u} \cdot \nabla \rho + \rho \, \underline{\nabla} \cdot u \right)}_{0 \text{ by MCE}} + \rho \left(\partial_t \Phi + \underline{u} \cdot \nabla \Phi \right) \Big] d^3x_t =$$

$$= \int_{B_t} \rho \, \frac{D\Phi}{Dt} \, d^3x_t . \qquad \text{QED.}$$

Of great importance is the computation of the time derivative of the linear momentum density (it will be the LHS of the momentum balance equation, i.e the translation in the mechanics of deformable bodies of the II Newton "principle").

We have

$$(1.10)\quad \frac{d}{dt}\int_{B_t} \rho \underline{u}\, d^3x_t \overset{\text{via the lemma}}{=} \int_{B_t} \rho \frac{D\underline{u}}{dt}\, d^3x_t \underset{\text{explicitly}}{=} \int_{B_t} \rho \left(\underline{u}_t + (\underline{u}\cdot\nabla)\underline{u}\right) d^3x_t .$$

For further use, we notice that ($\underline{e}_k$ being a Cartesian basis in $\mathbb{R}^3$),

$$(\underline{u}\cdot\nabla)\underline{u} = \sum_i u^i \partial_i \underline{u} = \sum_{ij} u^i \partial_i u^j \underline{e}_j =$$

$$= \sum_j \left(\sum_i (\partial_i u^j)\cdot u_i\right) \underline{e}_j = \sum_j \left(\sum_i J_{\underline{u}\,ji}\, u_i\right) \underline{e}_j$$

where $J_{\underline{u}}$ is the Jacobian matrix of the velocity vector field.

This somehow ends the kinematical description of the motion of a deformable body. In Section 2 we shall tackle the problem of dynamics.

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Intro to FD - Section 2 -

DYNAMICS.

A) The momentum balance equations.

We have to transplant the Newton's equation for systems of fluid particles

$$\frac{d}{dt}\sum_i p_i = \sum_i \underline{F}_i^{ext}$$

($\underline{F}^{ext}$ being the external forces acting on the system)

We have already computed the rate of variation of the linear momentum density (1.10)

$$\frac{d}{dt}\int_{B_t} \rho\,\underline{u}\, d^3x_t = \int_{B_t} \rho\left(\underline{u}_t + (\underline{u}\cdot\underline{\nabla})\,u\right) d^3x_t \,.$$

We have to equate this quantity to the Force exerted on B_t. This will be divided (as mentioned before) into a

$$(2.1)\quad \underline{F} = \boxed{\int_{B_t} \rho\,\underline{b}\; d^3x_t} + \boxed{\int_{\partial B_t} \tau \cdot dA_t}$$

Body Force (external)

short range force exerted by the "rest" of the fluid on B_t.

$\underline{b}$ in the first summand of (2.1) is the density per unit mass of the body force. E.g. in the case of gravity, $\underline{b} = \underline{g}$ - the gravity acceleration.

What is more subtle is the boundary force τ.

Since we do not have (or do not want) to give a microscopical modellization of such interactions we have to make some "phenomenological" assumptions.

The first one is the so called Cauchy hypothesis, that requires (let us "freeze" time) $\underline{\tau}$ to be a linear function of the normal $\underline{n}_x$ to B_t @ x.
(The Cauchy hypothesis rules out dependence on the curvature or other higher order geometric properties of B_t).
N.B.: we are not considering – in the Cauchy hyp – surface tension phenomena. They, in the case of fluids happen at the boundary, e.g., fluid – air).

As such, since $B_t \subset \Omega_t$ is open, there is no contradiction.

Hence, $\underline{\tau} = \underline{\Pi} \cdot \underline{n}_x$, i.e.

$$\underline{\tau} = \sum_k \tau^k \underline{e}_k = \sum_k \sum_j \Pi^{kj} n_j .$$

Such tensor Π is called "STRESS TENSOR"
(in Italian, "tensore degli sforzi").

Finally, the integrated form of the momentum balance equations are written as

$$\text{(2.2)}\qquad \boxed{\int_{B_t} \rho \frac{Du}{Dt}\, d^3x_t = \int_{B_t} \rho \underline{b}\, d^3x_t + \int_{\partial B_t} \underline{\Pi} \cdot \underline{n}_x \, dA(x_t) .}$$

To "localize" (2.2) we first use Gauss theorem on the boundary force term to write

$$\int_{\partial B_t} \mathbb{T}' \cdot \underline{n}_x \, dA(x_t) = \int_{B_t} (\nabla \cdot \mathbb{T}') \, d^3x_t \quad , \text{where}$$

$\nabla \cdot \mathbb{T}$ is the vector $\nabla \cdot \mathbb{T} = \sum_k \left(\sum_i \partial_i \mathbb{T}^{ki} \right) \underline{e}_k$

this transforms (2.2) into

$$\int_{B_t} \rho \frac{\Delta \underline{u}}{\Delta t} \, d^3x_t = \int_{B_t} \left(\rho \underline{b} + \nabla \cdot \mathbb{T} \right) d^3x_t .$$

Since this holds $\forall$ (regular enough) B_t we arrive at

$$(2.3) \quad \boxed{\rho \frac{\Delta \underline{u}}{\Delta t} = \rho \underline{b} + \nabla \cdot \mathbb{T}} \quad \leftarrow \text{"Cauchy equations".}$$

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The second fundamental equation of the dynamics ($\frac{d}{dt} L^{tot} = M^{ext}_{tot}$) can be analogously transplanted in the continuum setting as follows.

The angular momentum density will be.

$$L_t(B) = \int_{B_t} \underline{x} \wedge \rho \underline{u} \, d^3x_t \qquad , \underline{u} \text{ still being the Euler velocity.}$$

Hence

$$\frac{d L_t(B)}{dt} = \int \rho \frac{\Delta (\underline{x} \wedge \underline{u})}{\Delta t} \, d^3x_t$$

Using Leibnitz rule, $\frac{\Delta (\underline{x} \wedge \underline{u})}{\Delta t} = \frac{\Delta \underline{x}}{\Delta t} \wedge \underline{u} + \underline{x} \wedge \frac{\Delta \underline{u}}{\Delta t}$

Since x is an Eulerian coordinate, $\partial_t x = 0$, $\underline{\nabla}\,\underline{x} = \mathbb{I}$ and so $\frac{D x}{Dt} = \underline{u}$. So the integrand reduces to its second term, i.e.

$$(2.4) \quad \frac{d L_{\epsilon}(B)}{dt} = \int_{B_t} \rho \; \underline{x} \wedge \frac{D\underline{u}}{Dt} \; d^3 x_t .$$

The RHS of the angular momentum balance law is

$$(2.5) \quad \int_{B_t} \underline{x} \wedge \rho \underline{b} + \int_{\partial B_t} \underline{x} \wedge \mathbb{T} \cdot n \; dA$$

Let us manipulate the second term. In coordinates, its k-th component is

$$\sum_{i,j,e} \int_{\partial B_t} \varepsilon^{kij} x^i \mathbb{T}^{je} n_e \; dA$$ (ε^{kij} being the Levi-Civita completely antisymmetric tensor - or symbol).

