ES 1

5. $\{q, p\} = \{ \tilde{p}^{1/3} e^{\gamma \tilde{q}}, \tilde{p}^{\alpha} e^{-\beta \tilde{q}} \} =$

$= \left(\gamma e^{\gamma \tilde{q}} \tilde{p}^{1/3}\right)\left(\alpha \tilde{p}^{\alpha-1} e^{-\beta \tilde{q}}\right) - \left(\frac{1}{3} \tilde{p}^{-2/3} e^{\gamma \tilde{q}}\right)\left(-\beta \tilde{p}^{\alpha} e^{-\beta\tilde{q}}\right)$

$= e^{\gamma \tilde{q}} e^{-\beta \tilde{q}} \tilde{p}^{\alpha - \frac{2}{3}} \left(\alpha \gamma + \frac{\beta}{3}\right)$

$\gamma = \beta \quad \alpha = \frac{2}{3} \qquad \frac{2}{3}\beta + \frac{\beta}{3} = \beta$

$\beta = 1 \qquad \gamma = 1 \qquad \alpha = \frac{2}{3}$

$\rightsquigarrow \quad q = \tilde{p}^{1/3} e^{\tilde{q}} \qquad p = \tilde{p}^{2/3} e^{-\tilde{q}}$

6. $H(p, q) = \frac{1}{2} p^2 q^2 \xrightarrow[K_0 = 0]{c=1} K = H(p(\tilde{p}, \tilde{q}), q(\tilde{p}, \tilde{q})) =$

$= \frac{1}{2} \tilde{p}^{2/3} e^{2\tilde{q}} \tilde{p}^{4/3} e^{-2\tilde{q}} = \frac{1}{2} \tilde{p}^2$

$\rightarrow \quad \dot{\tilde{p}} = 0 \qquad \tilde{p}(t) = \tilde{p}_0$

$\dot{\tilde{q}} = \tilde{p} \qquad \tilde{q}(t) = \tilde{p}_0 t + \tilde{q}_0$

$p(t) = \tilde{p}_0^{2/3} e^{-\tilde{q}_0} e^{-\tilde{p}_0 t}$

$q(t) = \tilde{p}_0^{1/3} e^{\tilde{q}_0} e^{\tilde{p}_0 t}$

7. $p(0) = \tilde{p}_0^{2/3} e^{-\tilde{q}_0} \equiv p_0$

$q(0) = \tilde{p}_0^{1/3} e^{\tilde{q}_0} \equiv q_0$

$\rightarrow \quad \tilde{p}_0 = p_0 q_0$

$\tilde{q}_0 = \log \frac{q_0^{2/3}}{p_0^{1/3}}$

$$\Phi_t(p,q) = \begin{pmatrix} p e^{-pqt} \\ q e^{pqt} \end{pmatrix}$$

$$t \ll 1 \Rightarrow \Phi_t(p,q) = \begin{pmatrix} p - p^2 q t \\ q + p q^2 t \end{pmatrix}$$

$$\delta p = t\{p, \tfrac{1}{2}p^2q^2\} = -p^2 q t$$

$$\delta q = t\{q, \tfrac{1}{2}p^2q^2\} = p q^2 t$$

ES2

$x_p = R\,\text{sen}\,\theta$

$y_p = -R\cos\theta$

1)

$$T = \frac{M}{2}\dot{s}^2 + \frac{m}{2}R^2\dot{\theta}^2 \qquad a = \begin{pmatrix} M & 0 \\ 0 & mR^2 \end{pmatrix}$$

$$V = -mgR\cos\theta + \frac{1}{2}k\left(\underbrace{(R\,\text{sen}\,\theta - s)^2 + R^2\cos^2\theta}_{R^2 - 2Rs\,\text{sen}\,\theta + s^2}\right)$$

cost. irrilev. in eq. Lagr.

$$L = \frac{M}{2}\dot{s}^2 + \frac{m}{2}R^2\dot{\theta}^2 + mgR\cos\theta - \frac{1}{2}ks^2 + kRs\,\text{sen}\,\theta - \cancel{\frac{k}{2}R^2}$$

2)

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} &= M\ddot{s} \\ \frac{\partial L}{\partial s} &= -ks + kR\,\text{sen}\,\theta \end{aligned}\right\} \quad \ddot{s} = -\frac{k}{M}s + \frac{k}{M}R\,\text{sen}\,\theta$$

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= mR^2\ddot{\theta} \\ \frac{\partial L}{\partial \theta} &= -mgR\,\text{sen}\,\theta + kRs\cos\theta \end{aligned}\right\} \quad \ddot{\theta} = -\frac{g}{R}\text{sen}\,\theta + \frac{k}{m}\frac{s}{R}\cos\theta$$

