Corso di Laurea in Fisica - UNITS

Istituzioni di Fisica per il Sistema Terra

Momentum Analysis of Flow Systems

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Introduction

- Fluid flow problems can be analyzed using one of three basic approaches: differential, experimental, and integral (or control volume, CV).
- Control volume forms of the mass and energy equation were developed and used.
- In this section, we complete control volume analysis by presenting the integral momentum equation.
  - Review Newton's laws and conservation relations for momentum.
  - Use RTT to develop linear and angular momentum equations for control volumes.
  - Use these equations to determine forces and torques acting on the CV.

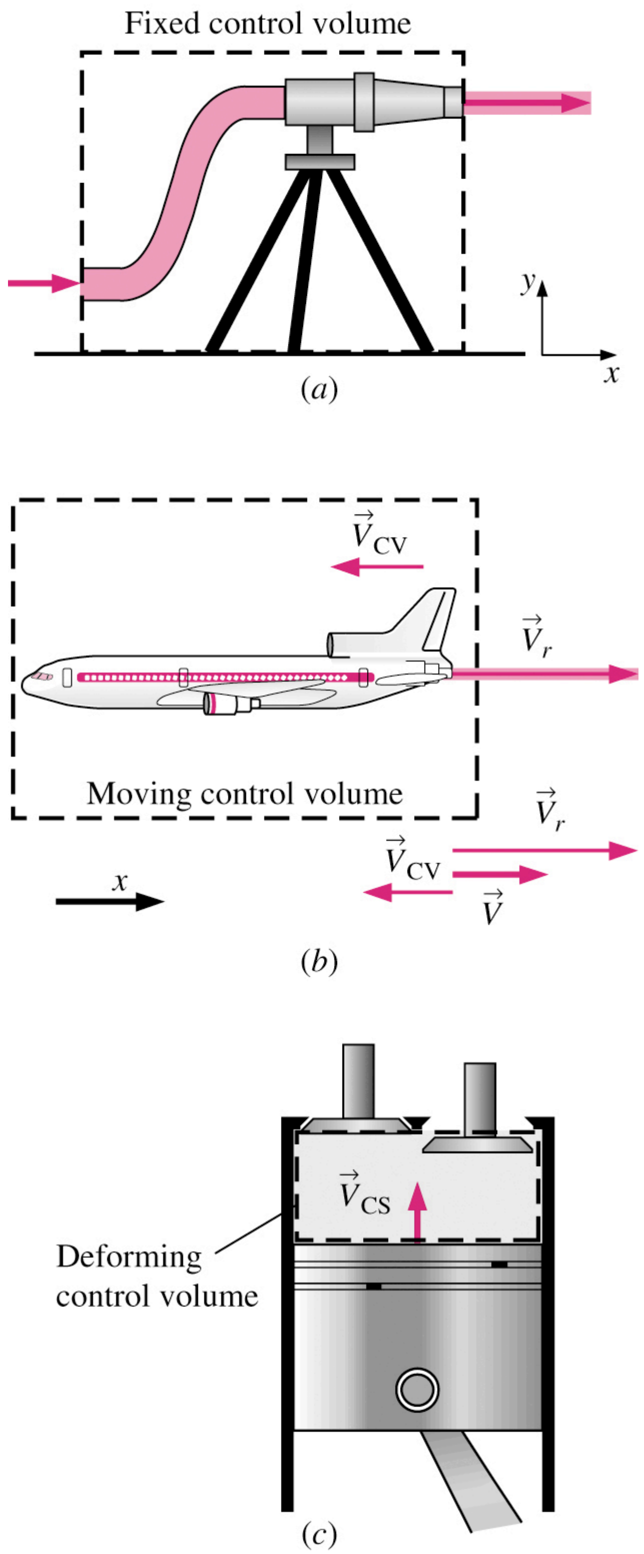
Newton's Laws

- Newton's laws are relations between motions of bodies and the forces acting on them.
  - First law: a body at rest remains at rest, and a body in motion remains in motion at the same velocity in a straight path when the net force acting on it is zero.
  - Second law: the acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass.

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d\left(m\vec{V}\right)}{dt}$$

  - Third law: when a body exerts a force on a second body, the second body exerts an equal and opposite force on the first.

