1 A number of preliminary results

1.0.1 Zorn's Lemma

Let X be a set with an order relation $\leq$. A totally ordered subset C (i.e. each two elements of C are comparable for $\leq$) of X is called a *chain*. We say that X is inductive if any chain C of X has an upper bound in X, that is there exists a $x \in X$ with $y \leq x$ for all $y \in C$. If C is a chain and $x \in C$ we set $P(C,x) := \{y \in C : y < x\}$. A set B is an initial segment of a chain C if $B = P(C,x)$ for some $x \in C$.

The following will play a repeated role in the sequel. The proof will be based on the *Axiom of Choice*.

Lemma 1.1 (Zorn's Lemma)**.** *Every nonempty ordered set that is inductive has a maximal element.*

Proof. Suppose that the statement is false. Then for any $x \in X$ there is a y with $x < y$. Now we claim that for any chain C there exists $x \in X$ with $y < x$ for all $y \in C$: indeed, just take an upper bound x_0 of the chain and then, since x_0 is not maximal, a new element with $x_0 < x$.
Using the axiom of choice, we define for every chain C such an element $f(C) := x$.
Given $C \subseteq X$ we say that C is conforming if the following three properties hold:

1. C is a chain;
2. C does not contain an infinite strictly decreasing sequence;
3. for any $x \in C$ we have $x = f\left(P(C,x)\right)$ for any $x \in C$.

By convention, $\emptyset$ is conforming. Furthermore, if C is conforming, also $C \cup \{f(C)\}$ is conforming.

We claim now the following.

Claim 1.2. Given two conforming sets A and B in X, if $A \neq B$ then one of the two is an initial segment of the other.

Proof. Let $C = \{c \in A \cap B : P(A,c) = P(B,c)\}$. We claim that either $C = A$ or $C = P(A,a)$ for some $a \in A$. If $C \neq A$, here exists an $a \in A \backslash C$ which is minimal (otherwise there would be an infinite strictly decreasing sequence in A). It follows that $P(A,x) \subseteq C$. If $P(A,a) \subsetneqq C$, there exists $c \in C \backslash P(A,a)$. We have $a \neq c$, since $c \in C$ and $a \notin C$. Then, since $c \in A$ and A is a chain, we have $a < c$. However, from the definition of C it is possible to see that if $c \in C$ then $P(A,c) \subseteq C$. So, since $a \in P(A,c) \subseteq C$, we conclude $a \in C$, which is a contradiction. So $P(A,a) = C$. Similarly, either $C = B$ or $C = P(B,b)$ for some $b \in B$. What is left to consider is the case $A \neq C$ and $B \neq C$. Then $C = P(A,a) = P(B,b)$. We have $a = f\left(P(A,a)\right) = f\left(P(B,b)\right) = b$. But then, by the definition of C, we have $a \in C$, giving a contradiction, because $C = P(A,a) \not\ni a$.

□

Let E be the union of all conforming subsets of X and let $a \in E$. Let A be a conforming set containing a. Then we claim that $P(A,a) = P(E,a)$. Indeed $P(A,a) \subseteq P(E,a)$ is obvious. On the other hand, if $x \in P(E,a)$ and B is a conforming set containing x, if B is equal or is an initial set of A, obviously $x \in A$, while, if $A = P(B,b)$, then $x < a < b$ and $x \in B$ implies $x \in P(B,b) = A$. That is, $P(A,a) = P(E,a)$.

Then, it can be shown that E itself is conforming, and obviously the largest conforming subset of X. However also $E \cup \{f(E)\} \supsetneqq E$ is conforming and strictly larger, and so we get a contradiction.

□

1.0.2 Complete metric spaces

Definition 1.3. A metric space is a set X and a function $d : X \times X \to \mathbb{R}$ such that the following properties hold:

1. $d(x,y) = 0 \Longleftrightarrow x = y$;
2. $d(x,y) = d(y,x)$ for any pair $x,y \in X$;
3. $d(x,y) \leq d(x,z) + d(z,y)$ for any choice of $x,y,z \in X$

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space (X,d) is Cauchy if for any $\epsilon > 0$ there exists a $n_\epsilon \in \mathbb{N}$ such that $n,m > n_\epsilon$ implies $d(x_n,x_m) < \epsilon$.