Using the Gauss thm we have that it equals

$$\sum_{ije} \int_{B_t} \partial_e \left(\varepsilon^{kij} x^i \mathbb{T}^{je} \right) d^3x =$$

$$= \underbrace{\sum_{ije} \int_{B_t} \varepsilon^{kij} x^i \partial_e \mathbb{T}^{je} \, d^3 x}_{\text{this is the } k\text{-th component of } \underline{x} \wedge \underline{\nabla} \cdot \mathbb{T}} + \sum_{ije} \int \varepsilon^{kij} \delta_{ie} \mathbb{T}^{je} d^3 x .$$

So using (2.4) and (2.5) we write the angular momentum balance law as

$$\int_{B_t} x \wedge \rho \frac{Du}{Dt} \, d^3 x = \int_{B_t} x \wedge \left(\rho \underline{b} + \underline{\nabla} \cdot \mathbb{T} \right) + \sum_k \underline{e}_k \int_{B_t} \sum_{ij} \varepsilon^{kij} \mathbb{T}^{ji} d^3 x .$$

i.e

(2.6) $$\int_{B_t} \underline{x} \wedge \Big[\underbrace{\rho \frac{D\underline{u}}{Dt} - \rho \underline{b} - \underline{\nabla} \cdot \Pi}_{0} \Big] d^3x = \sum_k \underline{e}_k \int_{B_t} \sum_{ij} \varepsilon^{kij} \Pi^{ji} \, d^3x$$

thanks to Cauchy's eq.

$\Rightarrow$ For $k = 1,2,3$ and $\forall B_t$ $\displaystyle\int_{B_t} \sum_{ij} \varepsilon^{kij} \Pi^{ji} \, d^3x = 0 \Rightarrow$

$\Pi^{ij} = \Pi^{ji}$, i.e Π is a **SYMMETRIC** tensor.

Summing up, so far we have the system

2.7 $$\begin{cases} \rho_t + \nabla(\rho \underline{u}) = 0 \\ \rho \dfrac{D\underline{u}}{Dt} = \rho \underline{b} + \underline{\nabla} \cdot \Pi \\ \Pi^t = \Pi \end{cases}$$

Supposing $\underline{b}$ given, so far this is undetermined. Indeed we have 1 degree of freedom for ρ, 3 for $\underline{u}$ and 6 (using the last algebraic relation) for the stress tensor Π. MCE and the Cauchy eq. are 4 equations.

To try and close the system one has to make suitable assumptions ("Ansätze") on

1) the form and functional dependence of Π

2) Relations between "thermodynamical quantities".

Assumptions 1) are called "constitutive equations"

2) "Equations of state".

A fluid can be defined as a continuum whose stress tensor is proportional to the identity when $\underline{u}=0$ (static configuration)

This means that the constitutive relation is

$$\mathbb{T} = -p\,\mathbb{I}d + \mathbb{T}^{dyn} .$$ p is the pressure, whence the minus sign in the relation.

Notice that $\mathbb{T} = -p\,\mathbb{I}d \Rightarrow$

$$\underline{\tau} = -p\,\underline{n} \quad \text{is normal to } B_t .$$

So, a continuum "is" a fluid if it cannot admit shear stresses <u>at rest</u>.

<u>Definition</u>: A fluid is an Euler (or "ideal") fluid if $\mathbb{T}$ is <u>always</u> proportional to the identity.

Writing $\mathbb{T} = -p(x,t)\,\mathbb{I}d$, it is immediate to see that $\nabla\cdot\mathbb{T} = -\nabla p$. Hence the fundamental system (2.7) becomes

$$\begin{cases} \rho_t + \nabla\cdot(\rho\underline{u}) = 0 \\ u_t + (\underline{u}\cdot\nabla)\,\underline{u} = -\dfrac{\nabla p}{\rho} + \underline{b} \end{cases}$$

(4 equations in 5 variables, $\rho, \underline{u}, p$).

To close the system one introduces an equation of state.

A fluid is called BAROTROPIC if such a relation can be written as $p = p_r(\rho)$ thus yielding the closed system (<u>gas dynamics</u>).

$$(2.8)\quad \begin{cases} \rho_t + \nabla\cdot(\rho\underline{u}) = 0 \\ u_t + (\underline{u}\cdot\nabla)\,\underline{u} = -p_r'(\rho)\,\dfrac{\nabla\rho}{\rho} + b . \end{cases}$$

(See the lecture on sound waves).

SECTION 3: INCOMPRESSIBILTY

Definition : The incompressibility regime for a fluid is the one in which the volume of any subportion of the fluid is constant. Hence, the characteristic equation is $\nabla \cdot \underline{u} = 0$ and so (2.8) closes as

$$
(2.8) \begin{cases} \rho_t + \underline{u} \cdot \nabla \rho = 0 \\ \nabla \cdot \underline{u} = 0 \\ u_t + \underline{u} \cdot \nabla u = -\dfrac{\nabla p}{\rho} + b \end{cases}
$$

1) As we shall see in the next page, the assumption of incompressibility is consistent for fluid motions with characteristic speeds much less than the speed of sound.

2) A particular case of "inc. fluids" is the case of homogeneous fluids, namely $\rho = \rho_0$.

3) While in gas dynamics p is a "thermodynamical quantity", in the incompressible regime p becomes a "mechanical quantity" and satisfies an "elliptic equation" with source depending on $\underline{u}$ and ρ. Indeed, take the divergence of the last of (2.8), i.e

$$
\underbrace{\partial_t (\nabla \cdot \underline{u})}_{=0} + \nabla \cdot (\underline{u} \cdot \nabla \underline{u}) = -\nabla \cdot \left(\frac{\nabla p}{\rho} \right) + \nabla \cdot b
$$

$$
= -\frac{\Delta p}{\rho} + \frac{1}{\rho^2} \nabla p \cdot \nabla \rho + \nabla \cdot b .
$$

In the homogeneous case ($\rho = \rho_0$) this simplifies to

$$
-\Delta p = \rho_0 \left[\nabla \cdot (u \cdot \nabla u) - \nabla \cdot b \right] .
$$

Let us discuss now point 1) of the previous page.

We have seen that, for a perfect fluid, the linearization around $u=0$, $\rho=\rho_0$ (constant) of the Euler equations is (with $u=\varepsilon v$, $\rho=\rho_0+\varepsilon\tilde\rho$)

$$(2.10)\quad \begin{cases} \tilde\rho_t + \rho_0 \nabla\cdot v = 0 \\ \rho_0 v_t + p'(\rho_0)\nabla\tilde\rho = 0 \end{cases} \qquad \text{with } p=p(\rho)$$

Let us remind that to obtain the D'Alembert equation for $\tilde\rho$ one takes ∂_t of the first one and the divergence of the second of (2.10) to get

$$\begin{cases} \tilde\rho_{tt} + \rho_0(\nabla\cdot v)_t = 0 \\ \rho_0(\nabla\cdot v)_t + p'(\rho_0)\Delta\tilde\rho = 0 \end{cases}$$

and then, by subtraction

$$\tilde\rho_{tt} - p'(\rho_0)\Delta\tilde\rho = 0$$

Thus $p'(\rho_0) = c_s^2$, the square of the speed of sound.

