FOGLIO 1 – SVOLGIMENTO

ES. 1

$$\frac{f(z)-f(0)}{z} = \frac{3ze^{iz}+|z|^2+4-4}{z} = 3e^{iz}+\bar{z} \xrightarrow[|z|\to 0]{} 3$$

ES. 2

$$f(x+iy) = (3/2)\,x^2 - xy + ixy^2$$

$$\frac{\partial f_1}{\partial x} = 3x-y \qquad \frac{\partial f_1}{\partial y} = -x$$

$$\frac{\partial f_2}{\partial x} = y^2 \qquad \frac{\partial f_2}{\partial y} = 2xy$$

Cauchy Riemann $\Rightarrow$ f derivabile in $x+iy$

$$\Leftrightarrow \begin{cases} 3x-y = 2xy \\ x = y^2 \end{cases}$$

$x+iy=0$ è ok.

se $x+iy \neq 0$, il sistema diventa

$$\begin{cases} 3y^2-y = 2y^3 \\ x=y^2 \end{cases} \Leftrightarrow \begin{cases} 2y^2-3y+1 \\ x=y^2 \end{cases}$$

$$y = \frac{3\pm\sqrt{9-8}}{4} = \frac{3\pm 1}{4} = \begin{cases} 1 \\ 1/2 \end{cases}$$

$\Rightarrow$ i punti in cui f è derivabile in senso complesso sono $0,\ 1+i,\ \frac{1}{4}+\frac{1}{2}i$

$$f'(0)=0$$

$$f'(1+i) = \frac{\partial f_1}{\partial x}(1,1) + i\frac{\partial f_2}{\partial x}(1,1) = 2+i$$

$$f'\left(\frac{1}{4}+\frac{1}{2}i\right) = \frac{\partial f_1}{\partial x}\left(\frac{1}{4},\frac{1}{2}\right) + i\frac{\partial f_2}{\partial x}\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{4}+\frac{1}{4}i$$

ES. 3

$f(0)=0 \qquad f'(0)=0$

$$\lim_{z\to 0} \frac{f(2z)}{z} = \lim_{z\to 0} \frac{f(2z)-f(0)}{2z} \cdot 2 = f'(0)\cdot 2 = 0.$$

$$\lim_{z\to 0} \frac{f(z^2)}{z} = \lim_{z\to 0} \frac{f(z^2)-f(0)}{z^2} \cdot z = f'(0)\cdot 0 = 0$$

$$\lim_{z\to 0} \frac{f(z^2-z)}{0} = \lim_{z\to 0} \frac{f(z^2-z)-f(0)}{z^2-z}\, \frac{z^2-z}{z} = f'(0)\cdot(-1) = 0$$

$$\lim_{z\to i} \frac{f(z^2+1)}{z-i} = \lim_{z\to i} \frac{f((z+i)(z-i))-f(0)}{(z-i)(z+i)}\,(z+i)$$

$$= 0 \cdot 2i = 0 .$$

ES. 4

$f_2(x+iy) \equiv 0$ Per Cauchy Riemann,

$$\frac{\partial f_2}{\partial x} \equiv 0 \implies \frac{\partial f_1}{\partial y} \equiv 0$$

$$\frac{\partial f_2}{\partial y} \equiv 0 \implies \frac{\partial f_1}{\partial x} \equiv 0$$

$\implies$ $Jf \equiv 0$ $\implies$ f costante.

ES. 5

$$\sum_{n=0}^{\infty} n^{1/n} (z-1)^n$$

$$\lim_{n\to\infty} \left(n^{1/n}\right)^{1/n} = \lim_{n\to\infty} n^{\frac{1}{n^2}} = \lim_{n\to\infty} e^{\frac{1}{n^2}\log n} = 1$$

$\implies R = 1$

$$\sum_{n=1}^{\infty} n^{-1} z^n$$

$$\lim_{n\to\infty} (n^{-1})^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}\log\frac{1}{n}} = 1$$

$$\Rightarrow \quad R = 1$$

$\sum_{n=1}^{\infty} n!\, n^{-n} z^n$ conviene usare il criterio del rapporto su $\sum_{n=1}^{\infty} n!\, n^{-n} |z|^n$

$$\lim_{n\to\infty} \frac{(n+1)!\,(n+1)^{-(n+1)}\,|z|^{n+1}}{n!\;\; n^{-n}\;\; |z|^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} |z| = \frac{1}{e}|z|$$

$$\Rightarrow \quad R = e$$

$$\boxed{\text{ES. 6}}$$

(1) $$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(i)^{2n}}{(2n)!} z^{2n} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (iz)^{2n}$$

$$i \sin z = i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{i\,(i)^{2n}}{(2n+1)!} z^{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (iz)^{2n+1}$$

$$\Rightarrow \quad \cos z + i \sin z = e^{iz} \quad \forall\, z \in \mathbb{C}.$$

(2) $\cos^2 z + \sin^2 z =$

$$\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)$$

$$= \frac{1}{4}\left(e^{2iz}+2+e^{-2iz}-e^{2iz}+2-e^{-2iz}\right) = 1$$

$\forall\ z \in \mathbb{C}$ grazie al punto 1.

(3) $$\cos(z+2\pi) = \frac{e^{i(z+2\pi)}+e^{-i(z+2\pi)}}{2}$$

$$= \frac{e^{2\pi i}e^{iz}+e^{-2\pi i}e^{-iz}}{2}$$

$$= \frac{e^{iz}+e^{-iz}}{2} = \cos z$$

grazie al punto (1).

analogo ragionamento per $\sin(z+2\pi)$

(4) Sia $z = x+iy$ t.c. $\cos z = 0$

cioè

$$0 = \frac{e^{iz}+e^{-iz}}{2} = \frac{e^{i(x+iy)}+e^{-i(x+iy)}}{2}$$

$$= \frac{e^{-y}e^{ix}+e^{y}e^{-ix}}{2}$$

$$\Rightarrow \quad e^{-y}e^{ix} = -e^{y}e^{-ix}$$

$$\Rightarrow \quad e^{2y} = -e^{2ix}$$

$$\Rightarrow \quad |e^{2y}| = e^{2y} = 1 \qquad \Rightarrow \quad y=0.$$

Analogo ragionamento per $\sin z = 0$ $\square$