A metric space (X,d) is complete if any Cauchy sequence in (X,d) is convergent in (X,d).

Definition 1.4. A completion of a metric space (X,d) is a pair consisting of a complete metric space $(\widehat{X},\widehat{d})$ and an isometry $j : X \to \widehat{X}$ such that $j(X)$ is dense in $\widehat{X}$.

Theorem 1.5. *Every metric space has a completion. The completion is unique, up to an isometric one to one and onto map.*

Proof. Consider the set M of Cauchy sequences in (X,d). Let us introduce a map $\mathbf{d} : M \times M \to \mathbb{R}$ defined by

$$\mathbf{d}\left(\{x_n\},\{y_n\}\right) = \lim_{n\to+\infty} d(x_n,y_n). \tag{1.1}$$

Notice that the above limit exists and is finite. Indeed, for $\epsilon > 0$ consider $n_\epsilon \in \mathbb{N}$ such that $n,m > n_\epsilon$ implies $d(x_n,x_m) < \epsilon$ and $d(y_n,y_m) < \epsilon$. Then, for $n,m > n_\epsilon$

$$|d(x_n,y_n) - d(x_n,y_m) + d(x_n,y_m) - d(x_m,y_m)| \leq |d(x_n,y_n) - d(x_n,y_m)| + |d(x_n,y_m) - d(x_m,y_m)|.$$

Now we check that

$$|d(x_n,y_n) - d(x_n,y_m)| \leq d(y_n,y_m). \tag{1.2}$$

Indeed $d(x_n, y_n) \leq d(x_n, y_m) + d(y_n, y_m)$ implies

$$d(x_n, y_n) - d(x_n, y_m) \leq d(y_n, y_m).$$

Similarly, $d(x_n, y_m) \leq d(x_n, y_n) + d(y_n, y_m)$ implies

$$d(x_n, y_m) - d(x_n, y_n) \leq d(y_n, y_m).$$

Hence we obtain (1.2). For the same reasons we obtain

$$|d(x_n, y_m) - d(x_m, y_m)| \leq d(x_n, x_m).$$

We conclude that

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m) < 2\epsilon.$$

So the sequence $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$, and hence the limit in (1.4) exists and is finite.

It is easy to see that $\mathbf{d} : M \times M \to [0, +\infty)$ is symmetric and satisfies the triangular inequality. We define the relation

$$\{x_n\} \sim \{y_n\} \Longleftrightarrow \mathbf{d}(\{x_n\}, \{y_n\}) = 0. \tag{1.3}$$

It is easy to see that it is an equivalence relation. Let $\widehat{M} := M/\sim$ with natural projection $\pi : M \to \widehat{M}$. It remains defined a metric $d_M : \widehat{M} \times \widehat{M} \to [0, +\infty)$. There is a natural immersion $j : X \hookrightarrow M$ given by $j(x) = \{x, x, x, ...\}$ such that $d(x, y) = \mathbf{d}(j(x), j(y)) = d_M(\pi \circ j(x), \pi \circ j(y))$. It is also easy to see that $\pi \circ j(X)$ is dense in M since for any given $\pi(\{x_n\}) \in M$, we have $\pi \circ j(x_{n_0}) \xrightarrow{n_0 \to \infty} \pi(\{x_n\})$ in (M, d_M). Indeed, since we know that

$$\text{for any } \epsilon > 0 \text{ there is } N(\epsilon) \text{ s.t. } n, m \geq N(\epsilon) \Longrightarrow |x_n - x_m| < \epsilon,$$

we obtain that for any $n_0 \geq N(\epsilon)$ we have

$$\mathbf{d}(\{x_{n_0}, x_{n_0}, x_{n_0}, ...\}, \{x_n\}) = \lim_{n \to +\infty} d(x_n, x_{n_0}) \leq \epsilon \tag{1.4}$$

which implies $\mathbf{d}(\{x_{n_0}, x_{n_0}, x_{n_0}, ...\}, \{x_n\}) \xrightarrow{n_0 \to \infty} 0$, which, in turn, implies $\pi \circ j(x_{n_0}) \xrightarrow{n_0 \to \infty} \pi(\{x_n\})$ in $(\widehat{M}, d_M)$.

