Corso di Laurea in Fisica - UNITS

Istituzioni di Fisica per il Sistema Terra

Mass, Bernoulli, and Energy Equations

FABIO ROMANELLI

Department of Mathematics & Geosciences
University of Trieste
romanel@units.it

Introduction

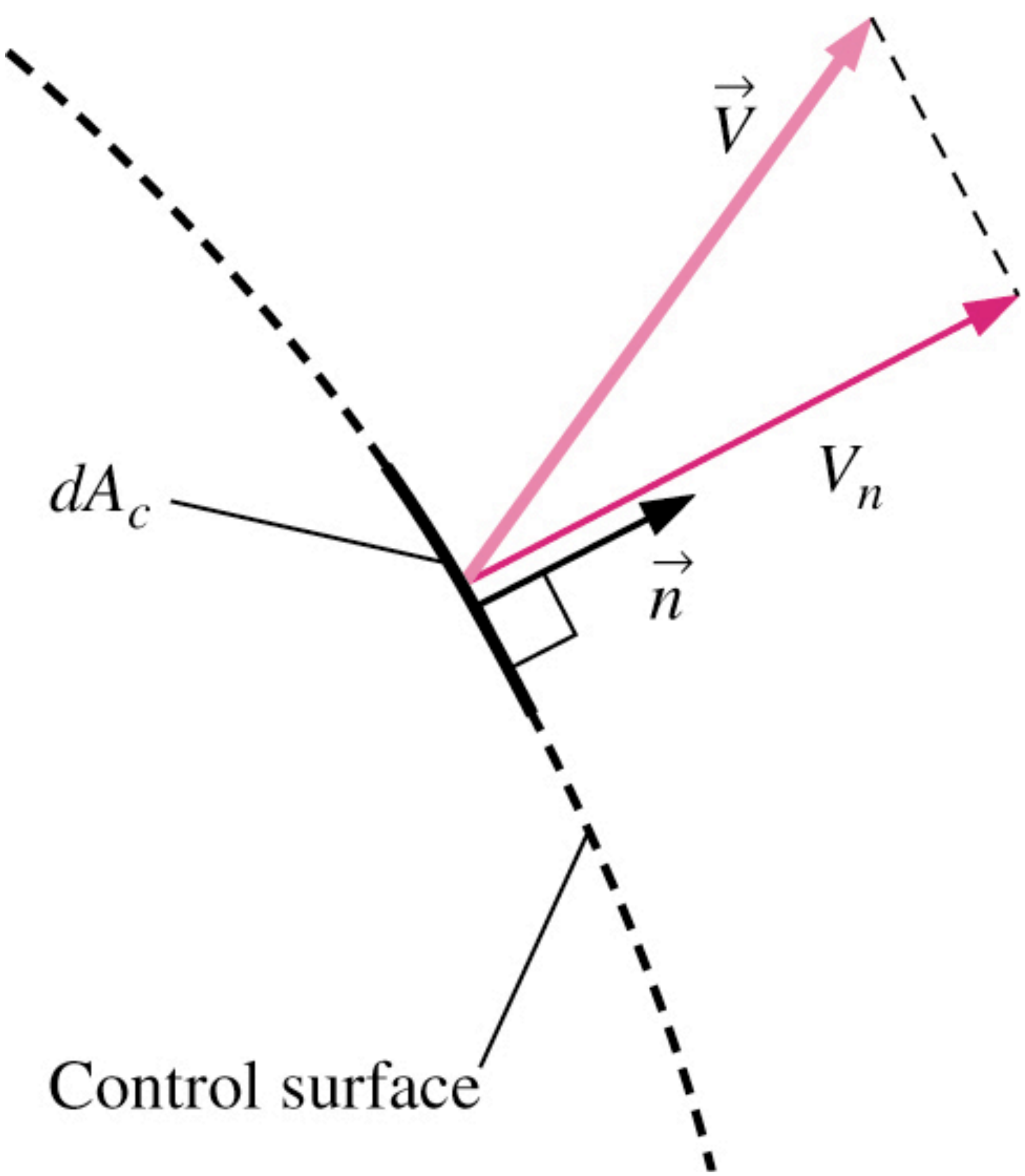
This section deals with 3 equations commonly used in fluid mechanics:

- The mass equation is an expression of the conservation of mass principle.
- The Bernoulli equation is concerned with the conservation of kinetic, potential, and flow energies of a fluid stream and their conversion to each other.
- The energy equation is a statement of the conservation of energy principle.

Conservation of Mass

- Conservation of mass principle is one of the most fundamental principles in nature.
- Mass, like energy, is a conserved property, and it cannot be created or destroyed during a process.
- For closed systems mass conservation is implicit since the mass of the system remains constant during a process.
- For control volumes, mass can cross the boundaries which means that we must keep track of the amount of mass entering and leaving the control volume.

