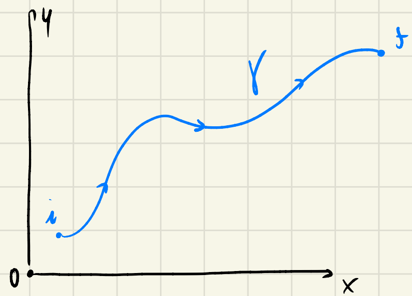
INTEGRALI CURVILINEI

CONSIDERIAMO UNA TRAIETTORIA IN $\mathbb{R}^2$ (SUL PIANO), COME POTREBBE ESSERE LA TRAIETTORIA DI UNA PALLINA CHE SI MUOVE SUL TAVOLO.



$$\gamma \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

t È IL TEMPO; AD OGNI ISTANTE SO DOVE SI TROVA LA PALLINA.

γ È DETTA <u>CURVA</u>.

<u>DEF.</u> <u>CURVA REGOLARE</u>:

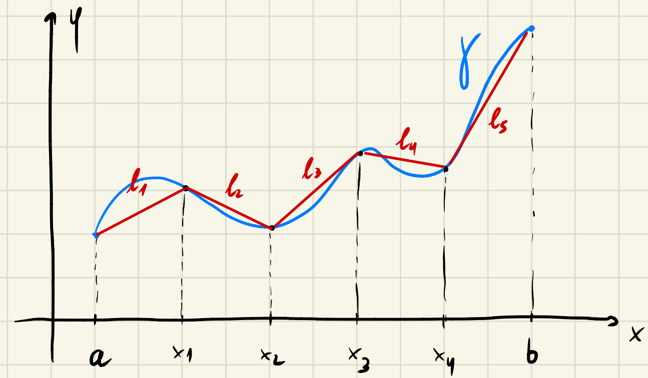
1) $x(t), y(t)$ SONO <u>DERIVABILI</u> IN TUTTO L'INTERVALLO IN CUI VARIA t.

2) $x'(t), y'(t)$ SONO <u>CONTINUE</u>.

3) LA CURVA <u>NON SI CHIUDE SU SE STESSA</u>, OVVERO: NON $\exists\, t' \neq t$ T.C. $\begin{cases} x(t) = x(t') \\ y(t) = y(t') \end{cases}$

COME POSSIAMO CALCOLARE LA LUNGHEZZA DI UNA CURVA? LA POSSIAMO APPROSSIMARE CON DEI SEGMENTI RETTILINEI.

$$L_\gamma = \sum_i^N l_i \quad \text{con} \quad l_i = \sqrt{(y(x_i) - y(x_{i-1}))^2 + (x_i - x_{i-1})^2}$$



SE PRENDO POCHI SEGMENTI NON APPROSSIMO BENE, IMMAGINIAMO DI PRENDERE TANTISSIMI PUNTI TRA x_0 E x_N, IN MODO CHE DIVENTINO FITTI, COSÌ MIGLIORIAMO LA NOSTRA APPROSSIMAZIONE.

$$L_\gamma = \sum_i^N \sqrt{(f(x_i) - f(x_{i-1}))^2 + \Delta x^2} = \sum_i^N |\Delta x| \cdot \sqrt{\frac{\Delta f_i^2}{\Delta x^2} + 1} \quad \text{con} \quad \Delta x = \frac{b-a}{N}$$

COSA SUCCEDE SE $\lim_{N \to \infty}$?

- $\sum_{i=1}^{\infty}$
- $\Delta x = \frac{b-a}{N} \to dx$ DIVENTA <u>INFINITESIMO</u>
- $\frac{\Delta f_i}{\Delta x} \to \frac{df}{dx}$

LA SOMMA DIVENTA UN INTEGRALE: $\sum_i^N |\Delta x| \sqrt{\frac{\Delta f_i^2}{\Delta x^2} + 1} \longrightarrow \int_a^b dx \sqrt{\left(\frac{df}{dx}\right)^2 + 1} = L_\gamma$

