ES. 1)

6) $\{\tilde{q},\tilde{p}\} = \{q,p\} - \varepsilon\{q, \frac{\partial G}{\partial q}\} + \varepsilon\{\frac{\partial G}{\partial p}, p\} + \cancel{\mathcal{O}(\varepsilon^2)}$

$= 1 - \varepsilon\left(\frac{\partial^2 G}{\partial p \partial q}\right) + \varepsilon\left(\frac{\partial^2 G}{\partial q \partial p}\right)$ trascuriamo poichè $\varepsilon \ll 1$

$= 1$ ✓

7) $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \simeq \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{pmatrix}$

$\Rightarrow \delta\vec{q} = \alpha \begin{pmatrix} q_2 \\ -q_1 \\ 0 \end{pmatrix} = \alpha\{q_1 p_2 - q_2 p_1, \vec{q}\}$

$\{\cancel{q_1 p_2} - q_2 p_1, q_1\} = q_2$

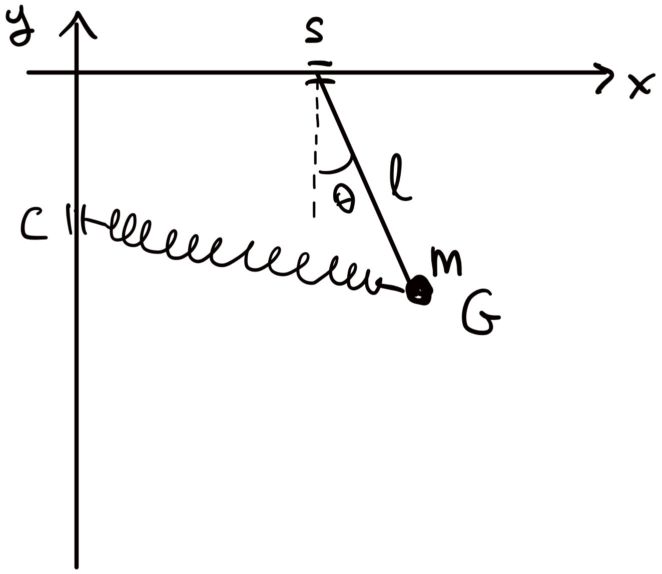
$\{q_1 p_2 - \cancel{q_2 p_1}, q_2\} = -q_1$

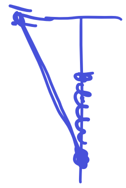
$\{q_1 p_2 - q_2 p_1, q_3\} = 0$

8) $\delta q = \varepsilon \cdot 1 = \varepsilon \frac{\partial G}{\partial p} \Rightarrow G(p,q) = p$

$\frac{dq}{d\varepsilon} = 1 \to q(\varepsilon) = q(0) + \varepsilon$

ES2





[C si muove con moto $y_C(t) = -h(1-\sin\omega t)$ h, ω costanti]

$y_C = -\frac{l}{2}$

$x_G = s + l\sin\theta$

$y_G = -l\cos\theta$

1) $T = \frac{m}{2}\left((\dot s + l\dot\theta\cos\theta)^2 + l^2\dot\theta^2\sin^2\theta\right)$

$V = -mgl\cos\theta + \frac{k}{2}\left[(s + l\sin\theta)^2 + (l\cos\theta + y_C)^2\right]$

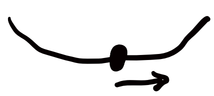
$L = \frac{m}{2}\dot s^2 + \frac{1}{2}ml^2\dot\theta^2 + ml\dot s\dot\theta\cos\theta - V(s,\theta)$

2) $A = \begin{pmatrix} m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{pmatrix}$

$\det A = (ml)^2(1 - \cos^2\theta) \geq 0 \quad (\cos^2\theta \leq 1)$

$\left.\mathrm{Tr}\,A > 0\right\} \Rightarrow A$ def. positiva.

→ A non è strettam. def. pos. p $\theta = 0, \pi$. In qle config. il sist. di coord. non è buono, in quanto non descrive tutte le possibili direzioni



$\frac{\partial \bar r}{\partial s} \propto \frac{\partial \bar r}{\partial \theta}$

3) $\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = \frac{d}{dt}\left(m\dot{s} + ml\dot{\theta}\cos\theta\right) =$

$= m\ddot{s} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2 \operatorname{sen}\theta$

$\frac{\partial L}{\partial s} = -ks - kl\operatorname{sen}\theta$

$\rightarrow \quad \ddot{s} + l\ddot{\theta}\cos\theta = l\dot{\theta}^2\operatorname{sen}\theta - \frac{k}{m}s - \frac{kl}{m}\operatorname{sen}\theta$

4) $k=0 \Rightarrow s$ è coord. ciclica

$\Rightarrow p_s = m\dot{s} + ml\dot{\theta}\cos\theta$ è cost. del moto.