If, instead, we take the laplacian of the first of (2.10) we get

$$(\bullet)\quad (\Delta\tilde\rho)_t + \rho_0\Delta\cdot(\nabla\cdot v) = 0$$

hence, taking ∂_t of the divergence of the second in 2.10

$$(\bullet\bullet)\quad \rho_0(\nabla\cdot v)_{tt} + p'(\rho_0)(\Delta\tilde\rho)_t = 0$$

so $(\Delta\tilde\rho)_t = -\dfrac{\rho_0}{p'(\rho_0)}(\nabla\cdot v)_{tt}$ and, substituting in $(\bullet)$

we get

$$\rho_0 \left[-(\underline{\nabla}\cdot\underline{v})_{tt} + p'(\rho_0)\,\Delta\cdot(\underline{\nabla}\cdot\underline{v}) \right] = 0$$

which shows that $\underline{\nabla}\cdot\underline{v}$ satisfies the same equation of $\tilde{\rho}$, i.e., perturbations of $\underline{\nabla}\cdot\underline{u}$ $(= \varepsilon\,\underline{\nabla}\cdot\underline{v})$ propagate with the speed of sound.

Now, suppose we want to describe phenomena with a characteristic speed U. Fix an (arbitrary) length scale L $\Rightarrow$ L/U is a time scale.

Consider the D'Alembert equation for $\underline{\nabla}\cdot\underline{v}$

(2.11) $$(\nabla\cdot\underline{v})_{tt} - c_s^2\,\Delta(\underline{\nabla}\cdot\underline{v}) = 0$$

and set

$$x = L x^* \quad \Rightarrow \quad \partial_{x_i} = \frac{1}{L}\,\partial_{x_i^*}$$

$$t = \frac{L}{U} t^* \quad \Rightarrow \quad \partial_t = \frac{U}{L}\,\partial_{t^*} .$$

(also, $v = U v^*$)

In the starred variables (2.11) becomes

$$\frac{U^2}{L^2}(\nabla\cdot v^*)_{t^*t^*}\cdot\frac{U}{L} - c_s^2\,\frac{U}{L^2}\,\Delta^*(\underline{\nabla}^*\cdot\underline{v}^*)\cdot\frac{1}{L} = 0$$

simplifying and dividing by c_s^2 we get

$$\frac{U^2}{c_s^2}(\nabla\cdot v^*)_{t^*t^*} - \Delta^*(\underline{\nabla}^*\cdot v^*) = 0 .$$

For C^2 functions we see that in the "asymptotic limit" $\frac{U}{c_s} \to 0$ $\underline{\nabla}^*\cdot\underline{v}^*$ will be a harmonic function. With suitable boundary conditions, we can get the incompressibility regime as $\underline{\nabla}^*\cdot\underline{v}^* = 0$.

Section 4 : Homogeneous Euler Fluids.

Let us consider the Euler equations for a Homog. fluid

$$(4.1)\quad \begin{cases} \underline{\nabla}\cdot\underline{v} = 0 \\ \underline{v}_t + \underline{v}\cdot(\underline{\nabla}\cdot\underline{v}) = -\dfrac{\nabla p}{\rho_0} + b \end{cases}$$

whose natural b.c. are $\underline{v}\cdot\underline{n} = 0$ @ physical boundary ("no flux through the walls.").

If J_v denotes the Jacobian of the velocity,

$(J_v)_{ij} = \partial_j v_i$ we notice that the second of 4.1 can be written as

$$\partial_t v^k + \sum_e (J_v)_{ke} v_e = -\partial_e p/\rho_0 .$$

Adding and subtracting $\underline{J_v}^T\cdot\underline{v}$ (in components,

$\sum_e (J_v)_{ek}\, v_e$ we can write it as

$$\partial_t v_k + \sum_e v_e \underbrace{(\partial_e v_k - \partial_k v_e)}_{\substack{\shortparallel \\ (\underline{\nabla}\times\underline{v})\wedge\underline{v}}} + \underbrace{\sum_e v_e \partial_e v_k}_{\substack{\shortparallel \\ \nabla \frac{1}{2}\|v\|^2}} = -\partial_e p/\rho_0 .$$

Reverting to the vector notation, with $\underline{\omega} = \underline{\nabla}\times\underline{v}$ we arrive at the "Lamb form" of the Euler equation, viz.

$$(4.2)\qquad \partial_t\underline{v} + \underline{\nabla}\frac{\|\underline{v}\|^2}{2} + \underline{\omega}\wedge\underline{v} = -\frac{\nabla p}{\rho} + \underline{b} .$$

when $\underline{b}$ admits a potential, $\underline{b} = -\nabla \phi_b$ we have

$$(4.3) \quad \partial_t \underline{v} + \nabla \frac{\|v\|^2}{2} + \omega \wedge \underline{v} = -\nabla \left(p/\rho_0 + \phi_b \right).$$

1) Bernoulli's thm:

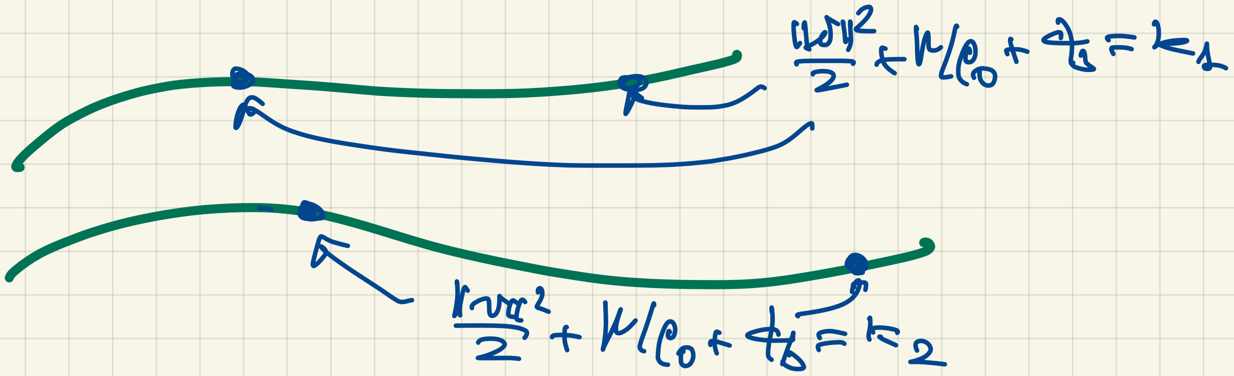
Let $\underline{v}$ be stationary, i.e., $v_t = 0$ $\Rightarrow$ (4.3) becomes

$$(4.4) \quad \nabla \left(\frac{\|\underline{v}\|^2}{2} + p/\rho_0 + \phi_b \right) = \underline{v} \wedge \underline{\omega}$$

so that, in the generic case,

$$\underline{v} \cdot \nabla \left(\frac{\|v\|^2}{2} + p/\rho_0 + \phi_3 \right) = 0$$

, i.e the quantity (energy per unit mass) $\frac{\|v\|^2}{2} + p/\rho_0 + \phi_b$ is CONSTANT ALONG THE STREAMLINES



While, in the special case $\underline{\omega} = 0$,

$\frac{1}{2}\|v\|^2 + p/\rho_0 + \phi_b$ is CONSTANT ON EVERY CONNECTED COMPONENT OF THE FLUID'S DOMAIN.

