$$\begin{cases} y_{xx} + y = x \\ y(0) = 0 \\ y(1) = 1 \end{cases}$$

Soluzione dell'omogenea associata:

$$\lambda^2 + 1 = 0 \Rightarrow y_{oh}(x) = c_1 e^{ix} + c_2 e^{-ix}$$

Metodo di variazione delle costanti:

$$y_p = u_1(x) e^{ix} + u_2(x) e^{-ix}$$

$$y_p' = u_1'(x) e^{ix} + u_2'(x) e^{-ix} + i\left[u_1 e^{ix} - u_2 e^{-ix}\right]$$

Ipotesi: $\boxed{u_1' e^{ix} + u_2' e^{-ix} = 0}$ (1)

$$y_p'' = i\left[u_1' e^{ix} - u_2' e^{-ix}\right] + i^2\left[u_1 e^{ix} + u_2 e^{-ix}\right]$$

$$= i\left[u_1' e^{ix} - u_2' e^{-ix}\right] - \left[u_1 e^{ix} + u_2 e^{-ix}\right]$$

Sostituiamo nella ODE:

$$i\left[u_1' e^{ix} - u_2' e^{-ix}\right] \underbrace{- \left[u_1 e^{ix} + u_2 e^{-ix}\right] + u_1 e^{ix} + u_2 e^{-ix}}_{=0} = x$$

$$\Rightarrow (ie^{ix})u_1' - (ie^{-ix})u_2' = x$$

$$\begin{bmatrix} e^{ix} & e^{-ix} \\ ie^{ix} & -ie^{-ix} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}$$

$$u_1' = -\frac{1}{2} ixe^{-ix} \longrightarrow u_1 = \frac{1}{2}(x-i)e^{-ix}$$

$$u_2' = \frac{1}{2} ix\, e^{ix} \longrightarrow u_2 = \frac{1}{2}(x+i)e^{ix}$$

$$y_p(x) = u_1 e^{ix} + u_2 e^{-ix} = x$$

$$y(x) = c_1 e^{ix} + c_2 e^{-ix} + x$$

$$y(0) = 0 = c_1 + c_2 \Rightarrow c_2 = -c_1$$

$$y(1) = 1 = c_1 e^{i} + c_2 e^{-i} + 1$$

$$= c_1 (e^{i} - e^{-i}) + 1$$

$$= 2ic_1 \sin(1) + 1$$

$$\Rightarrow c_1 = 0 = c_2$$

$$\boxed{g(x) = x}$$

Se la soluz. generale dell'omogenea si esprime in funzione di $\sin x$ e $\cos x$:
$C_1 \cos x + C_2 \sin x$

$$y = u_1 \cos x + u_2 \sin x$$

$$y' = -u_1 \sin x + u_2 \cos x \qquad \boxed{u_1' \cos x + u_2' \sin x = 0}$$

$$y'' = -u_1' \sin x + u_2' \cos x - u_1 \cos x - u_2 \sin x$$

$$\boxed{-u_1' \sin x + u_2' \cos x = x}$$

ortogonale

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix}$$

$$= \begin{bmatrix} -x \sin x \\ x \cos x \end{bmatrix}$$

$$u_1 = -\left\{ -x\cos x + \int \cos x \right\} =$$

$$= x\cos x - \operatorname{sen} x$$

$$u_2 = x\operatorname{sen} x - \int \operatorname{sen} x = x\operatorname{sen} x + \cos x$$

$$y_p = (x\cos x - \operatorname{sen} x)\cos x +$$

$$+ (x\operatorname{sen} x + \cos x)\operatorname{sen} x$$

$$= x\,c^2 - \cancel{sc} + x\,s^2 + \cancel{sc}$$

$$= x\,(c^2 + s^2) = 1.$$

etc. etc.