What is a **wave**?

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

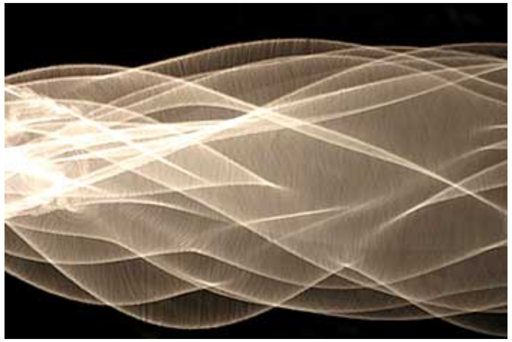
Coupling of **harmonic oscillators** → the disturbances can **propagate**, **superpose** and **stand**

General form of LWE

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} = v^2 \frac{\partial^2 \psi(x,t)}{\partial x^2}$$

WAVE: organized **propagating imbalance**, satisfying differential equations of motion

Separation of variables: string



$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2} = 0$$

and if it has separable solutions:

$$y(x,t) = X(x)T(t)$$

$$\frac{d^2X(x)}{dx^2} + k^2X(x) = 0$$

$$X(x) = A\cos(kx) + B\sin(kx)$$

$$T''(t) + c^2k^2T(t) = 0$$

$$T(t) = C\cos(\omega t) + D\sin(\omega t)$$

$\omega = ck$

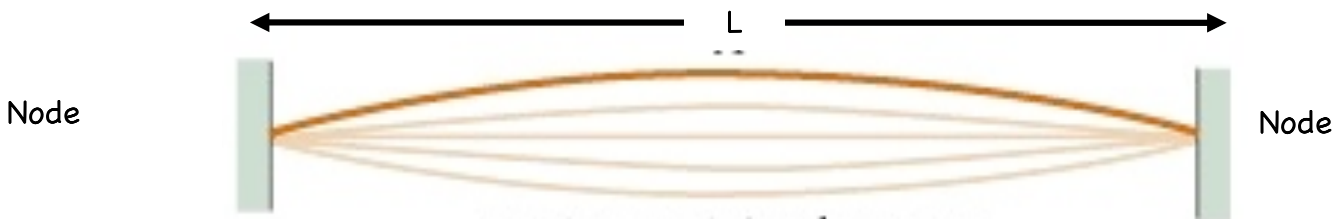
To be determined by **initial** and **boundary** conditions

Standing waves in a string fixed at both ends

Consider a string of length L and fixed at both ends

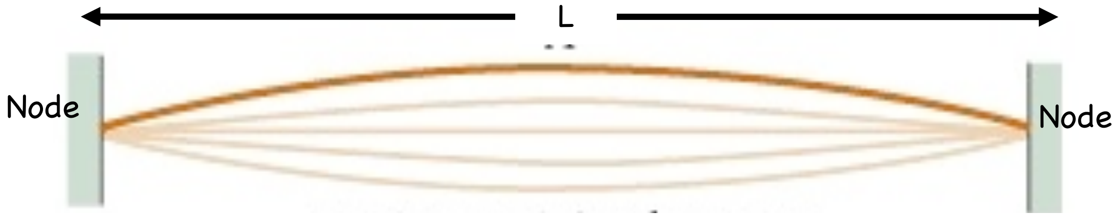
The string has a number of natural patterns of vibration called **NORMAL MODES**

Each normal mode has a characteristic frequency which we can easily calculate



When the string is displaced at its mid point the centre of the string becomes an antinode.

Standing waves in a string fixed at both ends



String is fixed at both ends $\therefore$ $y(x,t) = 0$ at $x = 0$ and L

$y(0,t)=0$ when $x = 0$ as $\sin(kx) = 0$ at $x = 0$

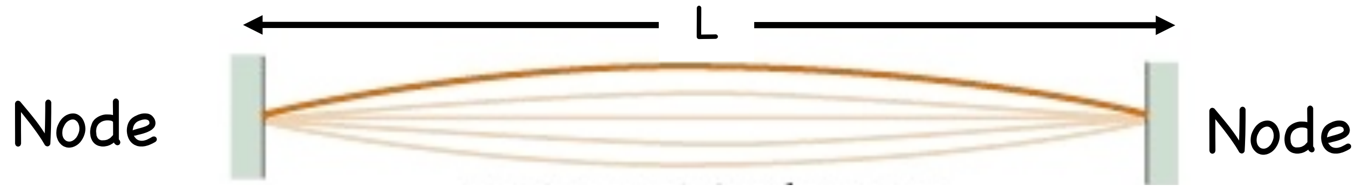
$$y(x,t) = 2A_0 \sin(kx)\cos(\omega t)$$

$y(L,t) = 0$ when $\sin(kL) = 0$ ie $k_n L = n\pi$ $n=1,2,3....$

but $k_n = 2\pi / \lambda$ $\therefore$ $(2\pi / \lambda_n)L = n\pi$ or $\lambda_n = 2L/n$