The constancy of the energy density on streamlines (and the special case for $\underline{\omega}=0$) are BERNOUILLI'S THM.

Still in the case of a homogeneous Euler fluid with body force density $\underline{b}$ admitting a potential, Lamb's form leads to the so-called Helmholtz system.

Consider ($\omega = \nabla \times v$) the equations

$$
(4.5)\quad \begin{cases} \nabla \cdot \underline{v} = 0 \\ v_t + \underline{\omega} \wedge \underline{v} = -\nabla\left(\frac{\|v\|^2}{2} + \mathcal{M}(\rho_0 + \phi_b) \right) \\ \nabla \cdot \underline{\omega} = 0 \end{cases}
$$

Taking the curl of the second equation we have

$$
\underline{\omega}_t + \nabla \wedge (\underline{\omega} \wedge \underline{v}) = 0
$$

Now, using the general vector identity

$$
(4.6)\quad \nabla \wedge (\underline{\omega} \wedge \underline{v}) = (\underline{v} \cdot \nabla)\underline{\omega} - (\omega \cdot \nabla)\underline{v} + (\nabla \cdot \underline{v}) \cdot \underline{\omega} - (\nabla \cdot \underline{\omega})\, \underline{v}
$$

we arrive, taking into account the equations $\nabla \cdot \underline{v} = \nabla \cdot \underline{\omega} = 0$,

$$
(4.7)\quad \begin{cases} \underline{\omega} = \nabla \times \underline{v} \\ \nabla \cdot \underline{v} = 0 \\ \nabla \cdot \underline{\omega} = 0 \\ \underline{\omega}_t + (\underline{v} \cdot \nabla)\,\underline{\omega} = -(\underline{\omega} \cdot \nabla)\,\underline{v} \end{cases}
$$

known as Helmholtz formulation (or equations).

In the sequel we shall use the fact that in 2D flows ($\underline{v} = (v_x(x,y,t), v_y(x,y,t), 0)$) ω is $(0, 0, \omega_z = \partial_x v_y - \partial_y v_x)$, and so the last of (4.7) say that 2D vorticity is advected:

$$
\omega^{(2)}_t + (\underline{v} \cdot \nabla)\,\omega^{(2)} = 0
$$

, i.e its material derivative vanish.

Section 5. The Kutta-Joukowski theory of airfoil. ("2D wing")

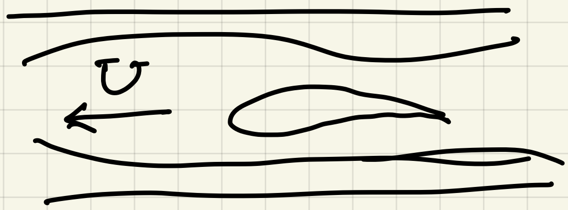
Background hypotheses:

a) The airfoil is infinitely long ⇒ from a 3D pb to a 2D pb.

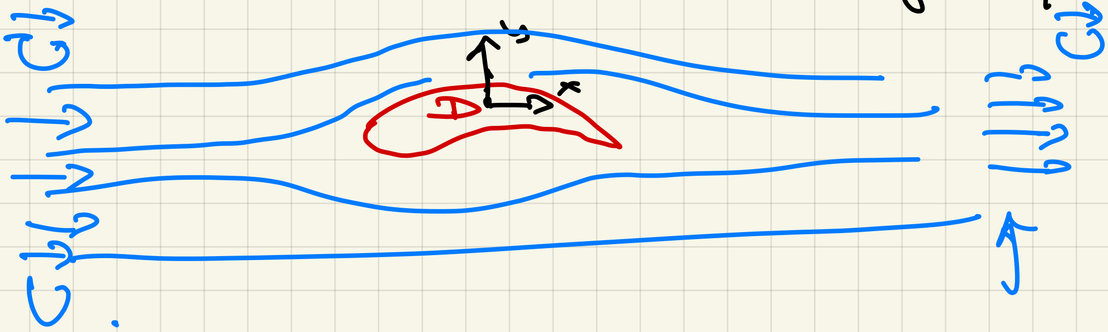
b) We shall set our study in the regime $V/c_s \ll 1$ so that air will be an "incompressible" homogeneous fluid

c) We shall study the stationary problem of a steady velocity $U = U_x$.

d) We shall not include gravity in the picture



Step (1): we shall use the wing's reference frame.



Call "downstream" the "region" $x \to +\infty$, $y \in \mathbb{R}$ and upstream the region $x \to -\infty$ $y \in \mathbb{R}$.
The wing (obstacle) occupies a fixed region D in $\mathbb{R}^2$. So the problem is

$$(5.1)\begin{cases} \underline{\nabla}\cdot\underline{v} = 0 \\ \underline{v}\cdot\underline{\nabla}(\underline{v}) = -\nabla p/\rho_0 \end{cases}$$ (gravity neglected)

with the following b.c.,

BC1) $\lim_{\sqrt{x^2+y^2}\to\infty} v(x,y) = U\cdot\underline{e}_1 \quad \forall y\in\mathbb{R}.$

"constant velocity" far away from the obstacle.

BC2) No-flux through the surface of D, i.e

$$\lim_{(x,y)\to\partial D} \underline{v}\cdot\underline{n} = 0$$

BASIC PROPERTY: since $v\to U\cdot\underline{e}_1$ for $x\to\pm\infty$ $\omega=\nabla\times\underline{v}\to 0$ $x\to\pm\infty$. From the Helmholtz eqn's (sec 4) we know that vorticity is advected in 2D $\Rightarrow \omega=0$ everywhere.

Hence, system (5.1) becomes

$$(5.2)\begin{cases} \boxed{\begin{matrix}\underline{\nabla}\cdot\underline{v} = 0 \\ \underline{\nabla}\times\underline{v} = 0\end{matrix} \quad \text{linear problem !}} \\ (\underline{v}\cdot\underline{\nabla})\underline{v} = -\nabla p/\rho_0 \end{cases}$$

So the strategy is to solve the linear part of (5.2) supplemented by BC1 and BC2, and then use the last of 5.2 to determine the pressure p.
Notice that instead of the equation $(\underline{v}\cdot\nabla v) + \nabla p/\rho_0 = 0$ we can use its "integrated form" coming from

Bernoulli's theorem,

$\frac{1}{2}\|v\|^2 + p/\rho_0 = K$, since $\omega = 0$ implies that K is a global constant.

In particular we shall be able to compute the force exerted on the wing

$$F = F_x \underline{e}_1 + F_y \underline{e}_2 = \underbrace{\left(-\rho_0 \oint_{\partial D} \partial_x p \, dl\right)}_{\text{DRAG}} \underline{e}_1 - \underbrace{\left(\rho_0 \oint_{\partial D} \partial_y p \, dl\right)}_{\text{LIFT}} \underline{e}_2$$

The Kutta-Joukowski strategy is then divided in three steps:

Step 1: We solve the problem in the simplest possible geometry, namely D = disk ("round wing")

Step 2: We transform the problem into a problem in complex geometry

Step 3: we use a suitable conformal transformation to obtain an airfoil like obstacle.

Remark: No uniqueness in point 1. Uniqueness will be enforced in step 3 by means of a regularity requirement.