3) $s \to -s \quad \theta \to -\theta$

4) $V = -mgR\cos\theta + \frac{k}{2}s^2 - kRs\,\text{sen}\,\theta$

$$\frac{\partial V}{\partial s} = ks - kR\,\text{sen}\,\theta \xrightarrow{=0} s = R\,\text{sen}\,\theta$$

$$\frac{\partial V}{\partial \theta} = mgR\,\text{sen}\,\theta - kRs\cos\theta \xrightarrow{=0} s = \frac{mg}{k}\frac{\text{sen}\,\theta}{\cos\theta}$$

$$\rightarrow \quad R\,\frac{\operatorname{sen}\theta}{\cos\theta}\left(\cos\theta - \frac{mg}{KR}\right) = 0$$

($\theta = \pi$ non fa parte delle config.)

$$\theta = 0 \quad \rightsquigarrow \quad s = 0$$

$\exists$ se $\frac{mg}{KR} \leq 1$

$$\theta^*_{\pm} = \pm \arccos \frac{mg}{KR} \quad \rightsquigarrow \quad s^*_{\pm} = \pm R\sqrt{1 - \frac{mg}{KR}} = R\operatorname{sen}\theta^*_{\pm}$$

$$\partial^2 V = \begin{pmatrix} K & -KR\cos\theta \\ -KR\cos\theta & mgR\cos\theta + KRs\operatorname{sen}\theta \end{pmatrix}$$

$$\partial^2 V_{(0,0)} = \begin{pmatrix} K & -KR \\ -KR & mgR \end{pmatrix} \rightarrow \det = mg\,KR - (KR)^2 = $$

$$= (KR)^2\left(\frac{mg}{KR} - 1\right)$$

STAB se $\frac{mg}{KR} > 1$

INSTAB se $\frac{mg}{KR} < 1$

$$\partial^2 V_{(s^*_{\pm},\, \theta^*_{\pm})} = \begin{pmatrix} K & -mg \\ -mg & \cancel{(mg)^2 \frac{1}{K}} + KR^2\left(1 - \cancel{\left(\frac{mg}{KR}\right)^2}\right) \end{pmatrix} =$$

$$= \begin{pmatrix} K & -mg \\ -mg & KR^2 \end{pmatrix} \rightarrow \det = (KR)^2 - (mg)^2 =$$

$$= (KR)^2\left(1 + \frac{mg}{KR}\right)\left(1 - \frac{mg}{KR}\right)$$

STAB qndo esiste, cioè $\frac{mg}{KR} < 1$

5) $M = m$ $\qquad \underbrace{\frac{mg}{KR}} < \frac{1}{10} < 1$

pti stab. $\rightarrow (s,\theta) = (s^*_\pm, \theta^*_\pm)$

$$A = \begin{pmatrix} m & 0 \\ 0 & mR^2 \end{pmatrix} \qquad B = \begin{pmatrix} K & -mg \\ -mg & KR^2 \end{pmatrix}$$

$$\det(B - \lambda A) = (K - \lambda m) R^2 (K - \lambda m) - (mg)^2 = 0 \rightarrow \frac{1}{m^2R^2}$$

$$\rightarrow \left(\lambda - \frac{K}{m}\right)^2 - \left(\frac{g}{R}\right)^2 = 0$$

$$\lambda_{1,2} = \frac{K}{m} \pm \frac{g}{R}$$

$s = \pm s_* + \delta s$
$\theta = \pm\theta_* + \delta\theta$

$$L_{lin} = \frac{1}{2}(\delta\dot{s}, \delta\dot{\theta}) \begin{pmatrix} m & 0 \\ 0 & mR^2 \end{pmatrix} \begin{pmatrix} \delta\dot{s} \\ \delta\dot{\theta} \end{pmatrix} - \frac{1}{2}(\delta s, \delta\theta) \begin{pmatrix} K & -mg \\ -mg & KR^2 \end{pmatrix} \begin{pmatrix} \delta s \\ \delta\theta \end{pmatrix}$$

$$= \frac{m}{2}\delta\dot{s}^2 + \frac{mR^2}{2}\delta\dot{\theta}^2 - \frac{K}{2}\delta s^2 + mg\,\delta s\,\delta\theta - \frac{KR^2}{2}\delta\theta^2$$

$$\begin{pmatrix} \delta\ddot{s} \\ \delta\ddot{\theta} \end{pmatrix} = \begin{pmatrix} 1/m & \\ & 1/mR^2 \end{pmatrix} \begin{pmatrix} K & -mg \\ -mg & KR^2 \end{pmatrix} \begin{pmatrix} \delta s \\ \delta\theta \end{pmatrix} =$$

$$= \begin{pmatrix} K/m & -g \\ -g/R^2 & K/m \end{pmatrix} \begin{pmatrix} \delta s \\ \delta\theta \end{pmatrix} = \begin{pmatrix} \frac{K}{m}\delta s - g\,\delta\theta \\ -\frac{g}{R^2}\delta s + \frac{K}{m}\delta\theta \end{pmatrix}$$