L indip. da $t \Rightarrow E = \frac{m}{2}\left(\dot{s}^2 + l^2\dot{\theta}^2 + 2l\dot{s}\dot{\theta}\cos\theta\right)$
$- mgl\cos\theta + \frac{k}{2}\left(s^2 + 2sl\operatorname{sen}\theta - l^2\cos\theta\right)$

è cost. del moto.

5) $\partial_s V = k(s + l\,\text{sen}\,\theta) = 0 \qquad s = -l\,\text{sen}\,\theta$

$$\partial_\theta V = mgl\,\text{sen}\,\theta + k(s + l\,\text{sen}\,\theta)l\cos\theta - k(l\cos\theta + y_c)l\,\text{sen}\,\theta$$

$s + l\,\text{sen}\,\theta = 0$

$$mgl\,\text{sen}\,\theta - kl\left(l\cos\theta - \frac{l}{2}\right)\text{sen}\,\theta =$$

$$= kl^2\,\text{sen}\,\theta\left(\frac{mg}{kl} + \frac{1}{2} - \cos\theta\right) = 0$$

$\text{sen}\,\theta = 0 \quad \vee \quad \cos\theta = \frac{mg}{kl} + \frac{1}{2} \quad \exists$ se $\frac{mg}{kl} < \frac{1}{2}$

Soluz. eq.: $(s,\theta) = (0,0),\ (0,\pi),\ (-l\,\text{sen}\,\theta^*, \theta^*),\ (l\,\text{sen}\,\theta^*, -\theta^*)$

con $\cos\theta^* = \frac{mg}{kl} + \frac{1}{2}$

$$\partial^2 V = \begin{pmatrix} k & kl\cos\theta \\ kl\cos\theta & \left(mgl + \frac{kl^2}{2}\right)\cos\theta - kls\,\text{sen}\,\theta \end{pmatrix}$$

$s = -l\,\text{sen}\,\theta$: $\quad kl^2\left[\left(\frac{mg}{kl} + \frac{1}{2}\right)\cos\theta + 1 - \cos^2\theta\right]$

$$\partial^2 V_{(0,\,{}^0_\pi)} = \begin{pmatrix} k & \pm kl \\ \pm kl & \pm\left(mgl + \frac{kl^2}{2}\right) \end{pmatrix} \to \det = \pm (kl)^2\left(\frac{mg}{kl} + \frac{1}{2}\right) - (kl)^2$$

$(0,\pi)$ INSTAB.

$(0,0)$ STAB se $\frac{mg}{kl} - \frac{1}{2} > 0$, INSTAB altrimenti

$$\partial^2 V_{(s^*,\theta^*)} = \begin{pmatrix} k & kl\left(\frac{mg}{kl} + \frac{1}{2}\right) \\ kl\left(\frac{mg}{kl} + \frac{1}{2}\right) & kl^2\left[\cancel{\left(\frac{mg}{kl} + \frac{1}{2}\right)^2} + 1 - \cancel{\left(\frac{mg}{kl} + \frac{1}{2}\right)^2}\right] \end{pmatrix}$$

$$\det = (kl)^2 - (kl)^2\left(\frac{mg}{kl} + \frac{1}{2}\right) = (kl)^2\left(\frac{1}{2} - \frac{mg}{kl}\right)$$

STAB. per $\frac{mg}{kl} < \frac{1}{2}$ cioè qndo esistono.

6) Freq. piccole oscillaz. attorno (s^*, θ^*)

$$A = \begin{pmatrix} m & ml\left(\frac{mg}{kl}+\frac{1}{2}\right) \\ ml\left(\frac{mg}{kl}+\frac{1}{2}\right) & ml^2 \end{pmatrix} = m \begin{pmatrix} 1 & l\cos\theta^* \\ l\cos\theta^* & l^2 \end{pmatrix}$$

$$B = \begin{pmatrix} k & kl\left(\frac{mg}{kl}+\frac{1}{2}\right) \\ kl\left(\frac{mg}{kl}+\frac{1}{2}\right) & kl^2 \end{pmatrix} = k \begin{pmatrix} 1 & l\cos\theta^* \\ l\cos\theta^* & l^2 \end{pmatrix}$$

$$\det(B-\lambda A) = \det\begin{pmatrix} k-\lambda m & * \\ kl\left(\frac{mg}{kl}+\frac{1}{2}\right)-\lambda ml\left(\frac{mg}{kl}+\frac{1}{2}\right) & kl^2-\lambda ml^2 \end{pmatrix}$$

$$(ml)^2\left(\frac{k}{m}-\lambda\right)^2 - \left(\frac{mg}{kl}+\frac{1}{2}\right)^2\left(\frac{k}{m}-\lambda\right)^2(ml)^2 = 0$$

$$\lambda_{1,2} = \frac{k}{m}$$

$$B-\lambda A = (k-\lambda m)\begin{pmatrix} 1 & l\cos\theta^* \\ l\cos\theta^* & l^2 \end{pmatrix}$$

7) $\cos\theta = \frac{1}{2} \leadsto \theta^* = \pi/3 \Rightarrow s^* = -l\,\mathrm{sen}\,\theta^* = -\frac{l\sqrt{3}}{2}$

