Recap for SEISMIC RISK: SEISMIC WAVES

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Principles of mechanics applied to bulk matter: Mechanics of fluids Mechanics of solids **Continuum Mechanics**

A material can be called solid (rather than -perfect- fluid) if it can support a shearing force over the time scale of some natural process.

Shearing forces are directed parallel, rather than perpendicular, to the material surface on which they act.

When a material is loaded at sufficiently low temperature, and/ or short time scale, and with sufficiently limited stress magnitude, its deformation is fully recovered upon uploading: the material is **elastic**

If there is a permanent (plastic) deformation due to exposition to large stresses: the material is **elastic-plastic**

If there is a permanent deformation (viscous or creep) due to time exposure to a stress, and that increases with time: the material is **viscoelastic** (with elastic response), or the material is **visco-plastic** (with partial elastic response)

Normal stress acts perpendicular to the surface (F=normal force)

Tensile causes elongation Compressive causes shrinkage

$$
\sigma = \frac{\text{stretching force}}{\text{cross sectional area}}
$$

A time-dependent perturbation of an elastic medium (e.g. a rupture, an earthquake, a meteorite impact, a nuclear explosion etc.) generates elastic waves emanating from the source region. These disturbances produce local changes in stress and strain.

To understand the propagation of elastic waves we need to describe kinematically the deformation of our medium and the resulting forces (stress). The relation between deformation and stress is governed by elastic constants.

The time-dependence of these disturbances will lead us to the elastic wave equation as a consequence of conservation of energy and momentum.

Deformation

Let us consider a point P_0 at position r relative to some fixed origin and a second point Q_0 displaced from P_0 by dx

Unstrained state:

Relative position of point P_0 w.r.t. Q_0 is δx .

Strained state:

x

Relative position of point P_0 has been displaced a distance **u** to P₁ and point Q_0 a distance v to Q_1 .

Relative position of point P_1 w.r.t. Q1 is δ**y=** δ**x+** δ**u.** The change in relative position between Q and P is just δ**u.**

The relative displacement in the unstrained state is **u**(**r**). The relative displacement in the strained state is **v**=**u**(**r**+ δ**x**).

So finally we arrive at expressing the **relative displacement** due to strain:

```
δu=u(r+ δx)-u(r)
```
We now apply Taylor's theorem in 3-D to arrive at:

What does this equation mean?

The partial derivatives of the vector components

represent a **second-rank tensor** which can be resolved into a **symmetric** and anti-symmetric part:

The symmetric part is called the **strain tensor**

$$
\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
$$

and describes the relation between deformation and displacement in linear elasticity. In 2-D this tensor looks like

$$
\mathcal{E}_{ij} = \left[\begin{array}{ccc} \frac{\partial u_1}{\partial x} & \frac{1}{2} \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) & \frac{\partial u_2}{\partial y} \end{array} \right]
$$

Figure 2.3-12: Some possible strains for a two-dimensional element.

$$
u_1 = \lambda_1 y_1 \qquad \qquad u_2 = \lambda_2 y_2 \qquad \qquad u_3 = \lambda_3 y_3
$$

... in other words ... the eigenvalues are the relative change of length along the three coordinate axes

 u_1

$$
N_1 = \frac{1}{y_1}
$$

In arbitrary coordinates the diagonal elements are the relative change of length along the coordinate axes and the off-diagonal elements are the infinitesimal shear angles.

The trace of a tensor is defined as the sum over the diagonal elements. Thus:

$$
\varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}
$$

This trace is linked to the volumetric change after deformation. Before deformation the volume was V_0 . Because the diagonal elements are the relative change of lengths along each direction, the new volume after deformation is

$$
V = L_1 (1 + \varepsilon_{11}) L_2 (1 + \varepsilon_{22}) L_3 (1 + \varepsilon_{33})
$$

... and neglecting higher-order terms ...

$$
V = L_1 L_2 L_3 \left(1 + \varepsilon_{ii}\right) \text{ or } V_0 \left(1 + \varepsilon_{ii}\right)
$$

$$
\theta = \frac{\Delta V}{V_0} = \varepsilon_{ii} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \text{div}u = \nabla \bullet u
$$

In an elastic body there are restoring forces if deformation takes place. These forces can be seen as acting on planes inside the body. **Forces divided by an areas are called stresses.**

In order for the deformed body to remain deformed these forces have to compensate each other.