6) $k = \frac{mg}{R}$ $\quad \bar{F} = F\vec{e}_x$

a) $\delta V(s)$ t.c. $\quad -\delta V'(s) = F \rightsquigarrow \delta V(s) = -Fs$

$$L = L_{old} + Fs$$

$$V = V_{old} - Fs \qquad\qquad k = \frac{mg}{R}$$

$$\frac{\partial V}{\partial s} = \frac{\partial V_{old}}{\partial s} - F = \frac{mg}{R}s - mg\,\text{sen}\theta - F$$

$$\frac{\partial V}{\partial \theta} = \frac{\partial V_{old}}{\partial \theta} = mgR\,\text{sen}\theta - mgs\cos\theta$$

Voglio che $s=R$ sia pto di equil. $\rightsquigarrow$

$$\left.\frac{\partial V}{\partial s}\right|_{s=R} = mg(1-\text{sen}\theta) - F = 0 \rightsquigarrow (\#)$$

$$\left.\frac{\partial V}{\partial \theta}\right|_{s=R} = mgR\,\text{sen}\theta - mgR\cos\theta = 0$$

$\hookrightarrow \text{tg}\,\theta = 1 \rightarrow \theta = \pi/4$

$$(\#):\ mg\left(1 - \frac{\sqrt{2}}{2}\right) = F$$

$$\partial^2 V = \partial^2 V_{old} = \begin{pmatrix} \frac{mg}{R} & -mg\cos\theta \\ -mg\cos\theta & mgR\cos\theta + mgs\,\text{sen}\theta \end{pmatrix}$$

$$\downarrow \Big|_{\substack{s=R \\ \theta=\pi/4}} = \begin{pmatrix} \frac{mg}{R} & -mg\frac{\sqrt{2}}{2} \\ -mg\frac{\sqrt{2}}{2} & mgR\sqrt{2} \end{pmatrix} \rightarrow \det = (mg)^2\left(\sqrt{2} - \frac{1}{2}\right) > 0$$

STAB.

7) Troviamo soluz. di $\quad Bu = \lambda A u$

$$\frac{1}{2}(\delta\dot{s}, \delta\dot{\theta}) \underset{\uparrow \atop A}{\begin{pmatrix} m & 0 \\ 0 & mR^2 \end{pmatrix}} \begin{pmatrix} \delta\dot{s} \\ \delta\dot{\theta} \end{pmatrix} - \frac{1}{2}(\delta s, \delta\theta) \underset{\uparrow \atop B}{\begin{pmatrix} K & -mg \\ -mg & KR^2 \end{pmatrix}} \begin{pmatrix} \delta s \\ \delta\theta \end{pmatrix}$$

$$\lambda_\pm = \frac{K}{m} \pm \frac{g}{R}$$

$$\lambda = \lambda_\pm \ : \quad (B - \lambda_\pm A)u = \begin{pmatrix} \mp mg/R & -mg \\ -mg & \mp mgR \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} =$$

$$= -mgR \begin{pmatrix} \pm 1 & R \\ R & \pm R^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha \begin{pmatrix} R \\ \mp 1 \end{pmatrix}$$

$$\Rightarrow U = \begin{pmatrix} R & R \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \delta s \\ \delta\theta \end{pmatrix} = U \begin{pmatrix} x \\ y \end{pmatrix}$$