Next, $(\widehat{M}, d_M)$ is complete. Indeed, if $\mathbb{N} \ni m \to \pi(\{x_{n,m}\}_{n \in \mathbb{N}})$ is a Cauchy sequence in $(\widehat{M}, d_M)$, then for any $\epsilon > 0$ there is $\mathbf{N}(\epsilon)$ s.t. $m_1, m_2 \geq \mathbf{N}(\epsilon)$ implies

$$\mathbf{d}(\{x_{n,m_1}\}_{n \in \mathbb{N}}, \{x_{n,m_2}\}_{n \in \mathbb{N}}) = \lim_{n \to +\infty} d(x_{n,m_1}, x_{n,m_2}) < \epsilon.$$

Set $y_1 := x_{1,1}$ and for any $m > 1$ choose $N(m) \in \mathbb{N}$ s.t. $d(x_{N(m),m}, x_{n,m}) < 1/m$ for any $n \geq N(m)$ and set $y_n := x_{N(n),n}$. Let us see, first of all, that $\{y_n\}$ is a Cauchy sequence.

We have for any $j \geq \min(N(m), N(n))$ and for $n, m \geq \mathbf{N}(\epsilon)$

$$
\begin{aligned}
& d\left(x_{N(n),n}, x_{N(m),m}\right) \leq d\left(x_{N(n),n}, x_{j,n}\right) + d\left(x_{j,n}, x_{j,m}\right) + d\left(x_{j,m}, x_{N(m),m}\right) \\
& < 1/n + 1/m + d\left(x_{j,n}, x_{j,m}\right) \ \text{(for } j \geq \min(N(m), N(n))\text{)} \\
& < 1/n + 1/m + \epsilon \ \text{(for } n, m \geq \mathbf{N}(\epsilon) \text{ and } j \gg 1\text{)} \\
& < 2\epsilon \ \text{(for } n, m \gg 1\text{)}\ .
\end{aligned}
$$

Next, we need to show that

$$
\lim_{m \to +\infty} \mathbf{d}\left(\{x_{N(n),n}\}_{n\in\mathbb{N}}, \{x_{n,m}\}_{n\in\mathbb{N}}\right) = 0. \tag{1.5}
$$

Now, for any $\epsilon > 0$,

$$
\begin{aligned}
& \mathbf{d}\left(\{x_{N(n),n}\}_{n\in\mathbb{N}}, \{x_{n,m}\}_{n\in\mathbb{N}}\right) = \lim_{n\to+\infty} d(x_{N(n),n}, x_{n,m}) \\
& \leq \limsup_{n\to+\infty} d(x_{N(n),n}, x_{N(m),m}) + \limsup_{n\to+\infty} d(x_{N(m),m}, x_{n,m}) \\
& \leq \limsup_{n\to+\infty} d(x_{N(n),n}, x_{N(m),m}) + 1/m \leq \epsilon
\end{aligned}
$$

for n and m sufficiently large. This yields (1.5). To complete the proof of Theorem 1.5 see Exercise 1.6 below. □

Exercise 1.6. Show that if (X, d_X) and (Y, d_Y) are two complete metric spaces, if Z is a dense subspace of X and if $T : Z \to Y$ is a continuous map, which is uniformly continuous on bounded subsets of Z, then there is, and is unique, a continuous extension $\overline{T} : X \to Y$ of T.

1.0.3 Tychonov's Theorem

Definition 1.7. Let $\{X_a\}_{a\in A}$ be a family of sets. We consider the product

$$
\prod_{a\in A} X_a := \{(x_a)_{a\in A} : x_a \in X_a \text{ for all } a \in A\}.
$$

Suppose now that each X_a is a topological space. Then the product topology is the weaker topology containing as open sets, products of the form $\prod_{a\in A} U_a$, where $U_a \subseteq X_a$ is open for any $a \in A$ and where $U_a \subsetneq X_a$ for at most finitely many a.