K-J step 1: Flow around a disk

The problem is to find a vector field $\underline{v}$ s.t.

$$(5.3)\begin{cases} \nabla \cdot \underline{v} = 0 \\ \nabla \wedge \underline{v} = 0 \\ \lim_{r\to\infty} \underline{v} = v_0 = U \cdot \underline{e}_1 \\ \underline{v} \cdot \underline{e}_r = 0 \quad @\ r = a \end{cases}$$

(a is the radius of the disk).

Let us consider the equation $\nabla \wedge \underline{v} = 0$. If we think to $\underline{v}$ as a 1-form $v_x dx + v_y dy$ on the domain $\mathbb{R}^2 \setminus D$, then $\underline{v}$ is closed.

The point is that the topology of $\mathbb{R}^2 \setminus D$ is non trivial. Its homology group is generated by the one-form

$$\alpha = \frac{1}{x^2+y^2}(-y\,dx + x\,dy) = d\theta \quad , \theta \text{ being the}$$

angular coordinate. The corresponding vector field will be $\underline{v}_T = \frac{\underline{e}_\theta}{r}$.

Notice that $\underline{v}_T$ satisfies BC2 and vanishes at $r\to\infty$.

$\underline{v}_T$ is the only topological term existing on our problem.

So we can move to the "potential" part of the problem, where we write $\underline{v} = \nabla \tilde{\Phi}$.

In such a case, (5.3) becomes

$$\begin{cases} \Delta \tilde{\phi} = 0 & \text{on } \mathbb{R}^2 - D \\ \lim_{r\to\infty} \nabla\tilde{\phi} = U \cdot \underline{e}_x \\ \dfrac{\partial \tilde{\phi}}{\partial r} = 0 & r = a \end{cases}$$

We notice that the condition at $r \to \infty$ can be described by $\tilde{\phi}_\infty = U \cdot x = U \cdot r\cos\theta$

Writing $\tilde{\phi} = \tilde{\phi}_\infty + \Phi$, we see that ϕ must satisfy

$$(5.4)\begin{cases} \Delta\phi = 0 & \text{in } \mathbb{R}^2 - D \\ \nabla\phi \to 0 & r\to\infty \\ \left.\dfrac{\partial\phi}{\partial r}\right|_{r=a} = -U\cos\theta & \left(\text{i.e. } \left.\dfrac{\partial\tilde{\phi}}{\partial r}\right|_{r=a} = 0\right) \end{cases}$$

In the first part of the course we studied a similar problem (but looked for a solution inside the disk).

By writing the Laplacian in polar coordinates

$$\Delta\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}$$

and looking for periodic functions, we found solutions

$$f_n(r,\theta) = r^n(\alpha_n\cos n\theta + \beta_n \sin n\theta), \quad n\in\mathbb{Z}.$$

Since we want solutions going to zero for $r\to+\infty$, we have to consider the case $n<0$, i.e expand $\tilde{\Phi}$ as

$$\phi(r,\theta) = \sum_{n\geq 1} \frac{1}{r^n}\left(\alpha_n \cos n\theta + \beta_n \sin n\theta\right)$$

now $\displaystyle \frac{\partial \phi}{\partial r} = -\sum_{n\geq 1} \frac{n}{r^{n+1}} \left(\alpha_n \cos n\theta + \beta_n \sin n\theta\right)$

and hence $\displaystyle \frac{\partial \phi}{\partial r}\Big|_{r=a} = -\sum_{n\geq 1} \frac{n}{a^{n+1}} \left(\alpha_n \cos n\theta + \beta_n \sin n\theta\right)$

Thus the equation $\displaystyle \frac{\partial \phi}{\partial r}\Big|_{r=a} = -U\cos\theta$ fixes

$\beta_n = 0 \ \forall n$

$\alpha_n = 0$, $n>1$, $\alpha_1 = a^2 U$. This yields the solution to 5.4 to be

$$\phi(r,\theta) = U a^2 \frac{\cos\theta}{r},$$

and so the solution to the potential problem (5.3) $\tilde{\Phi}$ to be

$$\tilde{\Phi} = U\cos\theta\left(r + \frac{a^2}{r}\right).$$

Finally, $\forall k \in \mathbb{R}$, the velocity field is $v = \nabla\tilde{\Phi} + k\,d\chi$ i.e

$$(5.5)\quad \underline{v} = \underbrace{U\cos\theta\left(1-\frac{a^2}{r^2}\right)}_{\frac{\partial \tilde{\Phi}}{\partial r}}\underline{e}_r - \Big(\underbrace{U\sin\theta\left(1+\frac{a^2}{r^2}\right)}_{\frac{1}{r}\frac{\partial \tilde{\Phi}}{\partial \theta}} + \frac{\Gamma}{2\pi}\frac{1}{r}\Big)\, e_\theta$$

v_T with $k = -\frac{\Gamma}{2\pi}$

Computation of the force.

According to Bernoulli's Theorem we have

$$\rho \frac{\|v\|^2}{2} + p = \tilde{k} \quad \text{(constant)},$$ so that

$p = -\rho \frac{\|v\|^2}{2} + \tilde{k}$. On the boundary of the disk we get ($v_r = 0$ @ $r = a$ by the BC2)

$$\|v\|^2 = v_\theta^2 = \left(2U \sin\theta + \frac{\Gamma}{2\pi a}\right)^2. \quad (5.7)$$

On ∂D the normal $\underline{n}$ is $\underline{n} = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2$, $ds = a\, d\theta$ and so, writing $F = F_x \underline{e}_1 + F_y \underline{e}_2$ since $p = -\frac{\rho}{2}\|v\|^2 + \tilde{k}$ we have ($\underline{F} = -\int_{\partial D} p\, \underline{n}\, dl$)

$$F_x = \int_0^{2\pi} a\, d\theta \cdot \left[\rho \frac{v_\theta^2}{2} - \tilde{k}\right] \cdot \cos\theta =$$

$$= a\rho \int_0^{2\pi} \left(2U^2 \sin^2\theta + \frac{U\Gamma}{2\pi a} \sin\theta + \frac{\Gamma^2}{8\pi^2 a^2} - \frac{\tilde{k}}{\rho}\right) \cos\theta$$

$$= 0$$

while

$$F_y = \rho_0 \int_0^{2\pi} a\, d\theta \cdot \left[2U^2 \sin^2\theta + \frac{U\Gamma}{2\pi a} + \frac{\Gamma^2}{8\pi^2 a^2} - \frac{\tilde{k}}{\rho}\right] \sin\theta =$$

$$= \frac{\rho U \Gamma}{\pi} \int_0^{2\pi} \sin^2\theta\, d\theta = \rho U \Gamma \quad \left(\neq 0 \text{ if } \Gamma \neq 0\right).$$

Remark: According to the choice of signs we made (see 5.5) when Γ is positive, the "topological" summand of the velocity vector field is directed as $-\underline{e}_\theta$, i.e., clockwise.