Choosing a Control Volume



- CV is arbitrarily chosen by fluid dynamicist, however, selection of CV can either simplify or complicate analysis.
  - Clearly define all boundaries. Analysis is often simplified if CS is normal to flow direction.
  - Clearly identify all fluxes crossing the CS.
  - Clearly identify forces and torques *of interest* acting on the CV and CS.
- Fixed, moving, and deforming control volumes.
  - For moving CV, use relative velocity,

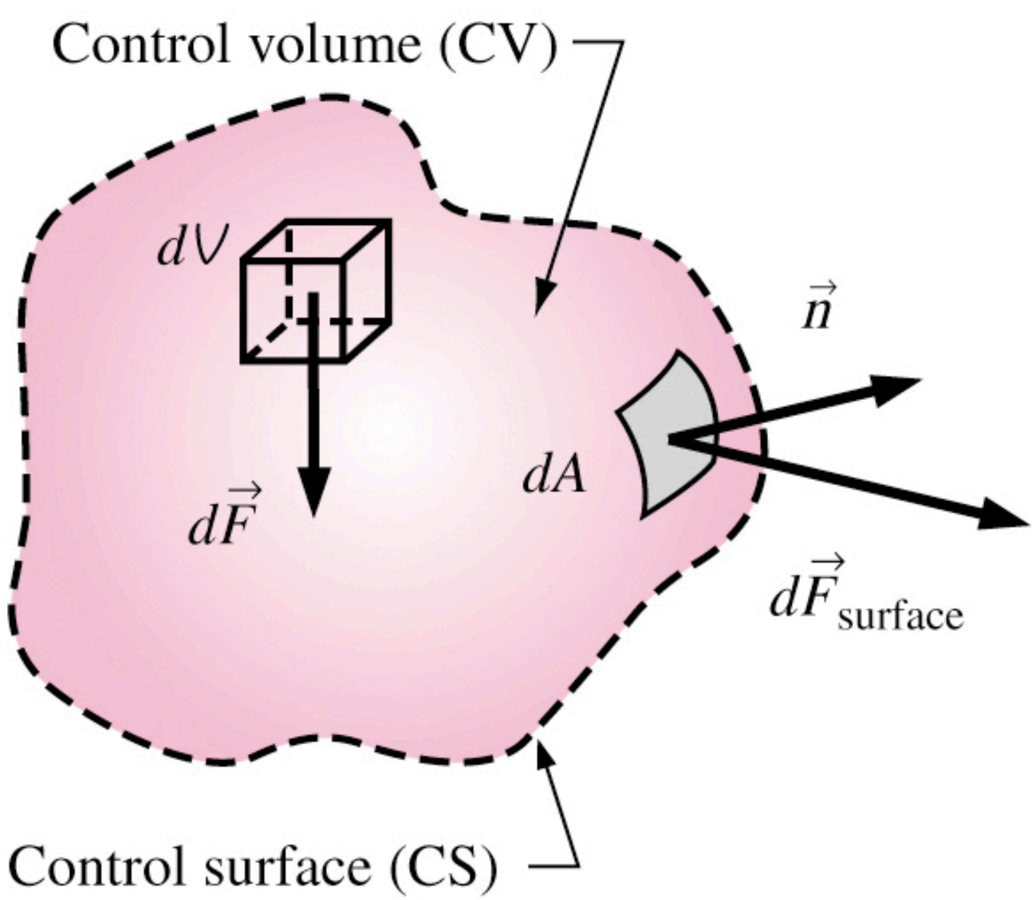
$$\vec{V}_r = \vec{V} - \vec{V}_{CV}$$