Theorem 1.8 (Tychonov's Theorem)**.** *The Cartesian product $\prod_{a\in A} X_a$ with the product topology is compact if and only if all the X_a are compact.*

The fact that if the product is compact all the X_a are compact follows easily from the fact that the projection function $\pi_{a_0} : X := \prod_{a\in A} X_a \to X_{a_0}$ is continuous for all $a_0 \in A$ and the continuous image of a compact space is compact. So the interesting part is showing that the product is compact. Consider a cover $X = \cup_{b\in B} \mathcal{A}_b$ with open sets. We need to show that there exists a finite subset B_f of B such that $X = \cup_{b \in B_f} \mathcal{A}_b$. An equivalent statement is that if $\cap_{b \in B} \mathcal{C}_b = \emptyset$, where the $\mathcal{C}_b$ are closed sets, then there exists a finite subset B_f of B such that $\cap_{b \in B_f} \mathcal{C}_b = \emptyset$.

Definition 1.9. A collection $\mathfrak{C} = \{\mathcal{C}_b : b \in B\}$ of distinct sets is said to have the finite collection property if for any finite subset B_f of B we have $\cap_{b \in B_f} \mathcal{C}_b \neq \emptyset$.

Notice that the following is elementary.

Exercise 1.10. A topological space X is compact if an only if any collection of closed subsets of X which has the finite collection property has non–empty intersection.

Given two such collections, $\mathfrak{C} = \{\mathcal{C}_b : b \in B\}$ and $\mathfrak{D} = \{\mathcal{D}_{b'} : b' \in B'\}$ of subsets of X, we can write that $\mathfrak{C} \preccurlyeq \mathfrak{D}$ if for any $\mathcal{C}_b$ there exists $\mathcal{D}_{b'} = \mathcal{C}_b$. The set formed by collections enjoying the finite collection property satisfies the *inductive property*, that is, if $\{\mathfrak{C}_j : j \in J\}$ is a totally ordered family of such collections, it has an upper bound, that is, a collection which is larger than all the $\mathfrak{C}_j$. Indeed just consider the collection $\mathfrak{C}$, which is formed by all the sets of all the collections $\mathfrak{C}_j$. Notice that it has the finite collection property, because if $\mathcal{C}_1, ..., \mathcal{C}_n \in \mathfrak{C}$, they belong $\mathcal{C}_1 \in \mathfrak{C}_{j_1}$,..., $\mathcal{C}_n \in \mathfrak{C}_{j_n}$, and since one of the $\mathfrak{C}_{j_1}, ..., \mathfrak{C}_{j_n}$, is the largest, for example $\mathfrak{C}_{j_1}$ is the largest, then $\mathcal{C}_1, ..., \mathcal{C}_n \in \mathfrak{C}_{j_1}$, and since $\mathfrak{C}_{j_1}$ has the finite intersection property, then $\mathcal{C}_1 \bigcap ... \bigcap \mathcal{C}_n \neq \emptyset$. So we conclude that $\mathfrak{C}$ has the finite intersection property.
Now, we apply Zorn's lemma and conclude that there exists a collection $\mathfrak{D} = \{\mathcal{D}_d : d \in D\}$ of distinct sets with the finite collection property and maximal. The following lemma is true.

Lemma 1.11. *Let X be a set and let $\mathfrak{D} = \{\mathcal{D}_d : d \in D\}$ be a collection of distinct subsets with the finite intersection property and maximal. Then the following holds:*

1. *every finite intersections of elements in $\mathfrak{D}$ is in $\mathfrak{D}$;*

2. *if A is a subset of X with non empty intersection with all elements of $\mathfrak{D}$, it is in $\mathfrak{D}$.*

Proof. Let $\mathcal{D}_{d_1} \cap .... \cap \mathcal{D}_{d_n}$ be a finite intersection of elements in $\mathfrak{D}$. It is elementary that if it is not an element of $\mathfrak{D}$, then $\mathfrak{D} \cup \{\mathcal{D}_{d_1} \cap .... \cap \mathcal{D}_{d_n}\}$ satisfies the finite intersection property and, since $\mathfrak{D}$ is a maximal collection with this property and $\mathfrak{D} \cup \{\mathcal{D}_{d_1} \cap .... \cap \mathcal{D}_{d_n}\} \supsetneqq \mathfrak{D}$, we get a contradiction. So $\mathcal{D}_{d_1} \cap .... \cap \mathcal{D}_{d_n} \in \mathfrak{D}$.