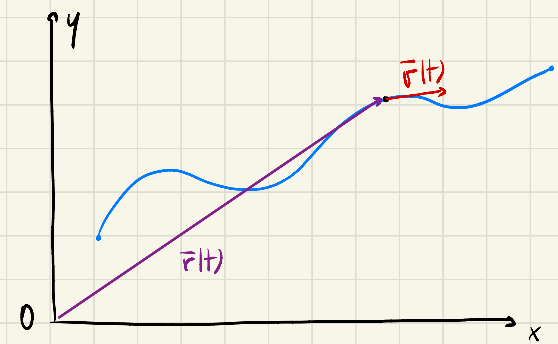
=> LA LUNGHEZZA DI γ È: $\boxed{L_\gamma = \int_a^b dx \sqrt{1 + (f'(x))^2}}$

NOTA: QUESTA FORMULA VALE SOLO SE SI PUÒ SCRIVERE y COME UNA FUNZIONE DI x!

SE y NON SI PUÒ SCRIVERE COME $f(x)$, COME IN UNA CIRCONFERENZA, COME SI FA?

↳ POSSO IDENTIFICARE UN PUNTO APPARTENENTE A γ CON UN VETTORE:

$\bar r(t) = [x(t), y(t)]$



DEFINISCO UNA VELOCITÀ: $\bar v(t) = \frac{d\bar r}{dt} = [\dot x(t), \dot y(t)]$

LA VELOCITÀ $\bar v$ È TANGENTE ALLA CURVA => POSSO DEFINIRE: $d\gamma = |\bar v| \cdot dt$ INCREMENTO INFINITESIMO DI CURVA AL VARIARE DI t.

($d\gamma$: SPAZIO, $|\bar v| \cdot dt$: VELOCITÀ × TEMPO)

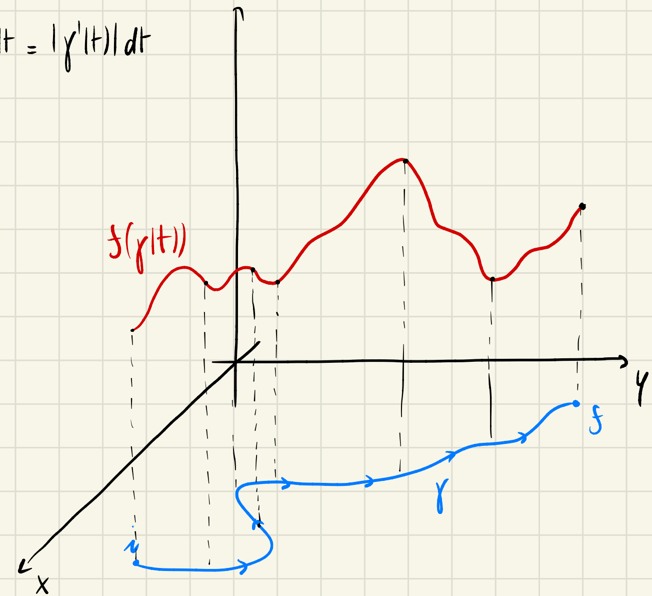
ADESSO SOMMO TUTTI GLI INCREMENTI INFINITESIMI: $\left[L_\gamma = \int d\gamma = \int dt\, |\bar{v}| = \int dt \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} \right]$

<u>**NOTA**</u>: IL PUNTO IN $\dot{x}$ E $\dot{y}$ INDICA LA DERIVATA RISPETTO A t.

ADESSO IMMAGINIAMO DI DOVER INTEGRARE UNA FUNZIONE SULLA TRAIETTORIA/CURVA $\gamma(t)$.

$\Rightarrow \int d\gamma\, f(\gamma(t))$ MA $\gamma(t) \to d\gamma = |\bar{v}| \cdot dt = |\gamma'(t)|\, dt$

$$\Rightarrow \left[\int_\gamma d\gamma\, f(\gamma(t)) = \int dt\, |\gamma'(t)|\, f(t) \right]$$