Config. stab: $(s,\theta) = (\pm s^*, \pm\theta^*)$

$$\alpha = \frac{1}{2} + \frac{mg}{k\ell} = \frac{1}{2} \quad \text{per } g=0$$

$$A = m \begin{pmatrix} L^2 & L\alpha \\ L\alpha & 1 \end{pmatrix} \qquad B = \frac{k}{m} \cdot m \begin{pmatrix} L^2 & L\alpha \\ L\alpha & 1 \end{pmatrix} = \frac{k}{m} A$$

$$A = m \begin{pmatrix} 1/L & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1/L & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} \rightsquigarrow \text{Autovalori} \quad \det \begin{pmatrix} 1-\lambda & \alpha \\ \alpha & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - \alpha^2$$

$$\lambda_{1,2} = 1 \pm \alpha$$

autovett.

$$\lambda_1 = 1+\alpha \qquad \begin{pmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\lambda_2 = 1-\alpha \qquad \begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

$$U = U^{-1}$$

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$$

$$\Rightarrow \quad A = m \begin{pmatrix} 1/L & 0 \\ 0 & 1 \end{pmatrix} U^T \begin{pmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} U \begin{pmatrix} 1/L & 0 \\ 0 & 1 \end{pmatrix}$$

$$= m \begin{pmatrix} 1/L & 0 \\ 0 & 1 \end{pmatrix} U^T \begin{pmatrix} \sqrt{1+\alpha} & \\ & \sqrt{1-\alpha} \end{pmatrix} \mathbb{1} \begin{pmatrix} \sqrt{1+\alpha} & \\ & \sqrt{1-\alpha} \end{pmatrix} U \begin{pmatrix} 1/L & 0 \\ 0 & 1 \end{pmatrix}$$

$$U' = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+\alpha} & 0 \\ 0 & \sqrt{1-\alpha} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/L & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{1+\alpha}}{L} & \sqrt{1+\alpha} \\ \frac{\sqrt{1-\alpha}}{2} & -\sqrt{1-\alpha} \end{pmatrix}$$

$$L_{lin} = \frac{1}{2}(\delta\dot{\theta}, \delta\dot{s})\, A \begin{pmatrix} \delta\dot{\theta} \\ \delta\dot{s} \end{pmatrix} - \frac{1}{2}(\delta\theta, \delta s)\, B \begin{pmatrix} \delta\theta \\ \delta s \end{pmatrix} =$$

$$\frac{k}{m} \equiv \omega^2 \qquad\qquad B = \frac{k}{m} A$$

$$= \frac{m}{2}(\delta\dot{\theta}, \delta\dot{s})\, U'^T U' \begin{pmatrix} \delta\dot{\theta} \\ \delta\dot{s} \end{pmatrix} - \frac{m}{2}\omega^2 (\delta\theta, \delta s)\, U'^T U' \begin{pmatrix} \delta\theta \\ \delta s \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \equiv U' \begin{pmatrix} \delta\theta \\ \delta s \end{pmatrix}$$

$$L_{lin} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} m\omega^2 (x^2 + y^2)$$

ES 3)

$$E_n^\theta = \frac{\hbar^2}{2mR^2}\left(n - \frac{\theta}{2\pi}\right)^2 \qquad n \in \mathbb{Z}$$

3) Sì, perché $\frac{e^{in\varphi}}{\sqrt{2\pi}} \in L^2(\underset{\text{cerchio}}{S^1})$ $(\neq \mathbb{R})$

4) $$\psi(\varphi) = \frac{1}{\sqrt{\pi}}\cos\varphi = \frac{1}{\sqrt{2}}\left(\underset{n=1}{\frac{e^{i\varphi}}{\sqrt{2\pi}}} + \underset{n=-1}{\frac{e^{-i\varphi}}{\sqrt{2\pi}}}\right)$$

$$E_{n=1}^\theta = \frac{\hbar^2}{2mR^2}\left(1 - \frac{\theta}{2\pi}\right)^2$$

$$E_{n=-1}^\theta = \frac{\hbar^2}{2mR^2}\left(1 + \frac{\theta}{2\pi}\right)^2$$

$$\psi(\varphi,t) = \frac{1}{\sqrt{2}}\left(\frac{e^{-iE_{n=1}^\theta t/\hbar}\, e^{i\varphi}}{\sqrt{2\pi}} + \frac{e^{-iE_{n=-1}^\theta t/\hbar}}{\sqrt{2\pi}}\right)$$

5) Stato fondamentale per $\theta = 0 \Longleftrightarrow n = 0$

$$\psi_0(\varphi) = \frac{1}{\sqrt{2\pi}}$$

$\hat{p}\,\psi_0(\varphi) = 0$ perché $\hat{p}$ è operatore derivata e $\psi_0(\varphi)$ è costante in φ.

$\Rightarrow \langle p^2 \rangle_{\psi_0} = 0$.