When $\Gamma = 0$ we have

$$\underline{v} = U\cos\theta\left(1-\frac{a^2}{r^2}\right)\underline{e}_r - U\sin\theta\left(1+\frac{a^2}{r^2}\right)\underline{e}_\theta$$

The phase portrait of such a vector field is (red lines)

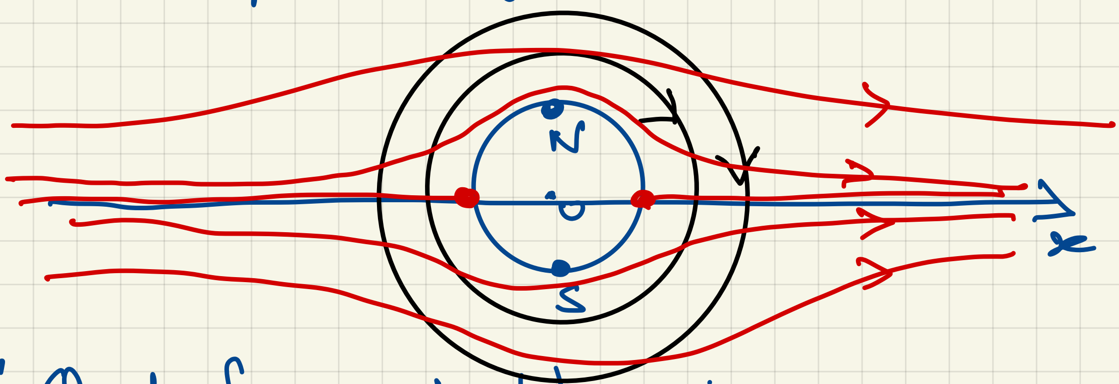


Fig 1.

(Check for exercise!).

The topological component is in black (for $\Gamma > 0$).

If N and S are the "north and south poles" of the disk, $\|v\|_N > \|v\|_S \Rightarrow$ by Bernoulli,

$p_N < p_S \Rightarrow$ there will be a force of lift in the vertical direction. This matches our last formula.

For steps 2 and 3 we refer the interested reader to Childress book or Acheson's book, Ch. 4.

Section 6: Water Waves: derivation of the equations.

Let us consider the Euler form of the equations for a homogeneous fluid in the gravitational field (set $\rho_0=1$)

$$(6.1)\quad v_t + \frac{1}{2}\nabla \|v\|^2 + \underline{\omega}\wedge \underline{v} = -\nabla p - g e_3$$

and suppose[1] $\underline{\omega}=0$, so that, assuming the domain be simply connected, $v=\nabla\phi$.

(6.1) becomes

$$(\nabla\phi)_t + \frac{1}{2}\nabla \cdot \|\nabla\phi\|^2 = -\nabla(p+gz)$$

$$\overset{\shortparallel}{\nabla(\phi_t)} \quad \Rightarrow \text{ "integrating w.r.t. } x\text{" we get}$$

$$(6.2)\quad \phi_t + \frac{1}{2}\|\nabla\phi\|^2 + gz = -(p+p_0)$$

where we called p_0 the integration constant.

So we get (since $\nabla\cdot\underline{v}=0$) the system

$$(6.3)\quad \begin{cases} \Delta\phi = 0 \\ \phi_t + \frac{1}{2}\|\nabla\phi\|^2 + gz = p - p_0 \end{cases}$$

---

1) It can be proven that in the Euler case, $\underline{\omega}(x,0)=0 \Rightarrow \omega(x,t)=0$.

The full Water-Wave problem consists in studying (6.2) "in a domain which varies in space & time", that is in a domain of the form:

$$(x,y) \in \mathbb{R}^2 \quad , \quad \underset{\substack{\shortparallel \\ \text{fixed bottom}}}{g(x,y)} \leq z \leq \underset{\substack{\shortparallel \\ \text{moving air-water} \\ \text{interface, i.e.} \\ \text{"wave profile"}}}{\eta(x,y,t)}$$

To this end we have to discuss the boundary conditions.

1) Kinematic boundary conditions translate the concept that, by the very definition of boundary, fluid particles cannot cross it.

If $\underline{x}(t)$ is the trajectory of a point particle on the boundary of the domain, and if the part of boundary is given by an equation

$f(\underline{x},t) = 0 \quad \Longrightarrow$

$0 = \frac{d}{dt} f(\underline{x},t) = \dot{\underline{x}} \cdot \nabla f + f_t \quad \Rightarrow (\dot{\underline{x}} = \underline{v}) \Longrightarrow$

$$\boxed{f_t + \underline{v} \cdot \nabla f = 0} \, .$$

If the bottom is given by $z - g(x,y) = 0$ we have

$v_3 - (v_1 \partial_x g + v_2 \partial_y g) = 0 \qquad \Longrightarrow \underline{v} = \nabla \phi \quad \Longrightarrow$

$$\phi_z - \phi_x g_x - \phi_y g_y = 0 \qquad @\ z = g(x,y)$$

In the case of a flat bottom $(g(x,y) = h_0)$, which is the one we shall stick to later, we simply have

(6.3) $\phi_z = 0 \qquad @\ z = h_0$

There is an analogous condition on the air-water interface $z = \eta(x,y,t)$ which reads (here $\eta = \eta(x,y,t)$)

(6.4)
$$\phi_t + \phi_x \eta_x + \phi_y \eta_y = \phi_z$$

We have also a further BC which comes from requiring that the pressure be continuous [we are assuming no surface tension(2)]. In the air-water case we have $\rho_{air} \simeq 0 \implies p_{air} = \text{constant} = p_0$.

[Exercise: why?]

So, substituting in the second of (6.3) we get

(6.5) $$\phi_t + \frac{\|\nabla\phi\|^2}{2} + gz = 0 \qquad @\ z = \eta(x,y,t)$$

This is called <u>Dynamical</u> boundary condition.

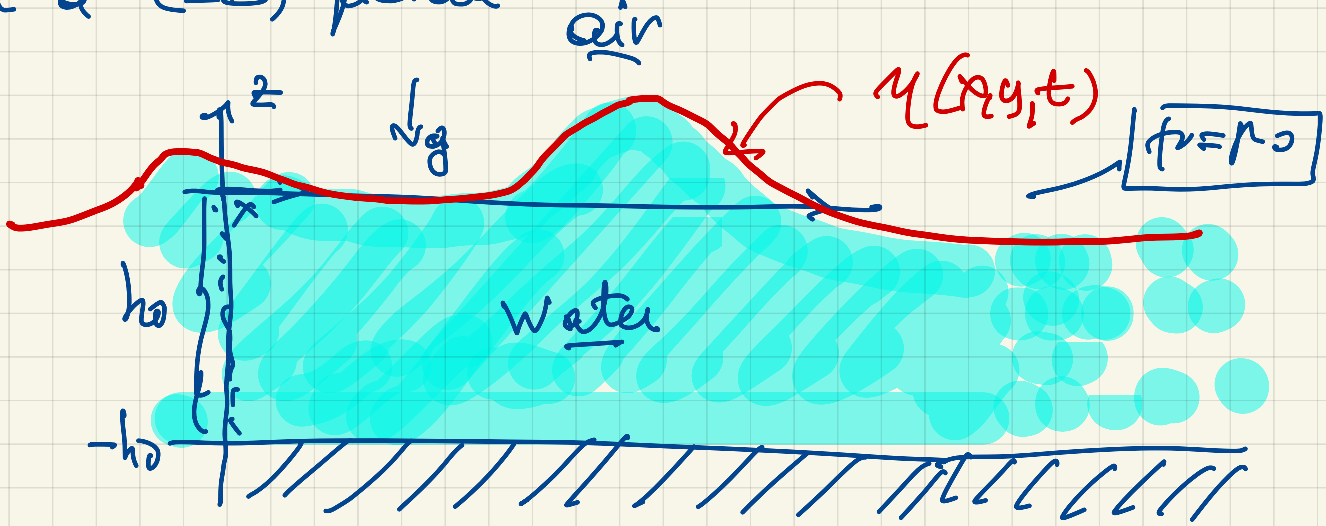
---

(2) The case of non-vanishing surface tension can be treated as well - See Salsa's book, Ch. 5.10