ES. 1: CALCOLARE LA LUNGHEZZA DELLA CICLOIDE $\gamma = \begin{cases} x(t) = r \cdot (t - \operatorname{sen} t) \\ y(t) = r \cdot (1 - \cos t) \end{cases}$

$$\Rightarrow \quad \dot{x} = r \cdot (1 - \cos t)$$

$$\dot{y} = r \cdot (0 + \operatorname{sen} t) = r \cdot \operatorname{sen} t$$

$$, \quad \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} = \sqrt{r^2(1-\cos t)^2 + r^2 \operatorname{sen}^2 t}$$

$$\Rightarrow \quad L_\gamma = \int_0^{2\pi} dt \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} = \int_0^{2\pi} dt \, r \sqrt{(1-\cos t)^2 + \operatorname{sen}^2 t} = r \int_0^{2\pi} dt \sqrt{1 + \cos^2 t - 2\cos t + \operatorname{sen}^2 t}$$

$$= r \int_0^{2\pi} dt \sqrt{2 - 2\cos(t)}$$

$$= \int_0^{2\pi} dt \, r \sqrt{\frac{2(1-\cos t) \cdot 2}{2}} = \int_0^{2\pi} dt \, 2r \overbrace{\sqrt{\frac{1-\cos t}{2}}}^{\operatorname{sen}\left(\frac{t}{2}\right)} = 2r \int_0^{2\pi} dt \operatorname{sen}\left(\frac{t}{2}\right)$$

$$f(t) = \frac{t}{2}$$

$$f'(t) = \frac{1}{2}$$

$$= 2 \cdot r \cdot \frac{2}{2} \int_0^{2\pi} dt \operatorname{sen}\left(\frac{t}{2}\right) = 4r \int_0^{2\pi} dt \underbrace{\frac{1}{2}}_{f'(t)} \cdot \underbrace{\operatorname{sen}\left(\frac{t}{2}\right)}_{\operatorname{sen}(f(t))} = 4 \cdot r \left(-\cos\left(\frac{t}{2}\right)\right)\Big|_0^{2\pi}$$

$$= -4r\left[\cos\left(\frac{2\pi}{2}\right) - \cos(0)\right] = -4r\left[-1-1\right] = \boxed{8r}\ \checkmark$$

ES. 2: CALCOLARE L'INTEGRALE CURVILINEO $\int_\gamma d\gamma\, y^2$ CON $\gamma = \begin{cases} x = t \\ y = e^t \end{cases}$ CON $t \in [0, \ln(2)]$

$$\dot{x}(t) = 1 \qquad \dot{y}(t) = e^t \quad \Rightarrow \quad \sqrt{1^2 + e^{2t}} = \sqrt{1+e^{2t}}$$

$$\int_\gamma d\gamma\, y^2 \longrightarrow \int_0^{\ln(2)} dt\, \sqrt{1+e^{2t}}\, \overset{y^2}{e^{2t}}$$

$$f(t) = 1 + e^{2t}$$

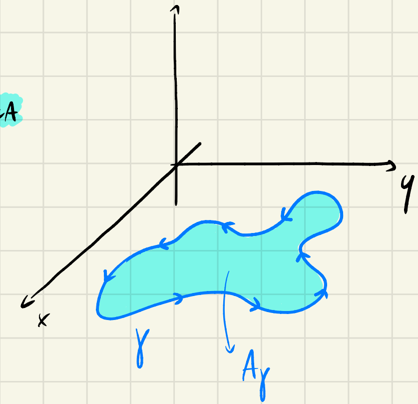
$$f'(t) = 0 + e^{2t} \cdot \frac{d}{dt}(2t) = 2e^{2t}$$

$$\Rightarrow \frac{1}{2}\int_0^{\ln(2)} dt\, \underbrace{(2e^{2t})}_{f'(t)}\, \underbrace{(1+e^{2t})^{\frac{1}{2}}}_{(f(t))^{\frac{1}{2}}} = \frac{1}{\cancel{2}} \cdot \left.\frac{(1+e^{2t})^{\frac{3}{2}}}{\frac{3}{\cancel{2}}}\right|_0^{\ln(2)} = \left.\frac{1}{3}(1+e^{2t})^{\frac{3}{2}}\right|_0^{\ln(2)}$$

$$= \frac{1}{3}\left[(1+e^{2\ln(2)})^{\frac{3}{2}} - (1+e^{2\cdot 0})^{\frac{3}{2}}\right] = \frac{1}{3}\left[5^{\frac{3}{2}} - 2^{\frac{3}{2}}\right] = \boxed{\frac{5\sqrt{5} - 2\sqrt{2}}{3}}\ \checkmark$$

