FOGLIO 7bis soluzioni

1) $\displaystyle\int_0^{+\infty} \frac{x^2+1}{x^4+1}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2+1}{x^4+1}$ perché l'integrando è pari.

Non ci sono poli in $\mathbb{R}$, e

$R(z) = \dfrac{z^2+1}{z^4+1} \sim \dfrac{1}{z^2}$ per $z \to +\infty$

Allora

$$\int_0^{+\infty} \frac{x^2+1}{x^4+1}\,dx = \frac{1}{2}\, 2\pi i \sum_{\substack{\alpha \text{ poli} \\ \operatorname{Im}\alpha>0}} \operatorname{Res}\left(\frac{z^2+1}{z^4+1}, \alpha\right)$$

Ci sono due poli con $\operatorname{Im}\alpha > 0$,

$z_1 = e^{\frac{\pi}{4}i}$ $z_2 = e^{\frac{3}{4}\pi i}$, entrambi semplici

$$\operatorname{Res}\left(\frac{z^2+1}{z^4+1}, z_1\right) = \left.\frac{z^2+1}{(z^4+1)'}\right|_{z_1} = \left.\frac{z^2+1}{4z^3}\right|_{z_1}$$

$$= \frac{i+1}{-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}} \cdot \frac{1}{4}$$

$$\operatorname{Res}\left(\frac{z^2+1}{z^4+1}, z_2\right) = \left.\frac{z^2+1}{4z^3}\right|_{z_2}$$

$$= \frac{-i+1}{\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}} \cdot \frac{1}{4}$$

$$\Rightarrow \int_0^{+\infty} \frac{x^2+1}{x^4+1}\,dx = \dots = \frac{\pi\sqrt{2}}{2}$$

2) $$\int_0^{+\infty} \frac{x^3 \sin x}{(x^2+4)^2}\, dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^3 \sin x}{(x^2+4)^2}\, dx$$

no poli su $\mathbb{R}$; $R(z) = \dfrac{z^3}{(z^2+4)^2} \sim \dfrac{1}{z}$ per $z \to +\infty$

Allora

$$\int_0^{+\infty} \frac{x^3 \sin x}{(x^2+4)^2}\, dx = \frac{1}{2} \operatorname{Im} 2\pi i \operatorname{Res}\left(\frac{z^3 e^{iz}}{(z^2+4)^2}, 2i\right)$$

$2i$ è polo di ordine 2

$$\Rightarrow \operatorname{Res}\left(\frac{z^3 e^{iz}}{(z^2+4)^2}, 2i\right) = \frac{d}{dz}\bigg|_{z=2i} \frac{z^3 e^{iz}}{(z+2i)^2}$$

$$= \dots = 0$$

$$\Rightarrow \int_0^{+\infty} \frac{x^3 \sin x}{(x^2+4)^2}\, dx = 0 .$$

3) $$\int_0^{+\infty} \frac{x \sin x \cos x}{x^4+1}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x \sin x \cos x}{x^4+1}\,dx$$

$$= \frac{1}{4}\int_{-\infty}^{+\infty} \frac{x \sin(2x)}{x^4+1}\,dx = \int_{-\infty}^{+\infty} \frac{t \sin t}{t^4+16}\,dt$$

$$= \operatorname{Im} 2\pi i \sum_{\substack{\alpha \text{ pôle} \\ \operatorname{Im}\alpha>0}} \operatorname{Res}\left(\frac{z e^{iz}}{z^4+16}, \alpha\right)$$

$$= \operatorname{Im} 2\pi i \left(\operatorname{Res}\left(\frac{z e^{iz}}{z^4+16}, \sqrt{2}+i\sqrt{2}\right) + \operatorname{Res}\left(\frac{z e^{iz}}{z^4+16}, -\sqrt{2}+i\sqrt{2}\right)\right)$$

$$= \operatorname{Im} 2\pi i \left(\left.\frac{z e^{iz}}{4z^3}\right|_{z=\sqrt{2}+i\sqrt{2}} + \left.\frac{z e^{iz}}{4z^3}\right|_{z=-\sqrt{2}+i\sqrt{2}} \right)$$