  - For deforming CV, use relative velocity all deforming control surfaces,

$$\vec{V}_r = \vec{V} - \vec{V}_{CS}$$

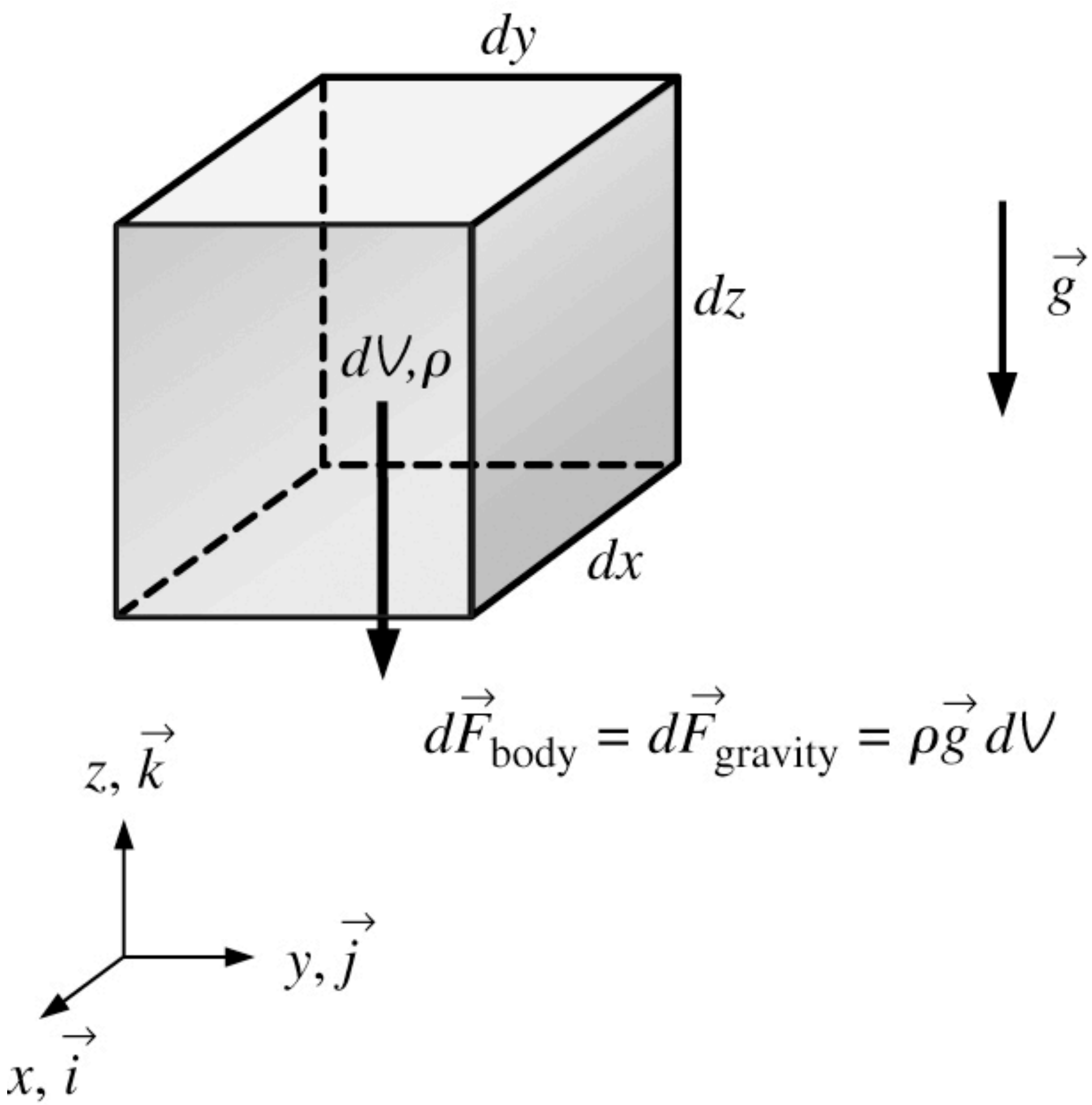
Forces Acting on a CV

- Forces acting on CV consist of **body forces** that act throughout the entire body of the CV (such as gravity, electric, and magnetic forces) and **surface forces** that act on the control surface (such as pressure and viscous forces, and reaction forces at points of contact).



- Body forces act on each volumetric portion *dV* of the CV.
- Surface forces act on each portion *dA* of the CS.