Mass and Volume Flow Rates



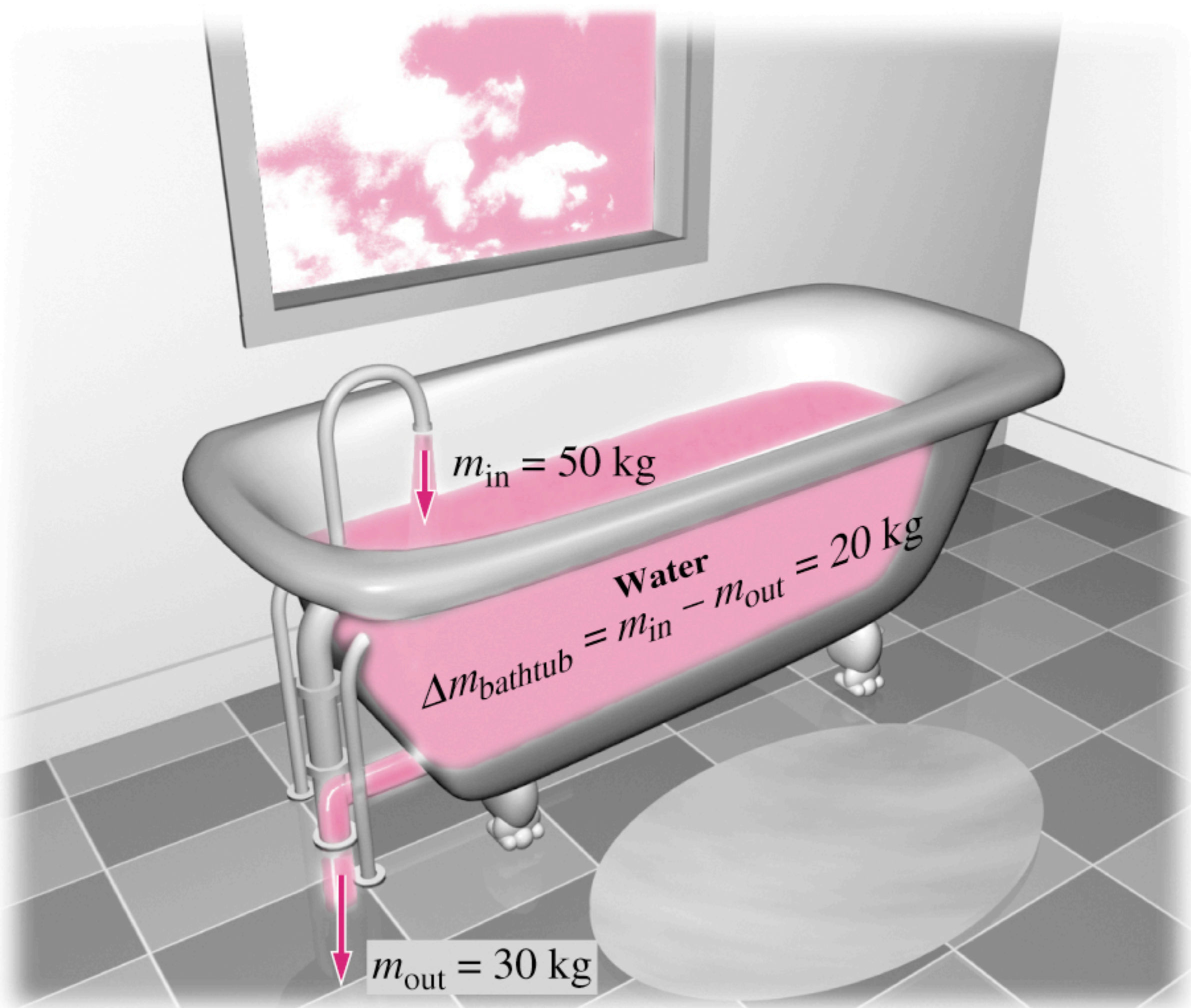
- The amount of mass flowing through a control surface per unit time is called the mass flow rate and is denoted $\dot{m}$
  - The dot over a symbol is used to indicate time rate of change.
- Mass flow rate across the entire cross-sectional area of a pipe or duct is obtained by integration

$$\dot{m} = \int_{A_c} \delta \dot{m} = \int_{A_c} \rho V_n \, dA_c$$

- While this expression for $\dot{m}$ is exact, it is not always convenient for engineering analyses.

Conservation of Mass Principle

$$\begin{pmatrix}\text{Total mass entering}\\ \text{the CV during } \Delta t\end{pmatrix} - \begin{pmatrix}\text{Total mass leaving}\\ \text{the CV during } \Delta t\end{pmatrix} = \begin{pmatrix}\text{Net change in mass}\\ \text{within the CV during } \Delta t\end{pmatrix}$$