$$= \dots = \frac{\pi}{2} e^{-\sqrt{2}} \sin\sqrt{2}$$

4) $\displaystyle\int_0^{2\pi} \frac{1}{2+\sin x}\,dx = \int_0^{2\pi} \frac{1}{2+\frac{e^{i\vartheta}+e^{-i\vartheta}}{2i}} \cdot \frac{1}{ie^{i\vartheta}} \cdot ie^{i\vartheta}\, d\vartheta$

$$= \oint_{\partial D(0,1)} \frac{1}{iz} \frac{1}{2+\frac{z-z^{-1}}{2i}}\, dz$$

$$= \oint_{\partial D} \frac{2}{z\,(4i+z-z^{-1})}\, dz$$

$$= \oint_{\partial D} \frac{2}{z^2+4iz-1}\, dz$$

$$= 4\pi i \sum_{\substack{\alpha \text{ poli} \\ |\alpha|<1}} \mathrm{Res}\left(\frac{1}{z^2+4iz-1}, \alpha\right)$$

$$= 4\pi i \; \mathrm{Res}\left(\frac{1}{z^2+4iz-1}, (\sqrt{3}-2)i\right)$$

$$= 4\pi i \left.\frac{1}{2z+4i}\right|_{(\sqrt{3}-2)i} = \frac{2\pi}{\sqrt{3}}$$

5) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt{x}\,(1+x^2)}\,dx = \int_0^{+\infty} \frac{\sqrt{x}}{x(1+x^2)}$

$R(x) = \dfrac{1}{x(1+x^2)} \sim \dfrac{1}{x}$ per $x \sim 0$

$R(x) = \dfrac{1}{x(1+x^2)} \sim \dfrac{1}{x^3}$ per $x \sim +\infty$

poli in $0, \pm i$.

Allora

$$(1-e^{\pi i}) \int_0^{+\infty} \frac{\sqrt{x}}{x\,(1+x^2)}\,dx =$$

$$= 2\pi i \left(\mathrm{Res}\left(\frac{z^{1/2}}{z(1+z^2)}, +i\right) + \mathrm{Res}\left(\frac{z^{1/2}}{z(1+z^2)}, -i\right)\right)$$

$$= 2\pi i \left(\frac{z^{1/2}}{3z^2+1}\bigg|_{z=+i} + \frac{z^{1/2}}{3z^2+1}\bigg|_{z=-i} \right)$$

$$= 2\pi i \left(\frac{e^{i\pi/4}}{-3+1} + \frac{e^{i\frac{3\pi}{4}}}{-3+1} \right)$$

$$= \pi\sqrt{2}$$

$$\Rightarrow \int_0^{+\infty} \frac{\sqrt{x}}{x(1+x^2)}\,dx = \frac{\pi\sqrt{2}}{2}$$

6) $\displaystyle\int_0^{2\pi} \frac{1}{(a+b\sin x)^2}\,dx \qquad a > |b|$

$$= \int_0^{2\pi} \frac{1}{\left(a + b\frac{e^{i\vartheta}-e^{-i\vartheta}}{2i}\right)^2} \frac{1}{ie^{i\vartheta}}\, ie^{i\vartheta}\, d\vartheta$$

$$= \oint_{\partial D(0,1)} \frac{1}{\left(a+b\,\frac{z-z^{-1}}{2i}\right)^2} \frac{1}{iz}\,dz .$$

$$= \oint_{\partial D} \frac{4iz}{(2aiz+bz^2-b)^2}\,dz =$$

$$= 2\pi i \sum_{\substack{\alpha \text{ polo} \\ |\alpha|<1}} \operatorname{Res}\left(\frac{4iz}{(2aiz+bz^2-b)^2}, \alpha\right)$$

$$= 2\pi i \operatorname{Res}\left(\frac{4iz}{(2aiz+bz^2-b)^2}, \frac{-a+\sqrt{a^2-b^2}}{b}i\right)$$

$$= 2\pi i \frac{d}{dz} \frac{4iz}{b^2\left(z - \frac{-a-\sqrt{a^2-b^2}}{|b|}i\right)^2} \Bigg|_{z=\frac{-a+\sqrt{a^2-b^2}}{|b|}i}$$

$$= \dots = \frac{2a\pi}{\sqrt{(a^2-b^2)^3}}$$