Body Forces



- The most common body force is gravity, which exerts a downward force on every differential element of the CV
- The differential body force

$$d\vec{F}_{body} = d\vec{F}_{gravity} = \rho\vec{g}d\mathcal{V}$$

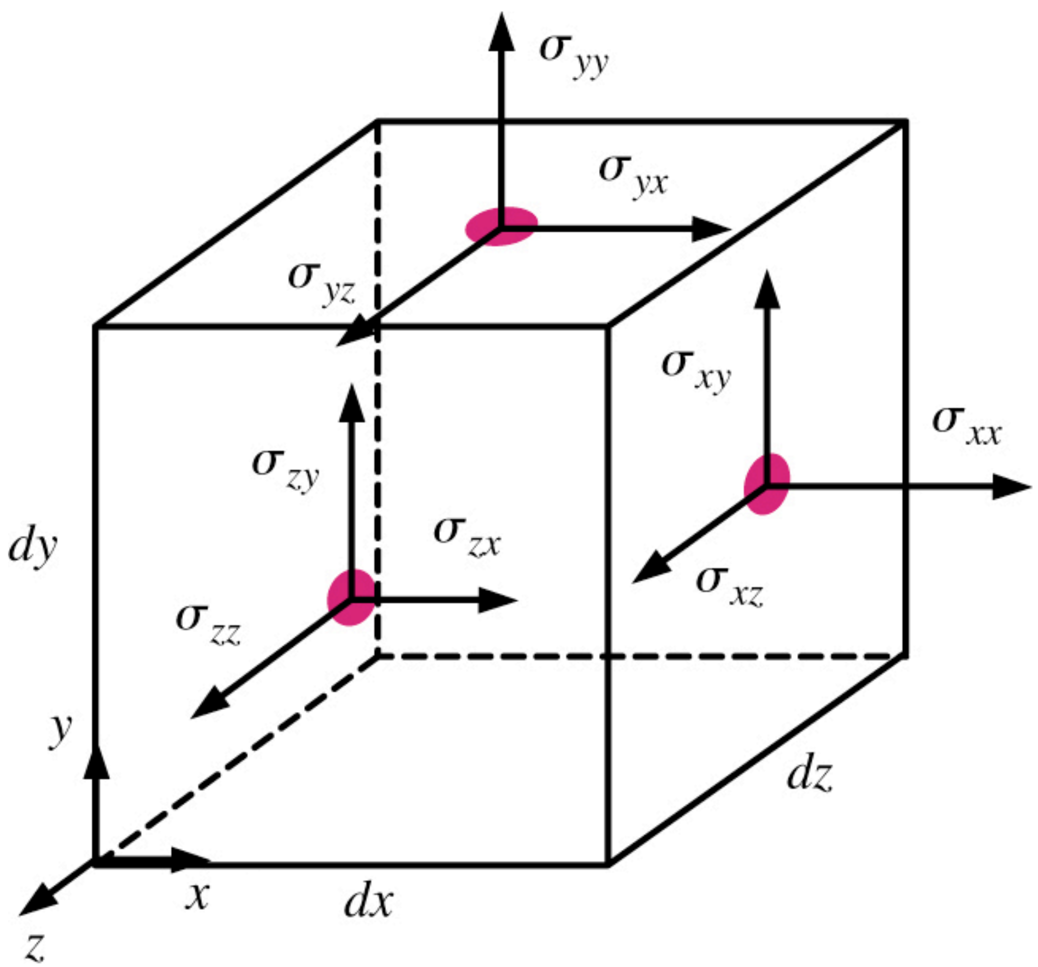
- Typical convention is that $\vec{g}$ acts in the negative *z*-direction,

$$\vec{g} = -g\vec{k}$$

- Total body force acting on CV

$$\sum \vec{F}_{body} = \int_{CV} \rho\vec{g}d\mathcal{V} = m_{CV}\vec{g}$$

Surface Forces



- Surface forces are not as simple to analyze since they include both normal and tangential components
- Diagonal components σ_{xx}, σ_{yy}, σ_{zz} are called **normal stresses** and are due to pressure and viscous stresses
- Off-diagonal components σ_{xy}, σ_{xz}, etc., are called **shear stresses** and are due solely to viscous stresses
- Total surface force acting on CS

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

$$\sum \vec{F}_{surface} = \int_{CS} \sigma_{ij} \cdot \vec{n} dA$$

Linear Momentum Equation

- Newton's second law for a system of mass *m* subjected to a force $\vec{F}$ is expressed as

$$\sum \vec{F} = \frac{d}{dt}\left(m\vec{V}\right) = \frac{d}{dt}\int_{sys} \rho \vec{V}\, d\mathcal{V}$$