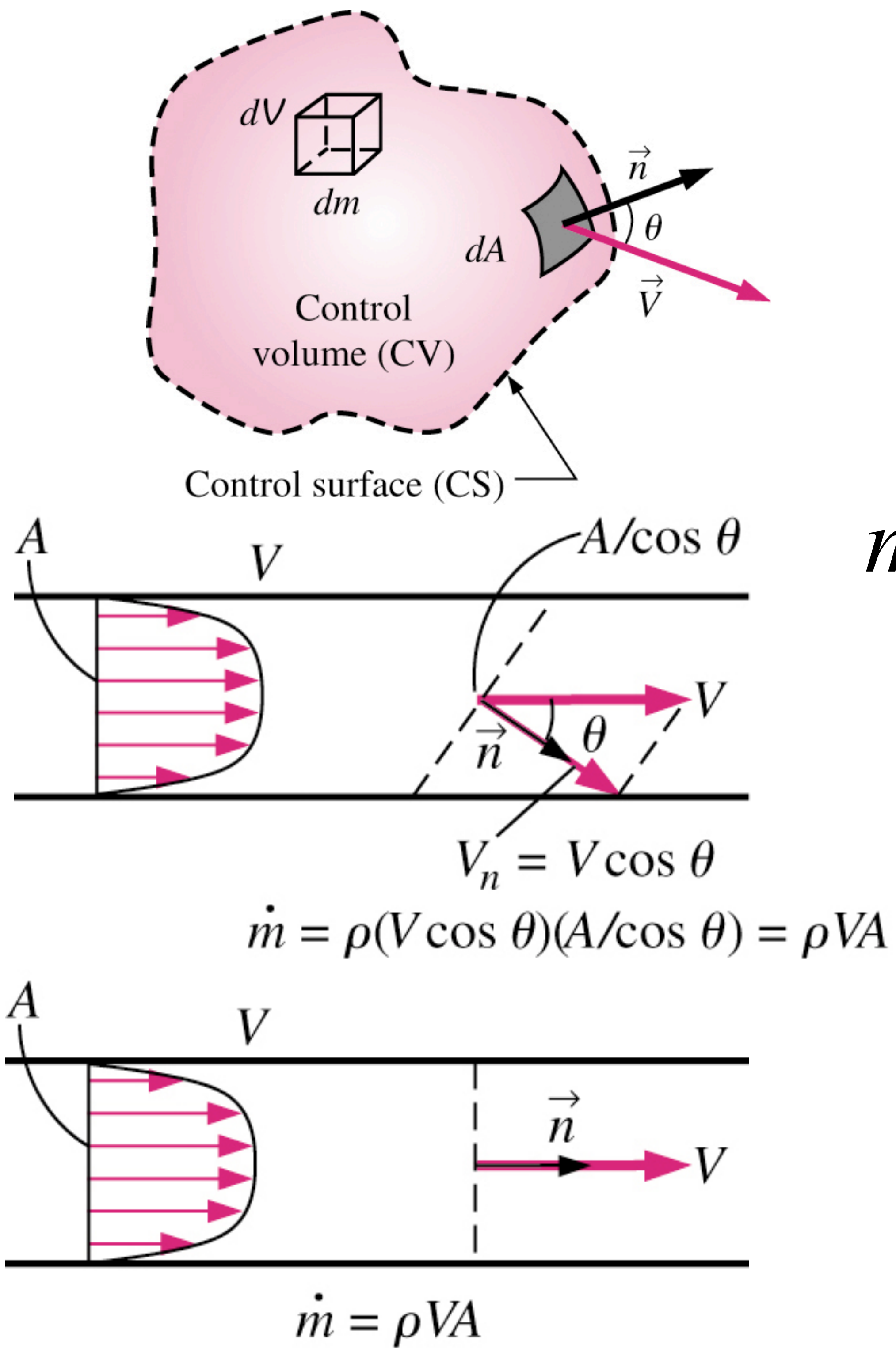


- The conservation of mass principle can be expressed as

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}$$

- Where $\dot{m}_{in}$ and $\dot{m}_{out}$ are the total rates of mass flow into and out of the CV,
and dm_{CV}/dt is the rate of change of mass within the CV.

Conservation of Mass Principle



- For CV of arbitrary shape:
  - rate of change of mass within the CV

$$\frac{dm_{CV}}{dt} = \frac{d}{dt}\int_{CV} \rho \, d\mathcal{V}$$

  - net mass flow rate

$$\dot{m}_{net} = \int_{CS} \delta \dot{m} = \int_{CS} \rho V_n \, dA = \int_{CS} \rho \left(\vec{V} \bullet \vec{n}\right) dA$$

- Therefore, general conservation of mass for a fixed CV is:

$$\frac{d}{dt}\int_{CV} \rho \, d\mathcal{V} + \int_{CS} \rho \left(\vec{V} \bullet \vec{n}\right) dA = 0$$

- It states that the time rate of change of mass within the control volume plus the net mass flow rate through the control surface is equal to zero.