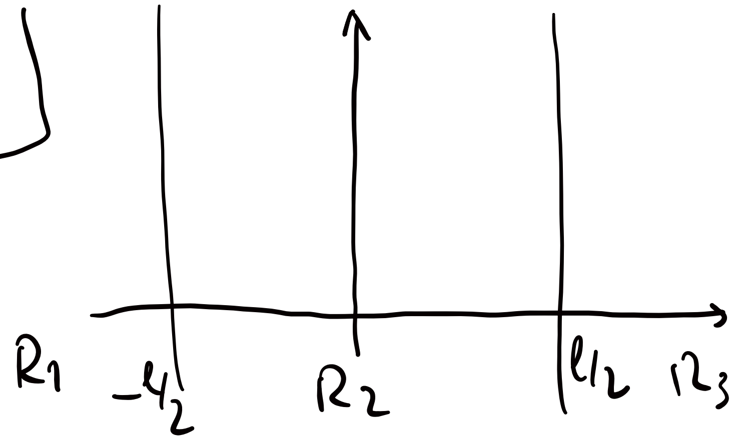
$$\begin{pmatrix} R & 1 \\ R & -1 \end{pmatrix} \begin{pmatrix} K & -mg \\ -mg & KR^2 \end{pmatrix} \begin{pmatrix} R & R \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} KR - mg & -mgR + KR^2 \\ KR + mg & -mgR - KR^2 \end{pmatrix} \begin{pmatrix} R & R \\ 1 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2KR^2 - 2mgR & 0 \\ 0 & 2KR^2 + 2mgR \end{pmatrix} = 2mR^2 \begin{pmatrix} \frac{K}{m} - \frac{g}{R} & 0 \\ 0 & \frac{K}{m} + \frac{g}{R} \end{pmatrix}$$

$$\begin{pmatrix} R & 1 \\ R & -1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & mR^2 \end{pmatrix} \begin{pmatrix} R & R \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} mR & mR^2 \\ mR & -mR^2 \end{pmatrix} \begin{pmatrix} R & R \\ 1 & -1 \end{pmatrix} =$$

$$= 2mR^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow L = 2mR^2 \left(\frac{1}{2}(\dot{x}\ \dot{y}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \frac{1}{2}(x\ y) \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

ES3)



1) Condiz. raccordo $\psi_2\left(-\frac{\ell}{2}\right)=0 \quad \psi_2\left(\frac{\ell}{2}\right)=0$

$\psi_2(x) = Ae^{ikx} + Be^{-ikx} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$

vedi note

$\boxed{n \neq 0} \Rightarrow m \neq 0$

$n=2m \quad \psi_{2m}(x) = \sqrt{\frac{2}{\ell}} \operatorname{sen}\left(\frac{2m\pi x}{\ell}\right)$

$n=2m+1 \quad \psi_{2m+1}(x) = \sqrt{\frac{2}{\ell}} \cos\left(\frac{(2m+1)\pi x}{\ell}\right)$

funzioni d'onda normalizzate

2) $E_n = \frac{\pi^2\hbar^2}{2m\ell^2} n^2 \qquad n = 1, 2, \dots \qquad (n \neq 0!)$

3) $\frac{(\psi_2, \hat{H}\psi_2)}{\|\psi_2\|^2} = E_2 \frac{(\psi_2, \psi_2)}{\|\psi_2\|^2} = E_2 = \frac{2\pi^2\hbar^2}{m\ell^2}$

4) $\psi_3(x) = \sqrt{\frac{2}{\ell}} \cos\left(\frac{3\pi x}{\ell}\right) \leftarrow$ funz. pari per $x \to -x$

$\Rightarrow \psi_3'(x) = -\sqrt{\frac{2}{\ell}} \frac{3\pi}{\ell} \operatorname{sen}\left(\frac{3\pi x}{\ell}\right) \leftarrow$ funz. dispari per $x \to -x$

$\frac{(\psi_3, \hat{P}\psi_3)}{\|\psi_3\|^2} = \int_{-\ell/2}^{\ell/2} \underbrace{\psi_3 \frac{\hbar}{i} \psi_3'}_{} dx = 0$

$\uparrow$ funz. dispari

5) $\psi_1(x) = \sqrt{\frac{2}{\ell}} \cos\left(\frac{\pi x}{\ell}\right)$ ← funz. normalizzata.

$$\text{Prob.}\left(x \in \left[0, \frac{\ell}{4}\right]\right) = \int_0^{\ell/4} |\underbrace{\psi_1}_{\text{normalizzata}}(x)|^2 dx =$$

$$= \int_0^{\ell/4} \left(\frac{2}{\ell}\right) \cos^2\left(\frac{\pi x}{\ell}\right) dx = \frac{1}{\ell} \int_0^{\ell/4} \left(1 + \cos\left(\frac{2\pi x}{\ell}\right)\right) dx =$$

$$= \frac{1}{\ell}\left(x + \frac{\ell}{2\pi} \operatorname{sen}\left(\frac{2\pi x}{\ell}\right)\right)\Bigg|_0^{\ell/4} =$$

$$= \frac{1}{4} + \frac{1}{2\pi} \operatorname{sen}\left(\frac{\pi}{2}\right) = \frac{1}{4} + \frac{1}{2\pi} \quad (\text{che è} < 1/2).$$