Traction vector cannot be completely described without the specification of the force (Δ**F**) and the surface (Δ**S**) on which it acts:

$$
T(n) = \lim_{\Delta s \to 0} \frac{\Delta F}{\Delta s} = \frac{dF}{dS}
$$

And from the linear momentum conservation, we can show that: $T(-n) = -T(n)$

Stress acting on a given internal plane can be decomposed in 3 mutually orthogonal components: one normal (direct stress), tending to change the volume of the material, and two tangential (shear stress), tending to deform, to the surface. If we consider an infinitely small cube, aligned with a Cartesian reference system:

Consider an infinitively small tethraedrum, whose 3 faces are oriented normally to the reference axes. The components of traction **T**, acting on the face whose normal is **n** can be written using the directional cosines referred to versor system **ê**

...and the stress state in a point of the material can be expressed with:

Figure 2.3-4: Stress components on the faces of a volume element.

The relation between stress and strain in general is described by the tensor of elastic constants c_{ijkl}

$$
\sigma_{ij} = c_{ijkl} \epsilon_{kl}
$$

Generalised Hooke's Law

From the symmetry of the stress and strain tensor and a thermodynamic condition if follows that the maximum number if independent constants of c_{ijkl} is 21. In an isotropic body, where the properties do not depend on direction, the relation reduces to

$$
\sigma_{ij} = \lambda \theta \delta_{ij} + 2\mu \epsilon_{ij}
$$

Hooke's Law

where λ and μ are the Lame parameters, θ is the dilatation and δ_{ii} is the Kronecker delta.

$$
\theta \delta_{ij} = \varepsilon_{kk} \delta_{ij} = \left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) \delta_{ij}
$$

The complete stress tensor looks like

$$
\sigma_{ij} = \begin{pmatrix}\n(\lambda + 2\mu)\varepsilon_{11} + \lambda(\varepsilon_{22} + \varepsilon_{33}) & 2\mu\varepsilon_{12} & 2\mu\varepsilon_{13} \\
2\mu\varepsilon_{21} & (\lambda + 2\mu)\varepsilon_{22} + \lambda(\varepsilon_{11} + \varepsilon_{33}) & 2\mu\varepsilon_{23} \\
2\mu\varepsilon_{31} & 2\mu\varepsilon_{32} & (\lambda + 2\mu)\varepsilon_{33} + \lambda(\varepsilon_{11} + \varepsilon_{22})\n\end{pmatrix}
$$
\nMean stress (invariant respect to the coordinate system)

\n
$$
M = \frac{(\sigma_{11} + \sigma_{22} + \sigma_{33})}{3} = \frac{\sum_{n=1}^{3} \lambda_n}{3}
$$
\nDeviatoric stress:

\n
$$
D_{ij} = \sigma_{ij} - M\delta_{ij}
$$

In the Earth the mean stress is essentially due to **lithostatic** load:

$$
P=-\int_{0}^{z} \rho(z) dz
$$

Rigidity is the ratio of pure shear strain and the applied shear stress component σ

ij

Bulk modulus of incompressibility is defined the ratio of pressure to volume change. Ideal fluid means no rigidity (μ = 0), thus λ is the incompressibility of a fluid. $2ε_{ij}$

 $\mu =$

$$
K=-\frac{P}{\theta}=\lambda+\frac{2}{3}\mu
$$

Consider a stretching experiment where tension is applied to an isotropic medium along a principal axis (say x).

$$
\begin{aligned}\n\text{Poisson's ratio} &= v = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = \frac{\lambda}{2(\lambda + 2\mu)} & \text{Young's modulus} &= E = -\frac{\sigma_{11}}{\varepsilon_{11}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \\
\lambda &= \frac{vE}{(1 + v)(1 - 2v)} & \mu &= \frac{E}{2(1 + v)}\n\end{aligned}
$$

For Poisson's ratio we have 0<ν<0.5. A useful approximation is $\lambda = \mu$ (Poisson's solid), then $\nu = 0.25$ and for fluids $\nu = 0.5$

As in the case of deformation the stress-strain relation can be interpreted in simple geometric terms:

Remember that these relations are a generalization of Hooke's Law:

$$
F = Kx
$$

Let us look at some examples for elastic constants:

We now have a complete description of the forces acting within an elastic body. Adding the inertia forces with opposite sign leads us from

$$
f_i + \frac{\partial \sigma}{\partial x_i} = 0
$$

to

the equations of motion for dynamic elasticity

$$
\rho \partial_{t}^{2} \mathbf{u} = \mathbf{f} + (\lambda + 2\mu)\nabla \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u}
$$

Let us apply the **div** operator to this equation, we obtain

$$
\rho \partial_{t}^{2} \theta = (\lambda + 2\mu) \Delta \theta
$$

where

 $\theta = \nabla \cdot \mathbf{u}$

$$
\rho \partial_{\tau}^{2} \mathbf{u} = \mathbf{f} + (\lambda + 2\mu)\nabla \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u}
$$

Let us apply the **curl** operator to this equation, we obtain

$$
\rho \partial_{t}^{2} \nabla \times \mathbf{u} = (\lambda + \mu) \nabla \times \nabla \theta + \mu \Delta (\nabla \times \mathbf{u})
$$

we now make use of
$$
\nabla \times \nabla \theta = 0
$$
 and define

to obtain

 μ

Shear wave S-wave velocity equation $\varphi = \nabla \times \mathbf{u}$

$$
\frac{1}{\beta^2}\partial_\tau^2\phi=\Delta\phi
$$

... what can we say about the direction of displacement, the **polarization** of seismic waves?

$$
u = \nabla \Phi + \nabla \times \Psi \Rightarrow u = P + S
$$

$$
P = \nabla \Phi \qquad S = \nabla \times \Psi
$$

... we now assume that the potentials have the well known form of plane harmonic waves

 $\Phi = A \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$ $\Psi = B \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$ $P = Akexp[i(k \cdot x - \omega t)]$ $S = k \times Bexp[i(k \cdot x - \omega t)]$

P waves are **longitudinal** as P is parallel to k

S waves are **transverse** because S is normal to the wave vector k

Wavefields visualization - body waves

P Wave

They are **spherical** waves and decay as (r)-1

https://www.iris.edu/hq/inclass/animation/seismic_wave_motions4_waves_animated

Seismic Velocities

Ray paths inside the Earth

Figure 3.5-2: Selection of body phases and their ray paths.

It seems that now we have the means to predict arrival times T_{pre} at a given the travel distance of a ray with a given emergence angle (ray parameter) and given structure. This is also termed a **forward (or direct) problem**.

We have recorded a set of travel times, T_{obs} , and we want to determine the structure of the Earth. Thus, what we really want is to solve the **inverse problem**.

In a very general sense we are looking for an Earth model that **minimizes** the difference between a theoretical prediction and the observed data:

$$
\sum \left(T_{obs} - T_{pre}(m) \right)
$$

where m is an Earth model.

Earth's crust

Andrija MOHOROVIČ**IĆ**

 Godišnje izvješće zagrebačkog meteorološkog opservatorija za godinu 1909. Godina IX, dio IV. polovina 1. Potres od 8. X. 1909

<http://www.gfz.hr/sobe-en/discontinuity.htm>

Rays in homogeneous sphere

Sphere with increasing velocity...

Earth's core

Richard Dixon Oldham

المستقطعة المستقطعة
1906 - مستقطعة المستقطعة المستقطعة المستقطعة المستقطعة المستقطعة المستقطعة المستقطعة المستقطعة المستقطعة المستق The Constitution of the Earth as revealed by earthquakes, 62, 456-475, 1906

Paths of seismic waves through the Earth assuming a core of radius $0.4R$, in which the speed is 3 km/sec, while the speed outside it is 6 km/sec. [From Oldham, 1906.]

Time curves of first and second phases of preliminary tremors. The marks surrounded by circles are averages. [From Oldham, 1906.]

1914 Über Erdbebenwellen VIIA. Nachr. Ges. Wiss. Göttingen Math. Physik. Kl, 166.

who calculated depth of the core as 2900km or 0.545R

Inge Lehmann

 Bureau Central Seismologique International, Series A, Travaux Scientifiques, 14, 88, 1936.

GNH7/GG09/GEOL4002 EARTHQUAKE SEISMOLOGY AND EARTHQUAKE HAZARD who discovered of the earth's inner core.