- Use RTT with $b = \vec{V}$ and $B = m\vec{V}$ to shift from system formulation to the control volume formulation

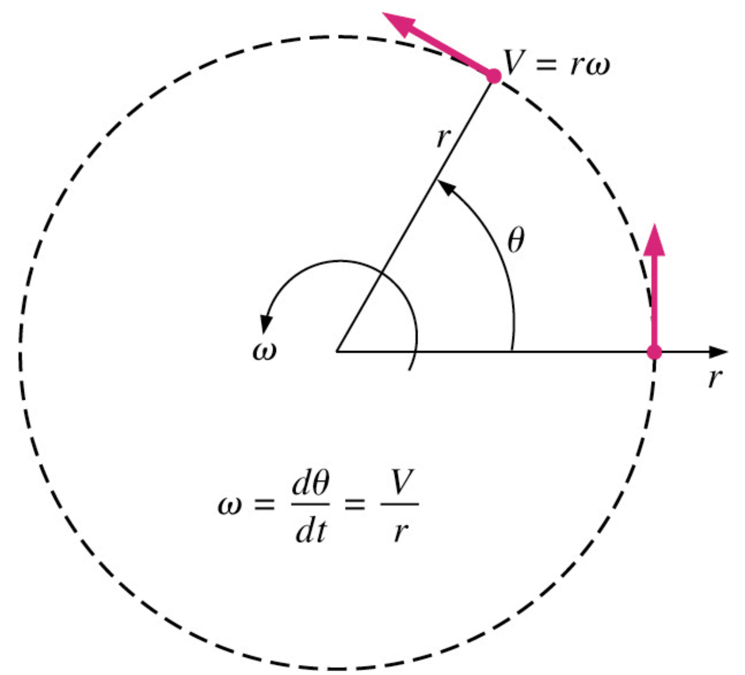
$$\frac{d}{dt}\left(m\vec{V}\right)_{sys} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, d\mathcal{V} + \int_{CS} \rho \vec{V}\left(\vec{V}_r \cdot \vec{n}\right) dA$$

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, d\mathcal{V} + \int_{CS} \rho \vec{V}\left(\vec{V}_r \cdot \vec{n}\right) dA$$

Angular Momentum

- Motion of a rigid body can be considered to be the combination of
  - the translational motion of its center of mass (U_x, U_y, U_z)
  - the rotational motion about its center of mass (ω_x, ω_y, ω_z)
- Translational motion can be analyzed with linear momentum equation.
- Rotational motion is analyzed with angular momentum equation.
- Together, the body motion can be described as a 6–degree–of–freedom (6DOF) system.

Review of Rotational Motion

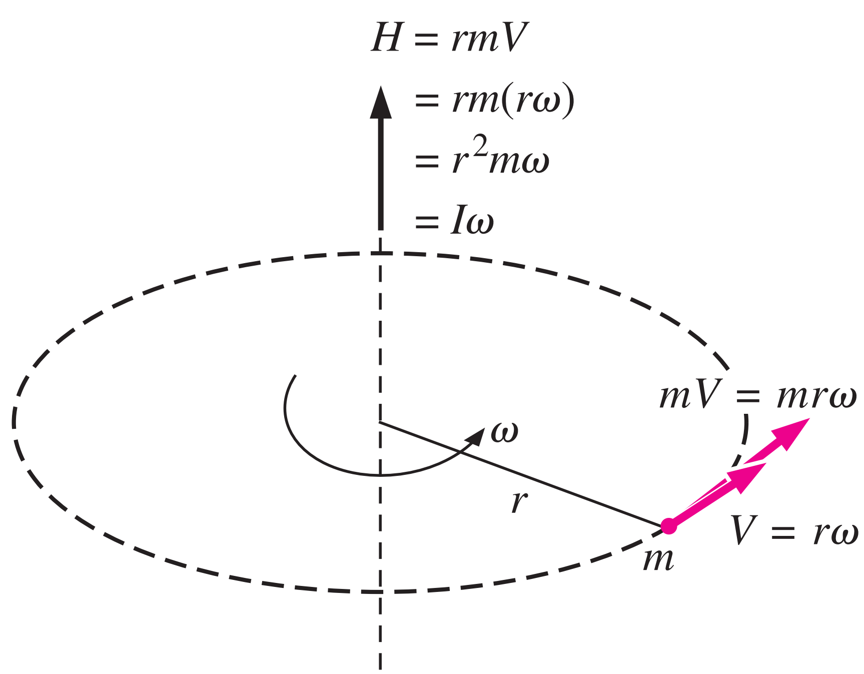


Angular velocity ω is the angular distance θ traveled per unit time, and angular acceleration α is the rate of change of angular velocity.