Standing waves in a string fixed at both ends

For first normal mode $L = \lambda_1 / 2$



The next normal mode occurs when the length of the string L = one wavelength, i.e. $L = \lambda_2$

The third normal mode occurs when $L = 3\lambda_3 /2$

Generally normal modes occur when $L = n\lambda_n /2$

ie $\lambda_n = \frac{2L}{n}$ where $n = 1,2,3........$

Standing waves in a string fixed at both ends

The natural frequencies associated with these modes can be derived from $f = v/\lambda$

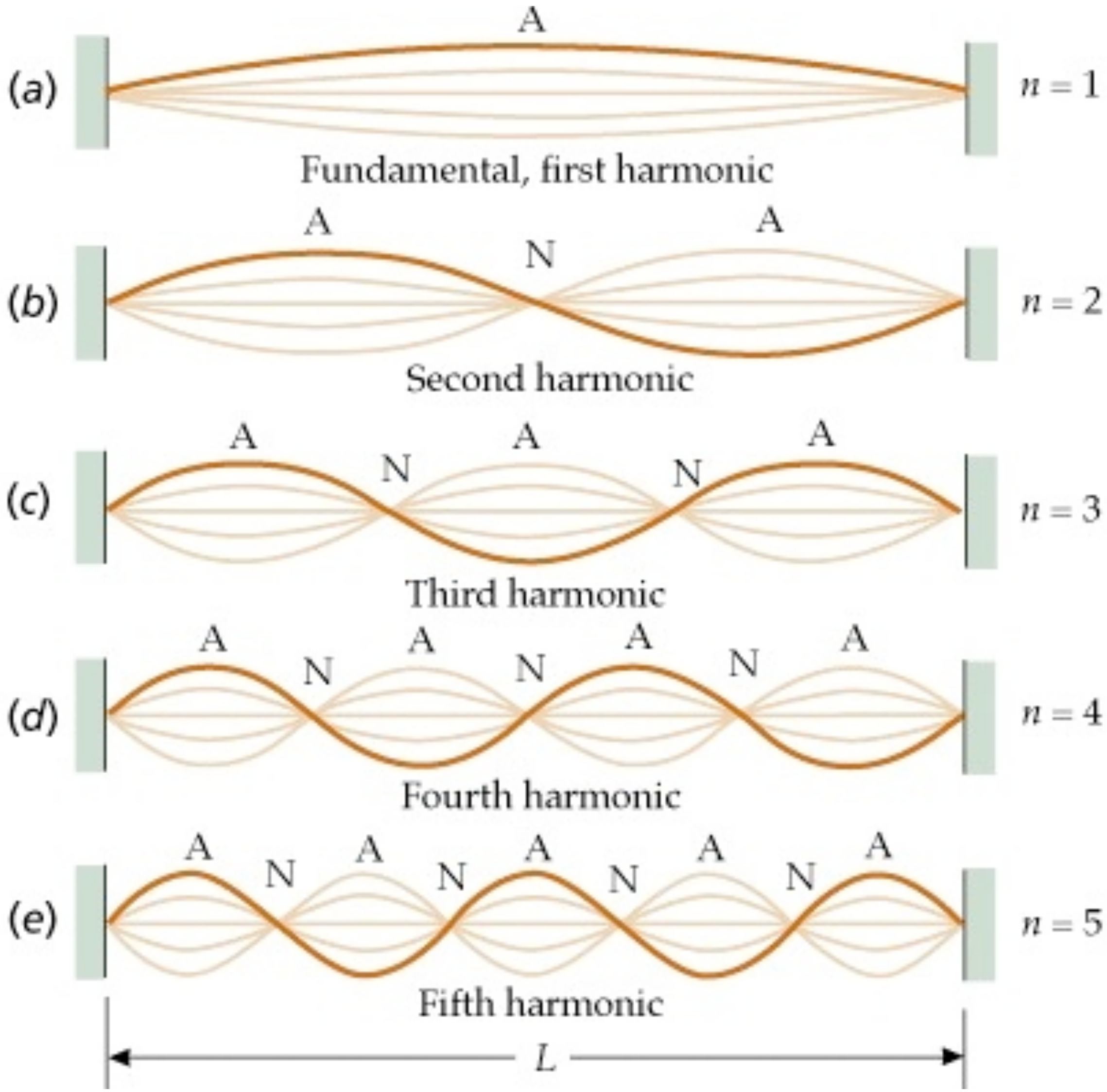
$$f = \frac{v}{\lambda} = \frac{n}{2L}v \quad \text{with } n = 1,2,3.....$$

