SCRITTO GEO 3/7/2023

SOLUZIONE

Risposte corrette alle domande a risposta multipla

1. (c)

2. (a)

3. (b) $(1+i)^6 = \left(\sqrt{2}\cdot\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]\right)^6$

$= 8\cdot\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right) = -8i$

4. (b)

5. (a) $$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

6. (a) $$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

7. (b) $\dim \operatorname{im}(f) \leq \dim(\mathbb{R}^2)$ sempre

$\Rightarrow \dim \operatorname{im}(f) \leq 2 < 3$.

8. (a) Perché la matrice che la rappresenta rispetto alla base canonica è la matrice nulla, che è simmetrica.

Esercizi

1. (a) $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right]$

$$\Rightarrow f(e_1) = \frac{1}{2}\left[\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} -1/2 \\ -1/2 \\ 0 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right]$$

$$\Rightarrow f(e_2) = \frac{1}{2}\left[\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} -1/2 \\ 1/2 \\ 2 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}\left[-\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right]$$

$$\Rightarrow f(e_3) = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$\Rightarrow M^{\mathcal{E}}_{\mathcal{E}}(f) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 \end{pmatrix} \quad \text{dove } \mathcal{E} = \{e_1, e_2, e_3\}$$

b) $$\operatorname{Ker}(f) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| -x - \cancel{y} + z = 0, \; -x + \cancel{y} + z = 0, \; \underline{2y = 0} \right\}$$

$$= \operatorname{Span} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \text{base di } \operatorname{Ker}(f) \text{ è } \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Per il teorema della dimensione: $\dim(\operatorname{im}(f)) = 2$

$\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \in \operatorname{im}(f)$ e sono l.i. $\Rightarrow \left\{ \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ è base di $\operatorname{im}(f)$

c) Osserviamo che $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \notin \mathrm{im}(f)$

$$\Rightarrow \ker(f) \cap \mathrm{im}(f) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow \mathbb{R}^3 = \ker(f) \oplus \mathrm{im}(f)$$

d) $$P_f(t) = \det \begin{pmatrix} -\frac{1}{2}-t & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}-t & \frac{1}{2} \\ 0 & 2 & -t \end{pmatrix} =$$

$$= \left(-\frac{1}{2}-t\right) \cdot \left[t^2 - \frac{1}{2}t - 1\right] + \frac{1}{2}\left[\frac{1}{2}t - 1\right]$$

$$= -t^3 \cancel{+t^2\left(-\frac{1}{2}+\frac{1}{2}\right)} + t\left(\frac{1}{4} + 1 + \frac{1}{4}\right) = -t\left(t^2 - \frac{3}{2}\right)$$

$\Rightarrow \mathrm{Sp}(f) = \left\{0, \pm\sqrt{\frac{3}{2}}\right\} \Rightarrow f$ è diagonalizzabile perché ha 3 autovalori distinti.

$$V_0 = \ker(f) = \mathrm{Span}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\frac{-1-\sqrt{6}}{2} =$$

$$V_{\sqrt{\frac{3}{2}}} = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \,\middle|\, \begin{pmatrix} -\frac{1}{2}-\sqrt{\frac{3}{2}} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}-\sqrt{\frac{3}{2}} & \frac{1}{2} \\ 0 & 2 & -\sqrt{\frac{3}{2}} \end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \right\}$$

$$\frac{1-\sqrt{6}}{2} \qquad 4 \quad -\frac{\sqrt{6}}{2}$$

$$= \mathrm{Span}\begin{pmatrix}\frac{-2+\sqrt{6}}{2}\\ \frac{\sqrt{6}}{2}\\ 2\end{pmatrix}$$

$$-\frac{1}{2}x + \frac{1-\sqrt{6}}{2}\cdot\frac{\sqrt{6}}{2} + 1 = 0$$

$$-2x + \sqrt{6} - 6 + 4 = 0 \Rightarrow x = \frac{-2+\sqrt{6}}{2}$$

$$V_{-\sqrt{\frac{3}{2}}} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \begin{pmatrix} -\frac{1}{2}+\frac{\sqrt{6}}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}+\frac{\sqrt{6}}{2} & \frac{1}{2} \\ 0 & 2 & \frac{\sqrt{6}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \operatorname{Span} \begin{pmatrix} \frac{2+\sqrt{6}}{2} \\ \frac{\sqrt{6}}{2} \\ -2 \end{pmatrix}$$

$$-\frac{1}{2}x + \frac{1+\sqrt{6}}{2} \cdot \frac{\sqrt{6}}{2} - 1 = 0$$

$$\Rightarrow -2x + \sqrt{6} + 6 - 4 = 0$$

$$\Rightarrow x = \frac{2+\sqrt{6}}{2}$$

$\Rightarrow$ base diagonalizzante: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{-2+\sqrt{6}}{2} \\ \frac{\sqrt{6}}{2} \\ 2 \end{pmatrix}, \begin{pmatrix} \frac{2+\sqrt{6}}{2} \\ \frac{\sqrt{6}}{2} \\ -2 \end{pmatrix} \right\}$

2\. a) g è simmetrica perché $M_e(g)$ è simmetrica.

$$g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = (x \quad y \quad z) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$= (x \quad y \quad z) \begin{pmatrix} x + y \\ x + 2y + z \\ y + 2z \end{pmatrix} = x^2 + xy + yx + 2y^2 + yz$$

$$+ zy + 2z^2 = (x+y)^2 + (y+z)^2 + z^2 \geq 0 \; \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

ed è $= 0$ se e solo se $x = y = z = 0$

$\Rightarrow$ g è anche definita positiva.

b)

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 ; \right.$$

$$\left. \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0 \right\} =$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x + y = 0 \; ; \; \cancel{2x} + \cancel{3}y + z = 0 \right\}$$

$$= \mathrm{Span} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \text{base di } W \text{ è } \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

c) $\left\|\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\| = \sqrt{g\left(\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)} = 1$

$\Rightarrow \left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}$ è base ortonormale di W.

$$\Rightarrow P_{W}^{\perp}\begin{pmatrix}0\\0\\1\end{pmatrix} = g\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 1 & 0\\1 & 2 & 1\\0 & 1 & 2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix}.$$

3.

Giacitura di π : $\begin{cases} x - y = 0 \\ x + z = 0 \end{cases} = \text{Span}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Giacitura di s : $\begin{cases} x + y - z = 0 \\ y + z = 0 \end{cases} = \text{Span}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\Rightarrow$ La giacitura di $\mathbb{P}$ $= \text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right)$

$$\Rightarrow \mathbb{P} : \begin{cases} x = 1 + t + 2s \\ y = 1 + t - s \\ z = 1 - t + s \end{cases}, \quad t, s \in \mathbb{R}$$