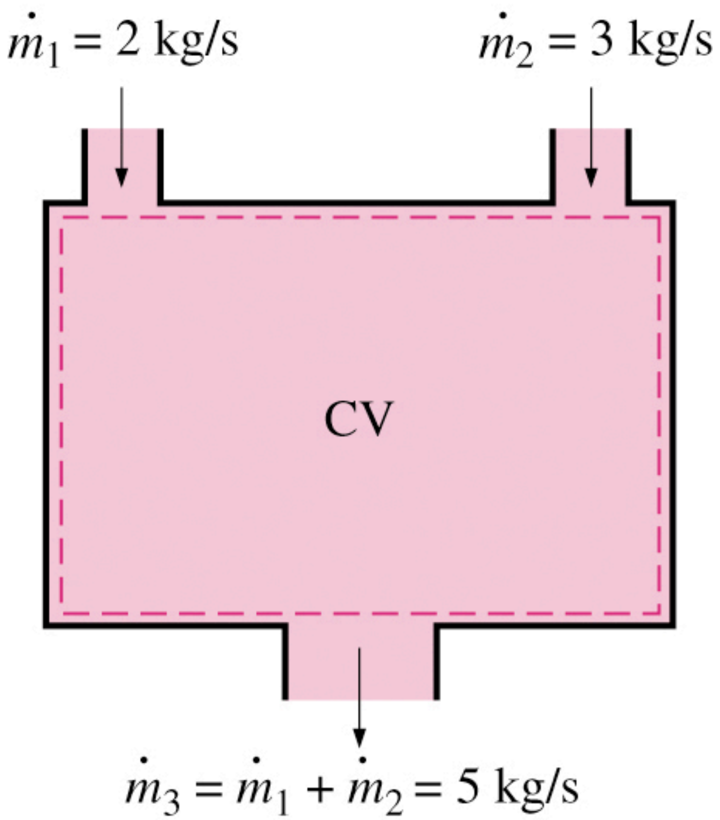
Conservation of Mass Principle

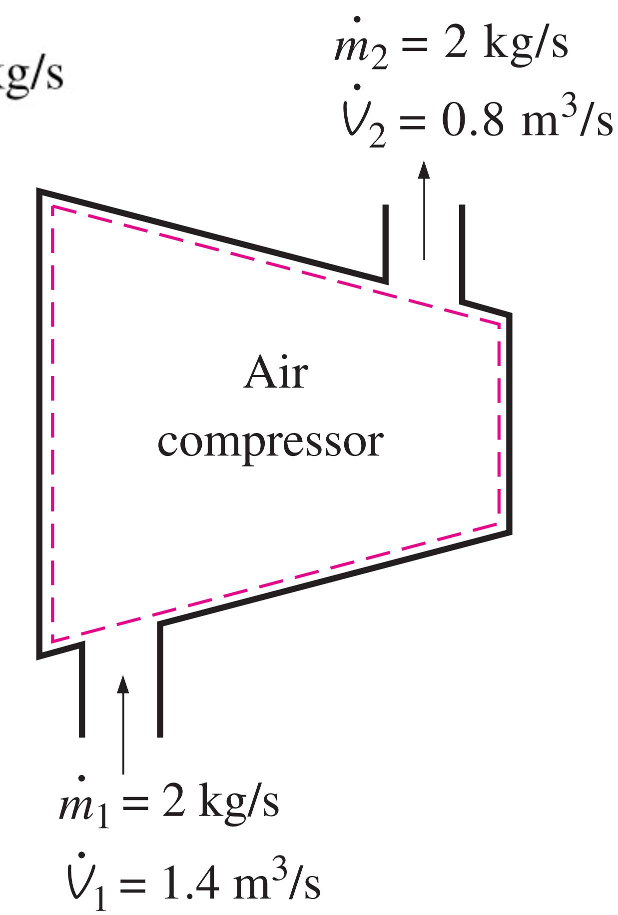
- The general conservation of mass relation for a control volume can also be derived using RTT by taking the property B to be the mass m and then b = 1
- Using the definition of mass rate:

$$\frac{d}{dt}\int_{CV}\rho\, d\mathcal{V} = \sum_{in}\dot{m} - \sum_{out}\dot{m} = \frac{dm_{CV}}{dt}$$

- It is also valid for moving or deforming control volumes provided that the absolute velocity is replaced by the relative velocity, which is the fluid velocity relative to the control surface.

Steady-Flow Processes





- For steady flow, the total amount of mass contained in CV is constant.
- Total amount of mass entering must be equal to total amount of mass leaving