Ray Paths in the Earth (2)

Copyright © 2005 Pearson Prentice Hall, Inc.

Ray Paths in the Earth (3)

Figure 3.5-10: Ray paths for additional core phases.

Spherically symmetric models

Velocity and density variations within Earth based on seismic observations. The main regions of Earth and important boundaries are labeled. This model was developed in the early 1980's and is called **PREM** for Preliminary Earth Reference Model.

Model **PREM** giving S and P wave velocities (light and dark green lines) in the earth's interior in comparison with the younger

IASP91 model (thin grey and black lines)

[http://ds.iris.edu/ds/products/emc-referencemodels/](#page-39-0)

Data example

Wavefields visualization - surface waves

Interference of **P-SV** waves at surfaces (e.g. free surface) and velocity is roughly 92% of β

Interference of multiply reflected

SH waves at surfaces (e.g. free surface) and velocity depends on β

They are **cylindrical dispersed** waves and decay as (r)-1/2

Effect of dispersion...

Demonstration: sum two harmonic waves with slightly different angular frequencies and wavenumbers:

 $u(x, t) = cos(\omega_1 t - k_1 x) + cos(\omega_2 t - k_2 x)$

$$
\omega_1 = \omega + \delta \omega \qquad \omega_2 = \omega - \delta \omega \qquad \omega \gg \delta \omega
$$

$$
k_1 = k + \delta k \qquad k_2 = k - \delta k \qquad k \gg \delta k
$$

Add the two cosines:

$$
u(x, t) = \cos(\omega t + \delta \omega t - kx - \delta kx)
$$

$$
+ \cos(\omega t - \delta \omega t - kx + \delta kx)
$$

$$
= 2\cos(\omega t - kx)\cos(\delta \omega t - \delta kx)
$$

The envelope (beat) has a group velocity: $U = \delta \omega / \delta k$

The individual peaks move with a phase velocity:

 $c = \omega/k$

Surface waves

Data example - 2

[http://ds.iris.edu/seismon/swaves/index.php](http://ds.iris.edu/seismon/swaves/index.php?)

Seismological (body waves) distance ranges **Numerous complete** texts and devoted to $\sqrt{2}$ different seismogram characteristics, $\frac{1}{2}$ u (DOQY WAVES) di

Strong motion seismology

Strong ground motion is an event in which an earthquake cause the ground to shake at least strongly enough for people to feel the motion or to damage or destroy man-made structures.

The goal of strong motion seismology is to be able to understand and predict seismic motions sufficiently well that the predictions can be used for engineering applications

The field of strong-motion seismology could initially be identified with a type of instrument, designed to remain on-scale and record the ground motion with fidelity under the conditions of the strongest ground motions experienced in earthquakes.

Anderson J.G Physical Processes That Control Strong Ground Motion. In: Gerald Schubert (editor-in-chief) Treatise on Geophysics, 2nd edition, Vol 4. Oxford: Elsevier; 2015. p. 505-557.

Strong motion seismology

- Early instruments were typically designed so that ground motions up to the acceleration of gravity (1g) would be on-scale.
- The lower limit of ground motion considered by the early strong motion seismology studies was roughly defined by the thickness of the light beam read until the edge of a recorded film. The minimum acceleration resolved is somewhat less than 0.01g, that approximately coincided with minimum ground motions that humans are able to feel.
- Since much smaller ground motions can be recorded on modern instruments, the distinction between strongmotion seismology and traditional seismology is blurred.

Example of Recordings

Ground acceleration, velocity and displacement, recorded at a strong-motion seismometer that was located directly above the part of a fault that ruptured during the 1985 Mw = 8.1 , Michaocan, Mexico earthquake.

The left panel is a plot of the three components of acceleration: strong, high-frequency shaking lasted almost a minute and the peak acceleration was about 150 cm/s2 (or about 0.15g). The middle panel shows the velocity of ground movement: the peak velocity for this site during that earthquake was about 20-25 cm/ sec. Integrating the velocity, we can compute the displacement, which is shown in the right-most panel: the permanent offsets near the seismometer were up, west, and south, for a total distance of about 125 centimeters.

"Attenuation" laws

Figure 1.2-5: Predicted strong ground motion in eastern and western U.S.

Seismic hazard