TEOREMA DI GREEN

CONSIDERIAMO UNA CURVA CHIUSA IN $\mathbb{R}^2$. QUESTA CURVA DELIMITA UN'AREA

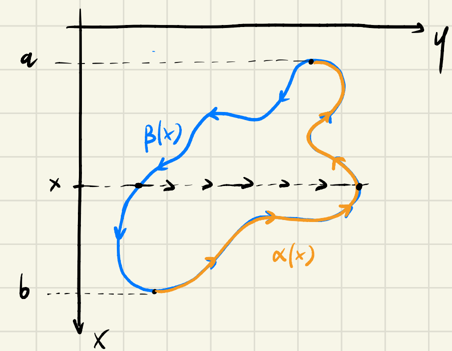


SIANO $f(x,y)$ E $g(x,y)$ DUE FUNZIONI DIFFERENZIABILI, ALLORA:

$$\oint_\gamma dx\, f(x,y) + \oint_\gamma dy\, g(x,y) = \int_{A_\gamma} dx\,dy \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)$$

IN MODO EQUIVAL.:

$$\underbrace{\oint_\gamma}_{\text{INTEGRALE SULLA CURVA } \gamma} \big(f(x,y)\,dx + g(x,y)\,dy \big) = \underbrace{\int_{A_\gamma}}_{\text{INTEGRALE SULL'AREA RACCHIUSA DA } \gamma} dx\,dy \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)$$

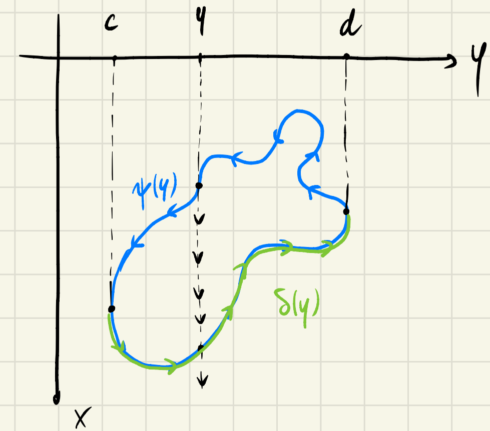


DIM. $$-\int_{A_\gamma} dx\,dy \frac{\partial f}{\partial y} = -\int_a^b dx \int_{\beta(x)}^{\alpha(x)} dy \left(\frac{\partial f}{\partial y} \right) = -\int_a^b dx\, f(x,y) \Big|_{y=\beta(x)}^{y=\alpha(x)} = -\int_a^b dx \left[f(x,\alpha(x)) - f(x,\beta(x)) \right]$$

$$= -\int_a^b dx\, f(x,\alpha(x)) + \int_a^b dx\, f(x,\beta(x)) = \int_b^a dx\, f(x,\alpha(x)) + \int_a^b dx\, f(x,\beta(x)) = \oint_\gamma dx\, f(x,y)$$

$$\Rightarrow \left[\int_{A_\gamma} dx\,dy \left(-\frac{\partial f}{\partial y}\right) = \oint_\gamma dx\, f(x,y) \right] \checkmark$$

SI PROVA ANCHE PER L'ALTRO PEZZO $\left(\int_{A_\gamma} dx\,dy\, \frac{\partial g}{\partial x} = \oint_\gamma dy\, g(x,y) \right)$

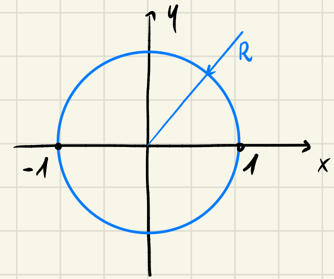


$$\int_{A_\gamma} dx\,dy\, \frac{\partial g}{\partial x} = \int_c^d dy \int_{\psi(y)}^{\delta(y)} dx\, \frac{\partial g}{\partial x} = \int_c^d dy\, \left[g(\delta(y),y) - g(\psi(y),y) \right]$$

$$\left[= \int_c^d dy\, g(\delta(y),y) + \int_d^c dy\, g(\psi(y),y) \right] \checkmark$$

[È UN TEOREMA FIGHISSIMO PERCHÉ CI PERMETTE DI MANDARE INTEGRALI SUL CAMMINO IN INTEGRALI SU AREE E VICEVERSA.]