$$\sum_{in} \dot{m} = \sum_{out} \dot{m}$$

- For incompressible flows,

$$\sum_{in} V_n A_n = \sum_{out} V_n A_n$$

Mechanical Energy

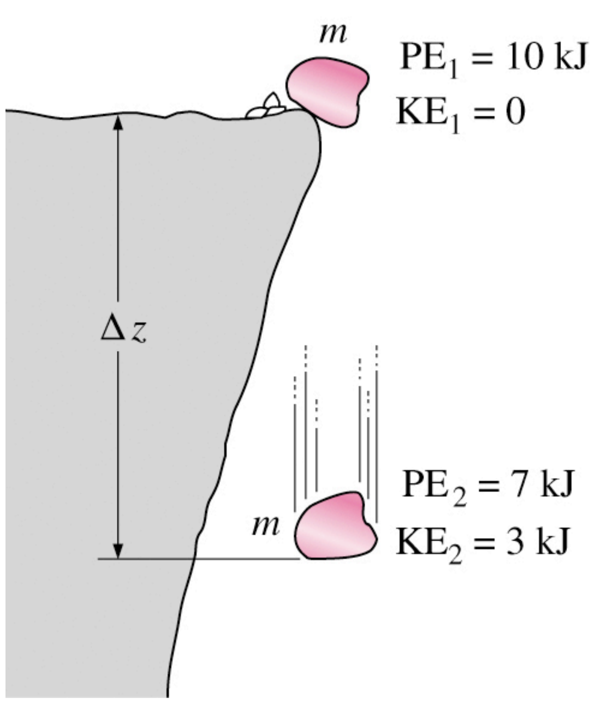
- Mechanical energy can be defined as the form of energy that can be converted to mechanical work completely and directly by an ideal mechanical device such as an ideal turbine.
- Flow P/ρ, kinetic V^2/g, and potential gz energy are the forms of mechanical energy $e_{mech} = P/\rho + V^2/g + gz$
- Mechanical energy change of a fluid during incompressible flow becomes

$$\Delta e_{mech} = \frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g\left(z_2 - z_1\right)$$

- In the absence of losses, Δe_{mech} represents the work supplied to the fluid ($\Delta e_{mech} > 0$) or extracted from the fluid ($\Delta e_{mech} < 0$).

General Energy Equation

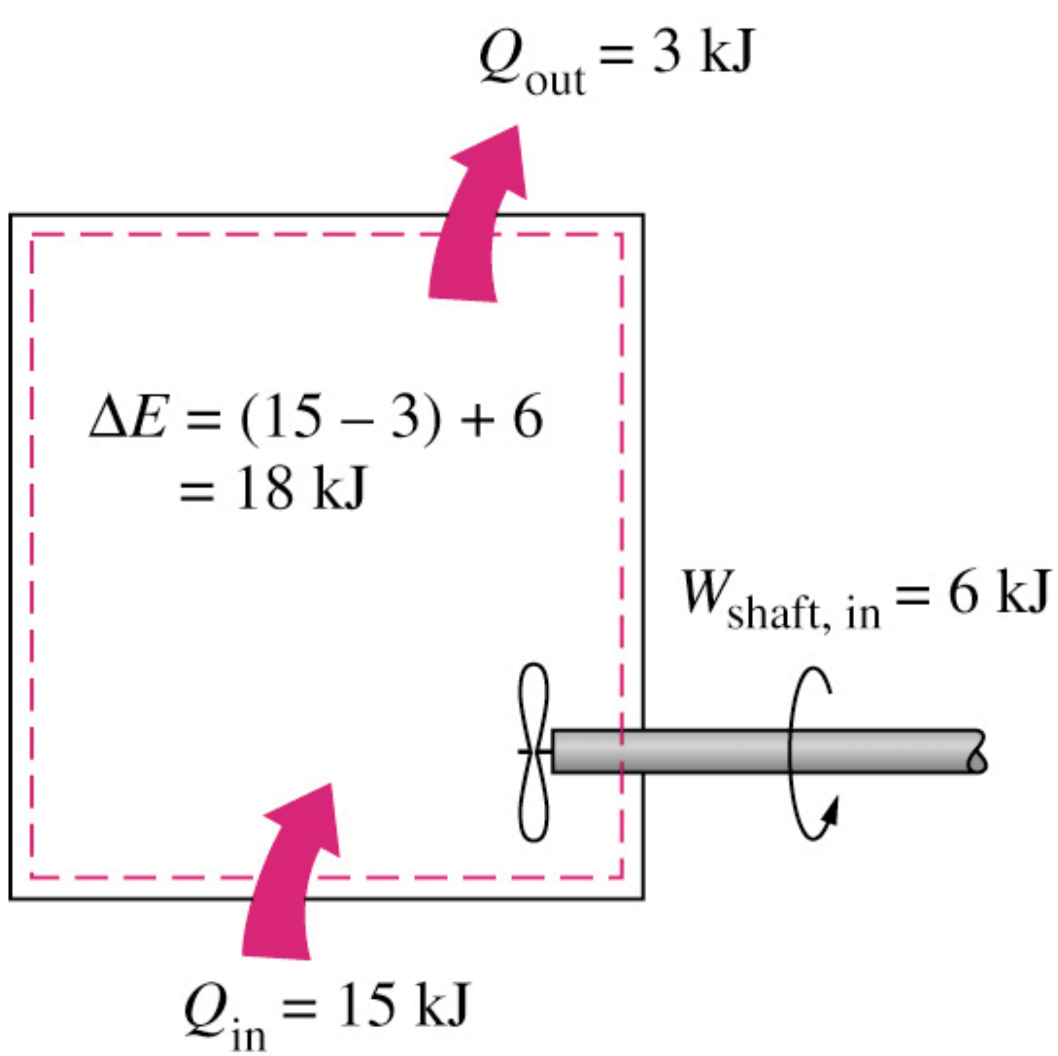
- One of the most fundamental laws in nature is the 1st law of thermodynamics, which is also known as the conservation of energy principle.
- It states that energy can be neither created nor destroyed during a process; it can only change forms
- The change in the energy content of a system is equal to the difference between the energy input and the energy output, and the conservation of energy principle for any system can be expressed simply as:



$$\Delta E_{sys} = E_{in} - E_{out}$$

Falling rock, picks up speed as PE is converted to KE.
If air resistance is neglected,
PE + KE = constant