Summing up we are facing the following problem (water wave problem with flat bottom):

$$(6.6)\quad \begin{cases} \Delta\phi = 0 & \text{in } \underbrace{x\in\mathbb{R}^3,\ t\in(0,T)} \\ \phi_t+\frac{1}{2}\|\nabla\phi\|^2+gz = p-p_0 & \text{"} \\ \eta_t+\phi_x\eta_x+\phi_y\eta_y=\phi_z & \Big\} \ x,y\in\mathbb{R}^2,\ z=\eta(x,y,t), \\ \phi_t+\frac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2)+gz=0 & \quad t\in(0,T) \\ \phi_z=0 & x,y\in\mathbb{R}^2,\ z=-h_0,\ t\in(0,T). \end{cases}$$

In a (2D) picture



The general strategy is to solve 6.6-1, 6.6-2, 6.6-3 and 6.6-4 for ϕ and η, and then use 6.6-4 to compute $p-p_0$. So we shall consider (easing a bit the notation)

$$(6.7):\quad \begin{cases} \Delta\phi=0 & -h_0<z<\eta \\ \phi_z=0 & @\ z=-h_0 \qquad \text{BKBC} \\ \eta_t+\phi_x\eta_x+\phi_y\eta_y=\phi_z & @\ z=\eta \quad \text{TKBC} \\ \phi_t+\frac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2)+g\eta=0 & @\ z=\eta,\ \text{TDBC} \end{cases}$$

Section 7: Water Waves: Linear theory.

Consider the solution to (6.7) $\phi = k$, $\eta = 0$

and perturb it: $\phi = k + \varepsilon\varphi + O(\varepsilon)$ $\quad \eta = \varepsilon\xi + O(\varepsilon)$

we get, @ $O(\varepsilon)$:

$$\begin{cases} \Delta\varphi = 0 & 0 < z < \varepsilon\xi \\ \varphi_z = 0 & @\ z = -h_0 \\ \xi_t - \varphi_z = 0 & @\ z = \varepsilon\xi \\ \varphi_t + g\xi = 0 & @\ z = \varepsilon\xi \end{cases}$$

Assuming regularity of φ we notice that the BC @ $z = \varepsilon\xi$ give, restoring ε

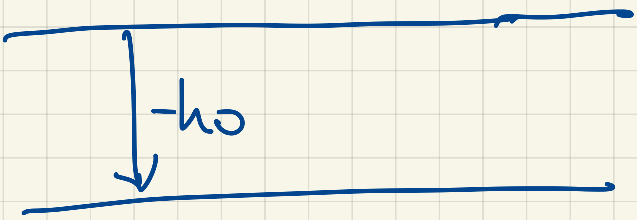
$$\begin{cases} \varepsilon\xi_t(x,y,t) - \varepsilon\varphi_z(x,y,\varepsilon\xi,t) = 0 \\ \varepsilon\varphi_t(x,y,\varepsilon\xi,t) + g\xi(x,y,t) = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \varepsilon\xi_t(x,y,t) - \varepsilon\varphi_z(x,y,0,t) - \varepsilon^2\varphi_{zz}(x,y,0,t)\cdot\xi + \dots = 0 \\ \varepsilon\varphi_t(x,y,0,t) + \varepsilon^2\varphi_{tz}(x,y,0,t)\,\xi + \dots + g\varepsilon\xi(x,y,t) = 0 \end{cases}$$

So that, @ $O(\varepsilon)$ we can "square up the domain" and write the problem for the linearized WW eqns as

$$\begin{cases} \Delta\varphi = 0 & -h_0 < z < 0 \\ \varphi_z = 0 & @\ z = -h_0 \\ \xi_t - \varphi_z = 0 & @\ z = 0 \\ \varphi_t + g\eta = 0 & @\ z = 0 \end{cases}$$



Let us shorten the discussion a bit and search for a plane wave solution: set $\underline{x} = (x, y)$, $\underline{k} = (k_1, k_2)$ and write the Ansatz

$$\begin{cases} \zeta = A \cos(\underline{k} \cdot \underline{x} - \omega t + \vartheta_0) \\ \varphi = y(z) \sin(\underline{k} \cdot \underline{x} - \omega t + \vartheta_0) \end{cases} \qquad (7.8) .$$

Let us first consider the φ-subsystem

$$\begin{cases} \Delta\varphi = 0 & -h_0 < z < 0 \\ \varphi_z = 0 & @\ z = -h_0 \end{cases}$$

this yields (setting $\psi := \underline{k} \cdot \underline{x} - \omega t + \varphi_0$

$$y''(z) \sin(\psi) - \underbrace{(k_1^2 + k_2^2)}_{\varkappa^2} \sin(\psi) = 0$$

$$y'(z) = 0 \qquad @\ z = -h_0$$

Simplifying the term $\sin\psi$ we have the system

$$\begin{cases} y'' = \varkappa^2 y & (\text{harmonic repulsor}) \\ y'(-h_0) = 0 \end{cases}$$

Let us write the general solution to the first one as

$$y(z) = B \operatorname{Ch}(\varkappa(z+h_0)) + C \operatorname{Sh}(\varkappa(z+h_0))$$

$$\Rightarrow y' = \varkappa B \operatorname{Sh}(\varkappa(z+h_0)) + \varkappa C \operatorname{Ch}(\varkappa(z+h_0))$$

$\Rightarrow y'(-h_0) = 0$ yields $C = 0$ so that

$y = B\,\mathrm{Ch}(\varkappa(z+h_0))$, and hence

- $\varphi = B\,\mathrm{Ch}(\varkappa(z+h_0))\sin(\underline{k}\cdot\underline{x}-\omega t+\vartheta_0)$

with ξ given by 6.8,

$(\cdot\cdot)\ \xi = A\cos(\underline{k}\cdot\underline{x}-\omega t+\vartheta_0)$ $\qquad (\underline{k}\cdot\underline{x}-\omega t+\vartheta_0 := \psi)$

let us now use the BC @ the interface and substitute $(\cdot)$ and $(\cdot\cdot)$ in

$$\begin{cases} \xi_t - \varphi_z = 0 & (@\ z=0) \\ \varphi_t + g\xi = 0 & (@\ z=0) \end{cases}$$

$$\begin{cases} A\omega\sin\psi - B\varkappa\,\mathrm{Sh}(\varkappa h_0)\sin\psi = 0 \\ gA\cos\psi - B\omega\,\mathrm{Ch}(\varkappa h_0)\cos\psi = 0 \end{cases}$$

simplifying we get the linear system in (A,B)

$$\begin{bmatrix} \omega & -\varkappa\,\mathrm{Sh}(\varkappa h_0) \\ g & -\omega\,\mathrm{Ch}(\varkappa h_0) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we want non-zero solutions $\Rightarrow$

$$\det\begin{pmatrix} \omega & -\varkappa\,\mathrm{Sh}(\varkappa h_0) \\ g & -\omega\,\mathrm{Ch}(\varkappa h_0) \end{pmatrix} = 0 \quad \text{i.e}$$

$$-\omega^2\,\mathrm{Ch}(\varkappa h_0) + \varkappa g\,\mathrm{Sh}(\varkappa h_0) = 0 \quad \text{i.e}$$

$$(6.9)\quad \boxed{\omega^2 = \varkappa g\,\mathrm{Th}(\varkappa h_0)}$$

(6.9) is the dispersion relation for linearized water waves.