ES. 3: AREA NEL CERCHIO $\int_A dx\,dy = A$ ⊛



$$\gamma = \begin{cases} x(t) = R \cdot \cos t \\ y(t) = R \cdot \operatorname{sen} t \end{cases}$$

$$\int_A dx\,dy \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) = \oint_\gamma \left(dx\, f(x,y) + dy\, g(x,y) \right)$$

⊛

TROVIAMO $g(x,y)$ E $f(x,y)$ T.C. $\int_A dx\,dy \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) = \int_A dx\,dy$

GUESS: $g(x,y) = \frac{1}{2}x \Rightarrow \frac{\partial g}{\partial x} = \frac{1}{2}$

$f(x,y) = -\frac{1}{2}y \Rightarrow \frac{\partial f}{\partial y} = -\frac{1}{2}$

$$\Rightarrow \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1 \Rightarrow \int_A dx\,dy \left(\frac{1}{2} - \left(-\frac{1}{2} \right) \right) = \int_A dx\,dy \ ✓$$

$$\Rightarrow A = \int_A dx\,dy = \oint_\gamma \left(dx \left(-\frac{1}{2}y \right) + dy \left(\frac{1}{2}x \right) \right) = \underbrace{\frac{1}{2}\oint_\gamma}_{1} \left(x \underbrace{dy}_{2} - y\,dx \right)$$

① $\frac{1}{2}\oint x\,dy$ $\quad$ $x(t) = R\cdot\cos(t)$, $y(t) = R\cdot \text{sen}(t) \Rightarrow dy = R\cdot\cos(t)\,dt$

$$\frac{R^2}{2}\int_0^{2\pi} dt\,\cos^2(t) = ?$$

$$\cos(2t) = \cos^2(t) - \text{sen}^2(t) = \cos^2(t) - (1-\cos^2(t)) = 2\cos^2(t) - 1$$

$$\Rightarrow 2\cos^2(t) = 1+\cos(2t) \Rightarrow \cos^2(t) = \frac{1}{2} + \frac{1}{2}\cos(2t)$$

$$\frac{R^2}{2}\int_0^{2\pi} dt\,\frac{1}{2}(1+\cos(2t)) = \frac{R^2}{4}\int_0^{2\pi} dt\,(1+\cos(2t)) = \frac{R^2}{4}\int_0^{2\pi} dt + \frac{R^2}{4}\int_0^{2\pi} dt\,\cos(2t)$$

$$= \frac{R^2}{4}t\Big|_0^{2\pi} + \frac{R^2}{4}\cdot\frac{1}{2}\int_0^{2\pi} dt\,\underbrace{2}_{f'(t)}\cdot\cos(\underbrace{2t}_{f(t)})$$

$$= \frac{2\pi R^2}{4} + \frac{R^2}{8}\cos(2t)\Big|_0^{2\pi} = \frac{2\pi R^2}{4} + \frac{R^2}{8}(\cos(4\pi) - \cos(0))$$

$$= \frac{\pi R^2}{2}$$

② $dx = -R\,\text{sen}(t)\,dt$ $\quad$ con $t \in [0, 2\pi]$ $\quad$ $y = R\,\text{sen}(t)$

$$\Rightarrow -\frac{1}{2}\oint_\gamma dx\, y = +\int_0^{2\pi} dt\, \frac{R^2}{2}\, \text{sen}^2(t) = \frac{R^2}{2}\int_0^{2\pi} dt\, \text{sen}^2(t)$$

$$\text{sen}(t) = \sqrt{\frac{1-\cos(2t)}{2}}$$

$$\text{sen}^2(t) = \frac{1}{2}\left(1-\cos(2t)\right)$$

$$= \frac{R^2}{2}\int_0^{2\pi} dt\,\left(1-\cos(2t)\right) = \frac{\pi R^2}{2}$$

$$1 + 2 = \pi R^2 \quad \Rightarrow \quad \left[\int_A dx\, dy = \pi R^2\right] \checkmark$$