General Energy Equation



- The energy content of a closed system can be changed by two mechanisms: heat transfer Q and work transfer W.
- Conservation of energy for a closed system can be expressed in rate form as

$$\dot{Q}_{net,in} + \dot{W}_{net,in} = \frac{dE_{sys}}{dt}$$

- Net rate of heat transfer to the system:

$$\dot{Q}_{net,in} = \dot{Q}_{in} - \dot{Q}_{out}$$

- Net power input to the system:

$$\dot{W}_{net,in} = \dot{W}_{in} - \dot{W}_{out}$$

General Energy Equation

- Recall general RTT

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b d\mathcal{V} + \int_{CS} \rho b \left(\vec{V}_r \bullet \vec{n}\right) dA$$

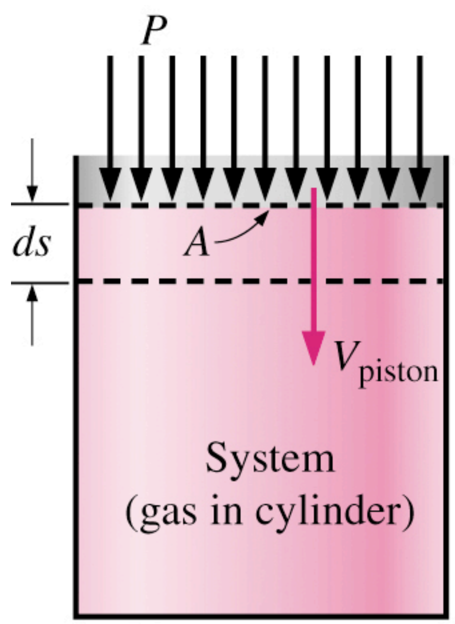
- "Derive" energy equation using B=E and b=e

$$\frac{dE_{sys}}{dt} = \dot{Q}_{net,in} + \dot{W}_{net,in} = \frac{d}{dt}\int_{CV} \rho e d\mathcal{V} + \int_{CS} \rho e \left(\vec{V}_r \bullet \vec{n}\right) dA$$

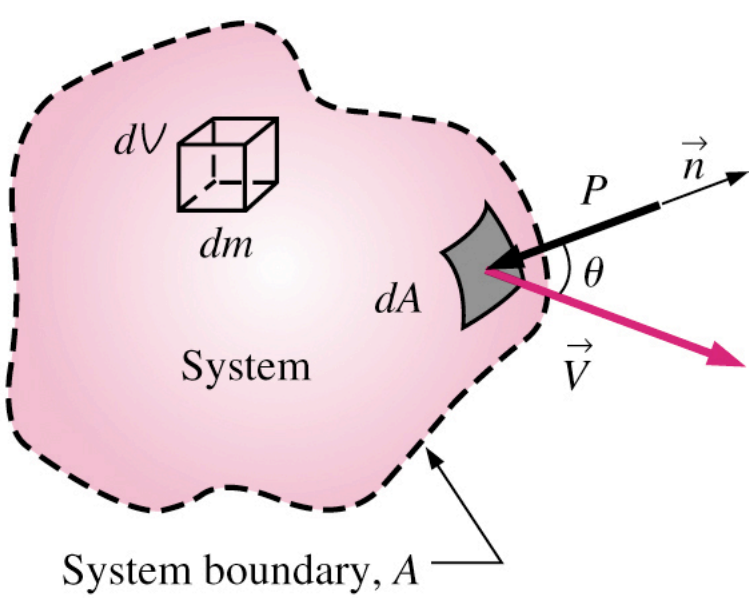
- Break power into rate of shaft and pressure work

$$\dot{W}_{net,in} = \dot{W}_{shaft,net,in} + \dot{W}_{pressure,net,in} = \dot{W}_{shaft,net,in} - \int P \left(\vec{V} \bullet \vec{n}\right) dA$$

Work done by Pressure Forces



(a)



(b)

- When piston moves down *ds* under the influence of *F=PA*, the work done on the system is $\delta W_{boundary}=PAds$.
- If we divide both sides by *dt*, we have

$$\delta \dot{W}_{pressure} = \delta \dot{W}_{boundary} = PA\frac{ds}{dt} = PAV_{piston}$$

- For generalized control volumes:

$$\delta \dot{W}_{pressure} = -PdAV_n = -PdA\left(\vec{V}\cdot\vec{n}\right)$$

- Note sign conventions:
  - $\vec{n}$ is outward pointing normal
  - Negative sign ensures that work done is positive when is done *on* the system.

