FOGLIO 9 - SVOLGIMENTO

ES. 1

$u : \mathbb{C} \to \mathbb{R}$ armonica $\Rightarrow \exists\, v : \mathbb{C} \to \mathbb{R}$ armonica t.c. $f = u + iv \in H(\mathbb{C})$.

Sia $u(z) \leq M \quad \forall z$.

Sia $g(z) = e^{f(z)} = e^{u(z)} e^{iv(z)}$

allora

$$|g(z)| \leq e^{u(z)} \leq e^{M}$$

Per Liouville $g(z) \equiv \text{cost}$.

Quindi

$$\begin{cases} g'(z) \equiv 0 \\ = f'(z) e^{f(z)} \end{cases}$$

$$\Rightarrow f'(z) \equiv 0 \Rightarrow f \equiv \text{cost.} \Rightarrow u \equiv \text{cost.}$$

Sia $m \leq u(z) \quad \forall z$

Sia $g(z) = e^{-f(z)} = e^{-u(z)} e^{-iv(z)}$

$$\Rightarrow |g(z)| = e^{-u(z)} \leq e^{-m}$$

$\Rightarrow$ per Liouville $g \equiv \text{cost}$

e si conclude come prima. $\blacksquare$

ES. 2

(1) $u(z) = xy(x^2+y^2)^{-2}$

$\Omega = \mathbb{C} \setminus \{0\}$

②

$$u_x = \dots = \frac{y^3 - 3x^2 y}{(x+y)^2}$$

$$u_{xx} = \dots = \frac{12\,xy(x^2 - y^2)}{(x^2+y^2)^4}$$

$$u_y = \dots = \frac{x^3 - 3y^2 x}{(x+y)^2}$$

$$u_{yy} = \dots = \frac{12\,xy\,(y^2 - x^2)}{(x^2+y^2)^4}$$

$\Longrightarrow \quad u_{xx} + u_{yy} = 0 \quad \Rightarrow$ u è armonica.

ora osserviamo che

$$u(z) = \frac{\frac{z+\bar z}{2}\,\frac{z-\bar z}{2i}}{z^2\bar z^2}$$

$$= \frac{1}{4i}\,\frac{z^2 - \bar z^2}{z^2 \bar z^2}$$

$$= \frac{1}{4i}\left(\frac{1}{\bar z^2} - \frac{1}{z^2}\right)$$

$$= \frac{1}{4i\bar z^2} - \frac{1}{4i z^2}$$

$$= \frac{1}{-4i z^2} + \overline{\left(\frac{1}{-4i z^2}\right)} = 2\,\mathrm{Re}\left(\frac{1}{-4i z^2}\right)$$

$$= \mathrm{Re}\left(\frac{1}{2}\,\frac{i}{z^2}\right)$$

$\Longrightarrow$ L'armonica coniugata è

(3)

$$v(z) = \operatorname{Im}\left(\frac{1}{2}\,\frac{i}{z^2}\right)$$

$$= \frac{1}{4i}\left(\frac{i}{z^2} - \frac{-i}{\bar{z}^2}\right)$$

$$= \frac{1}{4}\left(\frac{1}{z^2} + \frac{1}{\bar{z}^2}\right)$$

$$= \frac{1}{4}\,\frac{\bar{z}^2 + z^2}{z^2\bar{z}^2}$$

$$= \frac{1}{4}\,\frac{x^2 - 2ixy) - y^2 + x^2 + 2ixy - y^2}{(x^2+y^2)^2}$$

$$= \frac{1}{4}\,\frac{2x^2 - 2y^2}{(x^2+y^2)^2} =$$

$$= \frac{1}{2}\,\frac{x^2 - y^2}{(x^2+y^2)^2}$$

(2) $u(z) = \operatorname{Arg}(\sqrt{z}) \qquad \Omega = \mathbb{C}\ ]-\infty, 0]$