$$\omega = \frac{d\theta}{dt} = \frac{d(l/r)}{dt} = \frac{1}{r}\frac{dl}{dt} = \frac{V}{r}$$

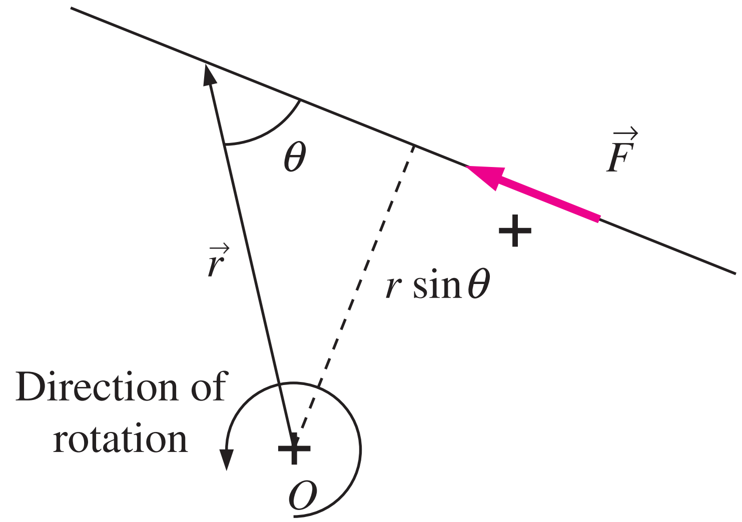
$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{1}{r}\frac{dV}{dt} = \frac{a_t}{r}$$

$$\boxed{V = r\omega \text{ and } a_t = r\alpha}$$



$$\vec{M} = I\vec{\alpha} = I\frac{d\vec{\omega}}{dt} = \frac{d(I\vec{\omega})}{dt} = \frac{d\vec{H}}{dt}$$

Review of Angular Momentum



- Moment of a force: $\vec{M} = \vec{r} \times \vec{F}$ $\quad M = Fr\sin\theta$ $$\vec{M} = \vec{r} \times \vec{F}$$
- Momentum of momentum: $$\vec{H} = \vec{r} \times m\vec{V}$$
- For a system: $$\vec{H}_{sys} = \int_{sys} (\vec{r} \times \vec{V})\rho \, d\mathcal{V}$$ $$\frac{d\vec{H}_{sys}}{dt} = \frac{d}{dt}\int_{sys} (\vec{r} \times \vec{V})\rho \, d\mathcal{V}$$
- Therefore, the angular momentum equation can be written as: $$\sum \vec{M} = \frac{d\vec{H}_{sys}}{dt}$$

Angular Momentum Equation for a CV

- To derive angular momentum for a CV, use RTT with $B = \vec{H}$ $\beta = \vec{r} \times \vec{V}$ .General form

$$\sum \vec{M} = \frac{d}{dt} \int_{CV} (\vec{r} \times \vec{V}) \rho \, d\mathcal{V} + \int_{CS} (\vec{r} \times \vec{V}) \rho \left(\vec{V}_r \cdot \vec{n} \right) \, dA$$

- Approximate form using average properties at inlets and outlets

$$\sum \vec{M} = \frac{d}{dt} \int_{CV} (\vec{r} \times \vec{V}) \rho \, d\mathcal{V} + \sum_{out} \vec{r} \times \dot{m} \vec{V} - \sum_{in} \vec{r} \times \dot{m} \vec{V}$$

- Steady flow

$$\sum \vec{M} = + \sum_{out} \vec{r} \times \dot{m} \vec{V} - \sum_{in} \vec{r} \times \dot{m} \vec{V}$$