General Energy Equation

$$\begin{pmatrix}\text{The net rate of energy}\\ \text{transfer into a CV by}\\ \text{heat and work transfer}\end{pmatrix} = \begin{pmatrix}\text{The time rate of}\\ \text{change of the energy}\\ \text{content of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of}\\ \text{energy out of the control}\\ \text{surface by mass flow}\end{pmatrix}$$

- Moving integral for rate of pressure work to RHS of energy equation results in:

$$\dot{Q}_{net,in} + \dot{W}_{shaft,net,in} = \frac{d}{dt}\int_{CV} \rho e \, d\mathcal{V} + \int_{CS}\left(\frac{P}{\rho} + e\right) e\left(\vec{V}_r \cdot \vec{n}\right) dA$$

- Recall that P/ρ is the flow work, which is the work associated with pushing a fluid into or out of a CV per unit mass.

General Energy Equation

- As with the mass equation, practical analysis is often facilitated as averages across inlets and exits

$$\dot{Q}_{net,in} + \dot{W}_{shaft,net,in} = \frac{d}{dt}\int_{CV} \rho e \, d\forall + \sum_{out} \dot{m}\left(\frac{P}{\rho} + e\right) - \sum_{in} \dot{m}\left(\frac{P}{\rho} + e\right)$$

$$\dot{m} = \int_{A_C} \rho\left(\vec{V} \cdot \vec{n}\right) dA_c$$

- Since e=u+ke+pe = u+V^2/2+gz

$$\dot{Q}_{net,in} + \dot{W}_{shaft,net,in} =$$

$$= \frac{d}{dt}\int_{CV} \rho e \, d\forall + \sum_{out} \dot{m}\left(\frac{P}{\rho} + u + \frac{V^2}{2} + gz\right) - \sum_{in} \dot{m}\left(\frac{P}{\rho} + u + \frac{V^2}{2} + gz\right)$$

The Bernoulli Equation

- If we neglect piping losses, and have a system without pumps or turbines, we get Bernoulli equation, in terms of pressure, velocity and elevation heads:

$$\frac{P_1}{\rho_1 g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2$$

- It can also be derived using Newton's second law of motion, and written as:

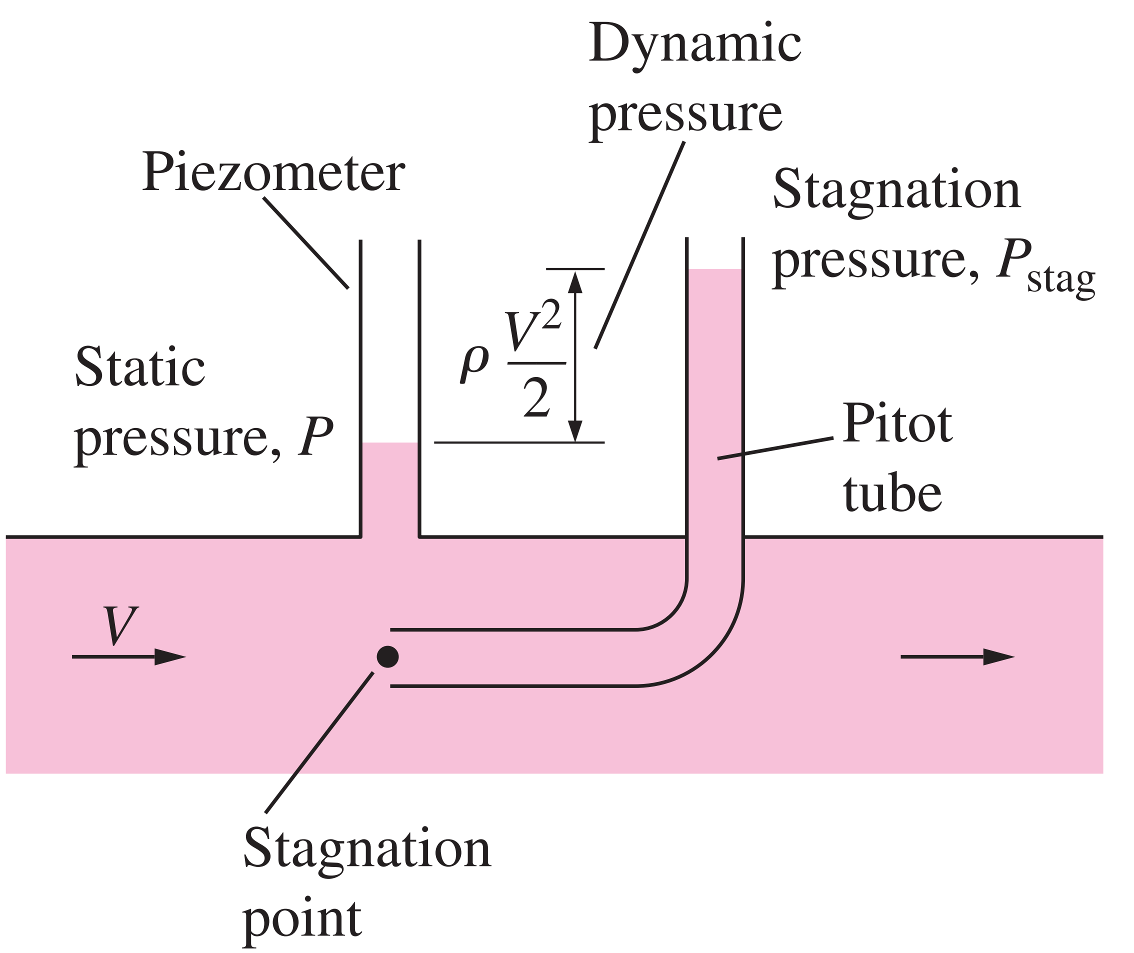
$$P + \rho\frac{V^2}{2} + \rho g z = \text{constant}$$