For a string of mass/unit length μ, under tension F we can replace v by $(F/\mu)^{1/2}$

$$f = \frac{n}{2L}\sqrt{\frac{F}{\mu}} \quad \text{with } n = 1,2,3.....$$

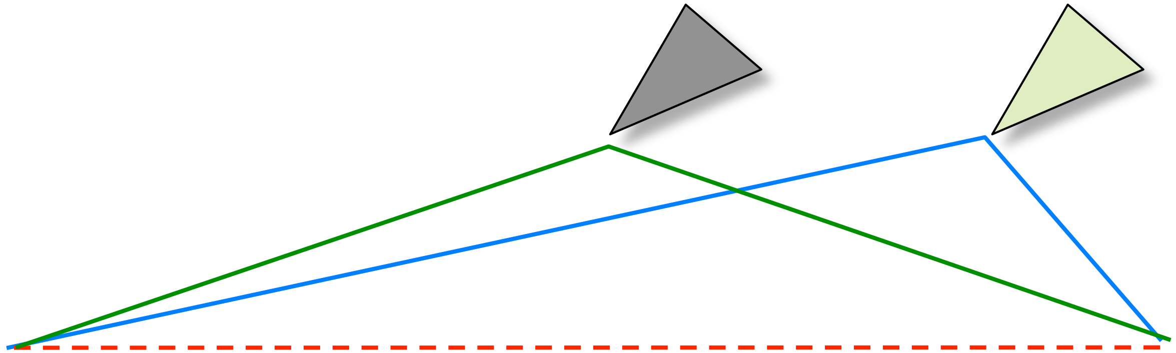
The lowest frequency (**fundamental**) corresponds to n = 1

ie $f = \frac{1}{2L}v$ or $f = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$


(a)
A
n = 1
Fundamental, first harmonic
(b)
A
N
A
n = 2
Second harmonic
(c)
A
N
A
N
A
n = 3
Third harmonic
(d)
A N A N A N A
n = 4
Fourth harmonic
(e)
A N A N A N A N A
n = 5
Fifth harmonic
L

Plucked string

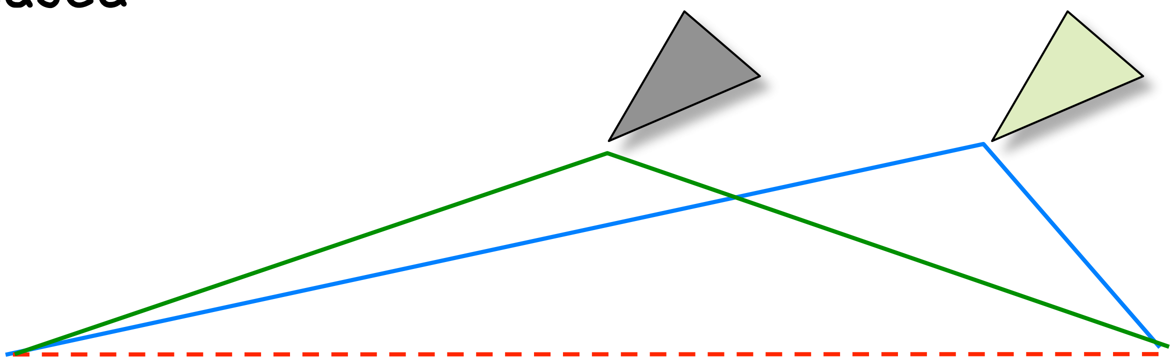
Can one predict the amplitude of each mode (overtone/harmonic?) following plucking?



Using the procedure to measure the Fourier coefficients it is possible to predict the amplitude of each harmonic tone.

Initial conditions

- You know the shape just before it is plucked.
- You know that each mode moves at its own frequency
- The shape when released
- We rewrite this as



$$\text{shape} = f(x, t = 0)$$

$$f(x, t = 0) = \sum_n A_n \sin(k_n x)$$

The plucked string (continued)

Each harmonic has its own frequency of oscillation, the m-th harmonic moves at a frequency $f_m = m f_0$ or m times that of the fundamental mode.

$$f(x, t = 0) = \sum_n A_n \sin(k_n x)$$

$$f(x, t) = \sum_n A_n \sin(k_n x) \cos(\omega_n t)$$

http://www.falstad.com/loadedstring/

Modal summation on a string

Recall modes on a string:

$$u(x, t) = \sum_{n=0}^{\infty} A_n \, U_n(x, \omega_n) \cos(\omega_n t)$$

This is the sum of standing waves or *eigenfunctions,* $U_n(x, \omega_n)$, each of which is weighted by the amplitude A_n and vibrates at its *eigenfrequency* ω_n.

The eigenfunctions and eigenfrequencies are constants due to the physical properties of the string.

The amplitudes depend on the position and nature of the source that excited the motion.

The eigenfunctions were constrained by the boundary conditions, so that

$$U_n(x, \omega_n) = \sin(n\pi x/L) = \sin(\omega_n x/\mathrm{v}) \qquad \omega_n = n\pi \mathrm{v}/L = 2\pi \mathrm{v}/\lambda$$

Source excitation

$$u(x, t) = \sum_{n=0}^{\infty} \sin(n\pi x_s/L)\, F(\omega_n)\, \sin(n\pi x/L)\, \cos(\omega_n t)$$

The source, at $x_s = 8$, is described by

$$F(\omega_n) = \exp[-(\omega_n \tau)^2/4]$$

with $\tau = 0.2$.