Remark that this is non linear, i.e

$$\frac{\omega^2}{\kappa^2} = \frac{g}{\kappa} \operatorname{Th}(\kappa h_0) \quad \text{is not a constant.}$$

Then we have plane waves solutions, but elementary waves travel with speeds that are a function of $\kappa = \sqrt{k_1^2 + k_2^2}$.

Remarks: If we set ourselves in the 1D case and take right moving waves ($k > 0$) the dispersion relation boils down to

$$\omega^2 = g k \operatorname{Th}(k h_0).$$

In the "infinite depth limit" ($h_0 \to \infty$) $\operatorname{Th}(k h_0) \to 1$ and so we have

$$\omega^2 = g k, \qquad \omega = \sqrt{g k} \implies$$

$$\frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g}{2\pi}} \sqrt{\lambda} \simeq 1.2 \sqrt{\lambda}.$$

In the opposite case ("$h_0 \to 0$") $\operatorname{Th} k h_0 \simeq k h_0$ and one gets

$$\omega^2 = g h_0 k^2 \implies$$

$$\frac{\omega}{k} = \sqrt{g h_0} = c_0$$

"Shallow – h_0 small – water linear waves are not dispersive"

Section 8: The Navier-Stokes equation (in a nutshell).

To introduce the Navier-Stokes equation (or, system) we have to reconsider the Cauchy equations (Section 2, eq. 2.3)

$$(8.1)\quad \rho \frac{D\underline{u}}{Dt} = \rho \underline{b} + \nabla\cdot\mathbb{T},$$

where $\underline{u}$ is the Euler velocity field and $\frac{D}{Dt}$ is the "material derivative",

$$\frac{D}{Dt} = \partial_t + (\underline{u}\cdot\nabla),$$

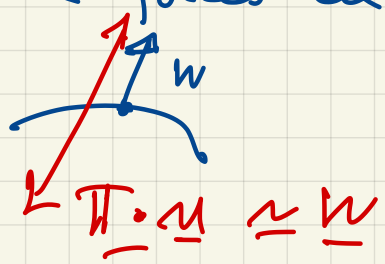
and $\mathbb{T}$ is the stress tensor, encoding surface forces as

$$f_{surf} = \int_{\partial B_t} \mathbb{T}\cdot\underline{n}\, dA,$$

$\underline{n}$ being the normal unit vector.

Remember that the Euler fluid hypothesis namely the Euler constitutive equation, was

$\mathbb{T} = -p\, Id$ (Id being the identity tensor), implying that surface forces are always directed along $\underline{n}$



The Navier-Stokes equations are obtained requiring that:

1) $\Pi = -p \cdot Id + \Pi^{vis}$, $\Pi^{vis} = 0$ if $\underline{u} = 0$

2) Π^{vis} depends linearly on the velocity gradient
$$J_v^{ij} = \partial_j u^i$$
(the linear dependence of Π^{vis} on J_v is called Newtonian setting).

3) Π^{vis} is rotationally invariant.

Remark: Π^{vis} cannot depend on $\underline{u}$ by Galilean invariance of the theory. It must thus depend on J_v.

To proceed further, 2) implies that $\exists$ a rank 4 tensor $C^{ab}{}_{cd}$ such that (in components)

(8.2) $$(\Pi^{vis})^{ij} = \sum_{a,b} C^{ij}{}_{ab} J_v^{ab}$$

$C^{ij}{}_{ab}$ can be, in principle, a function of the position but we still maintain it to be constant.

3) above means that, if $R \in SO(3)$ it must hold

(8.3) $$R\, \Pi^{vis}(J_v) R^T = \Pi^{vis}(R J_v R^T)$$

where the law $J_v \to \mathbb{T}^{vis}$ is given by 8.2.

In components (remember that in the Euclidean setting there is no difference in covariant and contravariant tensor indexes),

$$(8.4) \quad C^{ijkl} = \sum_{abcd} R^i_a R^j_b R^k_c R^l_d \, C^{abcd}$$

It can be shown that [exercise: check the sufficiency statement] rotational invariance yields

$$(8.5) \quad C \cdot J_v = \mu_1 J_v + \mu_2 J_v^T + \lambda(\mathrm{tr}\, J_v) \cdot \mathbb{I}d.$$

Since $\mathbb{T}$ must be symmetric (see Section 2)

$\mu_1 = \mu_2 = \mu$, and so, under the Newtonian setting,

$$[8.6] \quad \mathbb{T}^{vis} = \mu\left(J_v + J_v^T\right) + \lambda(\underbrace{\nabla \cdot v}_{\mathrm{Tr}\, J_v}) \cdot \mathbb{I}d.$$

Remark that the λ-term has the same form as of that of the Euler one (stress $\simeq$ normal) while the μ-term can be off-diagonal thus inducing shear stress (stress has component in the normal plane). We shall assume μ to be independent of x. (viscosity coeff.)

(8.2) The Navier Stokes equation

The Navier-Stokes equation (or system) are the equations for a

1) homogeneous
2) Newtonian
3) Non-Euler flow.

homogeneity means $\rho = \rho_0$ (constant) so

$\nabla \cdot \underline{v} = \operatorname{tr} J_v = 0$.

Hence $\mathbb{T}^{vis}$ (8.6) reduces to $\mu \cdot (J_v + J_v^T)$.

Writing the Cauchy equations

$$\rho_0 \frac{D\underline{v}}{Dt} = \nabla \cdot \mathbb{T} + \underline{b}$$ componentwise yields

$$(\mathbb{T} = -p\,\mathrm{Id} + \mu(J_v + J_v^T))$$

$$\rho_0\Big(v^k_t + \sum_j v^j \partial_j v^k\Big) = -\partial_k p + \mu \sum_j \partial_j\left[\partial_k v^j + \partial_j v^k\right] =$$

$$= -\partial_k p + \mu\Big(\sum_k \partial_k \big(\cancel{\partial_j v^j}\big)^{0} + \partial_j^2 v^k\Big),$$ i.e. finally setting $\nu := \mu / \rho_0$, the NAVIER-STOKES eq:

$$\boxed{(8.7)\quad \underline{v}_t + (\underline{v} \cdot \nabla)\underline{v} = -\frac{\nabla p}{\rho_0} + \nu \Delta \underline{v} + \underline{b}}$$