$$u(z) = \operatorname{Arg} e^{\frac{1}{2}\operatorname{Log} z} =$$

$$= \operatorname{Arg} e^{\frac{1}{2}(\log|z| + i\operatorname{Arg} z)}$$

$$= \frac{1}{2}\operatorname{Arg} z = \frac{1}{2}\operatorname{Im}(\operatorname{Log} z)$$

$$= \frac{1}{2}\operatorname{Re}(-i\operatorname{Log} z)$$

$\Rightarrow$ u è armonica e la sua armonica coniugata è

$$v(z) = \operatorname{Im}\left(-\frac{i}{2}\operatorname{Log} z\right) = -\frac{1}{2}\log|z|$$

$$= \log\frac{1}{x^2+y^2}$$

(3) $u(z) = \log|z^2-1| \qquad \Omega = \{\operatorname{Im} z > 0\}$

$$u(z) = \operatorname{Re}\left(\operatorname{Log}(z^2-1)\right)$$

$$= \operatorname{Re}\left(\log|z^2-1| + i\operatorname{Arg}(z^2-1)\right)$$

$\Rightarrow$ u è armonica e la sua coniugata è

$$v(z) = \operatorname{Arg}(z^2-1)$$

n.b. $\operatorname{Im} z > 0 \Rightarrow z^2 \in \mathbb{C} \setminus [0,+\infty[$

$\Rightarrow z^2 - 1 \in \mathbb{C} \setminus [-1,+\infty[$

$\Rightarrow \operatorname{Log}(z^2-1)$ è definita su Ω

(qui $\operatorname{Log} : \mathbb{C} \setminus [0,+\infty[\longrightarrow \mathbb{C}$)
$\quad\;\operatorname{Arg} : \mathbb{C} \setminus [0,+\infty[\longrightarrow]0,2\pi[$

ES. 3

(1) $\begin{cases} \Delta u = 0 & \text{in } D(0,1) \\ u(z) = h(z) & z \in \partial D(0,1) \end{cases}$

$h(z) = x^2 - 2y$

In coordinate polari,

$$h(e^{i\vartheta}) = (\cos\vartheta)^2 - 2\sin\vartheta$$
$$= \left(\frac{e^{i\vartheta} + e^{-i\vartheta}}{2}\right)^2 - 2\left(\frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}\right)$$
$$= \frac{1}{4} e^{2i\vartheta} + \frac{1}{4} e^{-2i\vartheta} + \frac{1}{2} + ie^{i\vartheta} - ie^{-i\vartheta}$$

Dalla dimostrazione del teorema di Poisson-Schwarz, si ha che la soluzione è

$$u(re^{i\vartheta}) = \frac{1}{4} r^2 e^{2i\vartheta} + \frac{1}{4} r^2 e^{-2i\vartheta} + \frac{1}{2} + ire^{i\vartheta} - ire^{-i\vartheta}$$

e quindi

$$u(z) = \frac{1}{4}(z^2 + \bar{z}^2) + \frac{1}{2} + i(z - \bar{z})$$
$$= \frac{1}{2} \operatorname{Re} z^2 + \frac{1}{2} - 2 \operatorname{Im} z$$
$$= \frac{1}{2}(x^2 - y^2) + \frac{1}{2} - 2y .$$

Verifica:

$u_{xx} = 1 \quad u_{yy} = -1 \implies u_{xx} + u_{yy} = 0$

Inoltre su $x^2+y^2=1$,

$$u(z) = \frac{1}{2}(x^2-y^2) + \frac{1}{2} - 2y =$$
$$= \frac{1}{2}(x^2+x^2-1) + \frac{1}{2} - 2y$$
$$= x^2 - 2y. \checkmark$$

(2) $\begin{cases} \Delta u = 0 & \text{in } D(0,2) \\ u(z) = h(z) & \text{in } \partial D(0,2) \end{cases}$

dove $h(z) = x^2 - 2y^2$.