- The three terms correspond to: static, dynamic, and hydrostatic pressure.

The Bernoulli Equation

- The sum of the static, dynamic, and hydrostatic pressures is called the total pressure. The Bernoulli equation states that the total pressure along a streamline is constant.
- The sum of the static and dynamic pressures is called the stagnation pressure, and it is expressed as

$$P_{stag} = P + \rho \frac{V^2}{2}$$

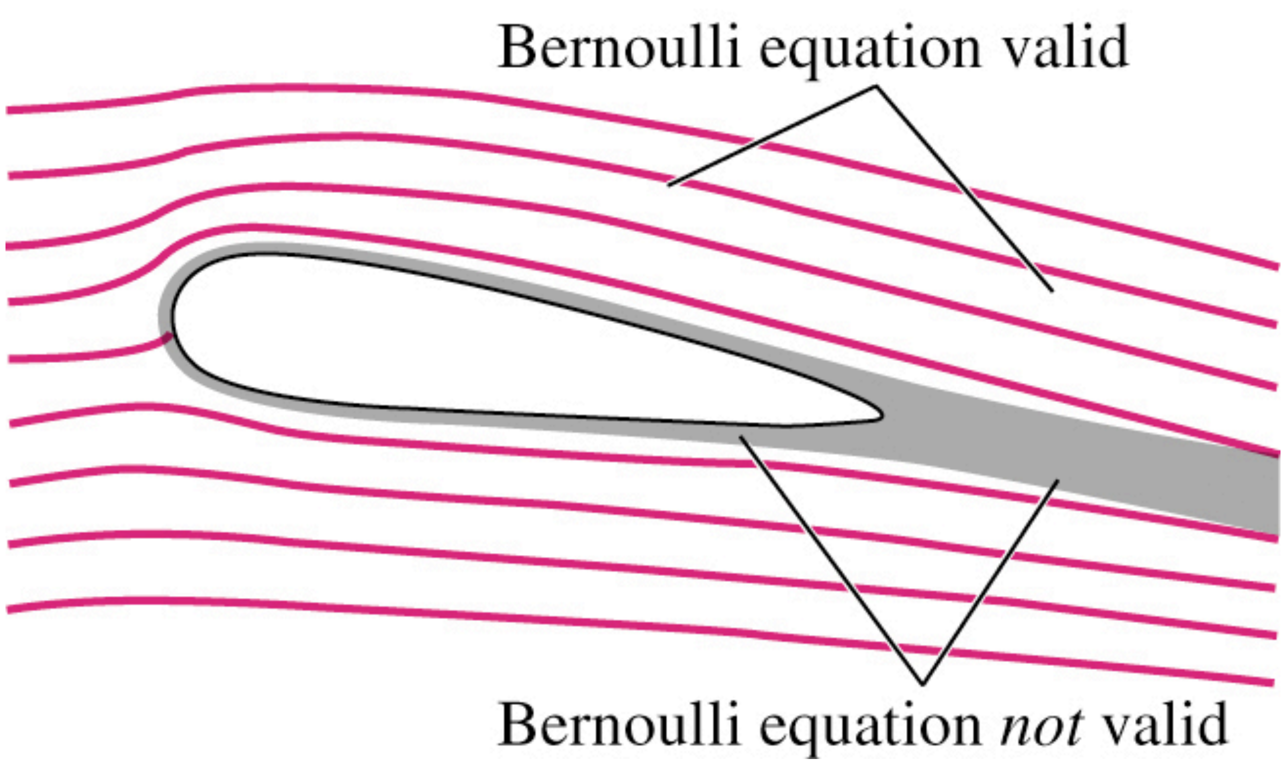


$$V = \sqrt{\frac{2(P_{stag} - P)}{\rho}}$$

The Bernoulli Equation

- Limitations on the use of the Bernoulli Equation
  - Steady flow: d/dt = 0
  - Frictionless flow
  - No shaft work: $w_{pump}=w_{turbine}=0$
  - Incompressible flow: ρ = constant
  - No heat transfer: $q_{net,in}=0$
  - Applied along a streamline (except for irrotational flow)

The Bernoulli Equation



- The Bernoulli equation is an approximate relation between pressure, velocity, and elevation and is valid in regions of steady, incompressible flow where net frictional forces are negligible.
- Equation is useful in flow regions outside of boundary layers and wakes.