Sia $v(z) := u(2z) \quad z \in D(0,1)$

v soddisfa

$\begin{cases} \Delta v = 0 & \text{in } D(0,1) \\ v = k(z) & \text{in } \partial D(0,1) \end{cases}$

$k(z) = h(2z) = 4x^2 - 8y^2$

In coordinate polari

$$k(e^{i\vartheta}) = 4(\cos\vartheta)^2 - 8(\sin\vartheta)^2$$
$$= 4\left(\frac{e^{i\vartheta}+e^{-i\vartheta}}{2}\right)^2 - 8\left(\frac{e^{i\vartheta}-e^{-i\vartheta}}{2i}\right)^2$$
$$= \dots = 6\left(\frac{e^{2i\vartheta}+e^{-2i\vartheta}}{2}\right) - 2$$

come prima, la soluzione è

$$v(re^{i\vartheta}) = 6\left(\frac{r^2 e^{2i\vartheta} + r^2 e^{-2i\vartheta}}{2}\right) - 2$$

$$= 6 \operatorname{Re} z^2 - 2 = 6(x^2 - y^2) - 2$$

quindi

$$u(z) = v\left(\frac{z}{2}\right) = \frac{3}{2}(x^2 - y^2) - 2$$

(3) $\begin{cases} \Delta u = 0 & \text{in } D(0, 1/2) \\ u(z) = h(z) & \text{in } \partial D(0, 1/2) \end{cases}$

$h(z) = 4x^2y - 8xy^2.$

Poniamo $v(z) = u(z/2)$. v soddisfa

$$\begin{cases} \Delta v = 0 & \text{in } D(0,1) \\ v(z) = k(z) & \text{in } \partial D(0,1) \end{cases}$$

$$k(z) = h(z/2) = \frac{1}{2}x^2y - xy^2$$

in coordinate polari

$$k(e^{i\vartheta}) = \frac{1}{2}\cos^2\vartheta \sin\vartheta - \cos\vartheta \sin^2\vartheta$$

$$= \frac{1}{2}\left(\frac{e^{i\vartheta} + e^{-i\vartheta}}{2}\right)^2 \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i} - \frac{e^{i\vartheta} + e^{-i\vartheta}}{2}\left(\frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}\right)^2$$

$$= \ldots\ldots = $$

$$= \frac{1}{8}\,\frac{e^{3i\vartheta}-e^{-3i\vartheta}}{2i} + \frac{1}{8}\,\frac{e^{i\vartheta}-e^{-i\vartheta}}{2i}$$

$$+ \frac{1}{4}\,\frac{e^{3i\vartheta}+e^{-3i\vartheta}}{2} - \frac{1}{4}\,\frac{e^{i\vartheta}+e^{-i\vartheta}}{2}$$

(8)

Quindi

$$v(re^{i\vartheta}) = \frac{1}{8}\,\frac{r^3e^{3i\vartheta}-r^3e^{-3i\vartheta}}{2i} + \frac{1}{8}\,\frac{re^{i\vartheta}-re^{-i\vartheta}}{2i}$$

$$+ \frac{1}{4}\,\frac{r^3e^{3i\vartheta}+r^3e^{-3i\vartheta}}{2} - \frac{1}{4}\,\frac{re^{i\vartheta}+re^{-i\vartheta}}{2}$$

$$v(z) = \frac{1}{8}\,\mathrm{Im}\, z^3 + \frac{1}{8}\,\mathrm{Im}\, z + \frac{1}{4}\,\mathrm{Re}\, z^3 - \frac{1}{4}\,\mathrm{Re}\, z$$

$$\overset{!}{=} \frac{3}{8}x^2y - \frac{1}{8}y^3 + \frac{1}{8}y + \frac{1}{4}x^3 - \frac{3}{4}xy^2 - \frac{1}{4}x$$

$$u(z) = v(2z) = 3x^2y - y^3 + \frac{1}{4}y - 2x^3 - 6xy^2 - \frac{1}{2}x$$