Since the 1st claim in the statement is true, moving to the 2nd statement, it is elementary to see that $\mathfrak{D} \cup \{A\}$ satisfies the finite intersection property. By the maximality of $\mathfrak{D}$, it follows that the 2nd statement is true.

□

Completion of the proof of Tychonov's Theorem. Let by contradiction assume that X is not compact. Then there exists a a collection $\mathfrak{C} = \{\mathcal{C}_b : b \in B\}$ formed by closed sets of X which has the finite collection property but which satisfies $\bigcap_{b \in B} \mathcal{C}_b = \emptyset$. We can consider a maximal collection $\mathfrak{D} = \{\mathcal{D}_d : d \in D\}$ with $\mathfrak{C} \preccurlyeq \mathfrak{D}$. If we consider finitely many elements $\mathcal{D}_1, ..., \mathcal{D}_n \in \mathfrak{D}$, by the finite collection property of $\mathfrak{D}$ the intersection $\mathcal{D}_1 \bigcap ... \bigcap \mathcal{D}_n$

is nonempty and has nonempty image $\pi(\mathcal{D}_1 \bigcap ... \bigcap \mathcal{D}_n) = \pi(\mathcal{D}_1) \bigcap ... \bigcap \pi(\mathcal{D}_n)$. So the family of sets $\mathfrak{D}_{(a)} := \{\pi_a(\mathcal{D}_d) : d \in D\}$ has the finite collection property in X_a for any $a \in A$. By the compactness of X_a, we have

$$\bigcap_{d \in D} \overline{\pi_a(\mathcal{D}_d)} \neq \emptyset,$$

so in particular this intersection contains an $x_a \in X_a$. Consider $X \ni \mathbf{x} = (x_a)_{a \in A}$. We claim that $\mathbf{x} \in \mathcal{C}_b$ for any $b \in B$. This will contradict $\bigcap_{b \in B} \mathcal{C}_b = \emptyset$.
An open neighborhood of $\mathbf{x}$ is of the form $\mathcal{U} = \prod_{a \in A} U_a$, where $U_a \subseteq X_a$ is open for any $a \in A$ and where $U_a \subsetneq X_a$ for at most finitely many a. Let $U_{a_1}, ..., U_{a_n}$ be the only factors which are proper subsets of the corresponding X_a's. Then $U_{a_j} \bigcap \pi_{a_j}(\mathcal{D}_d) \neq \emptyset$ for any $d \in D$. So $\pi_{a_j}^{-1}\left(U_{a_j}\right) \cap \mathcal{D}_d \neq \emptyset$ for any $d \in D$. But then $\pi_{a_j}^{-1}\left(U_{a_j}\right)$ is an element of $\mathfrak{D}$ by Lemma 1.11. So $\mathcal{U} = \bigcap_{j=1,\dots,n} \pi_{a_j}^{-1}\left(U_{a_j}\right)$ is an element of $\mathfrak{D}$. We conclude that $\mathcal{U} \cap \mathcal{C}_b \neq \emptyset$ for any $b \in B$ and for any neighborhood $\mathcal{U}$ of $\mathbf{x}$. This implies that $\mathbf{x} \in \mathcal{C}_b$ for any $b \in B$, completing the proof of Tychonov's Theorem.

1.0.4 Normal topological spaces

Recall that a topological space X is Hausdorff if given two distinct points $x, y \in X$ there exist a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$.

Definition 1.12. A topological space X is *regular* if for any $x \in X$ and for any closed subspace B of X with $x \notin B$, there exist a neighborhood U of x and a neighborhood V of B such that $U \cap V = \emptyset$.

A topological space X is *normal* if for any pair A and B of disjoint closed subspaces of X there exist a neighborhood U of A and a neighborhood V of B such that $U \cap V = \emptyset$.

Theorem 1.13. *Every metric space X is normal.*

Proof. Let A and B be two disjoint closed subspaces of X. For any $a \in A$ consider a ball of center a and radius $\epsilon_a > 0$ such that $D_X(a, \epsilon_a) \cap B = \emptyset$ and for any $b \in B$ consider a ball of center b and radius $\epsilon_b > 0$ such that $D_X(b, \epsilon_b) \cap A = \emptyset$. Set

$$U := \bigcup_{a \in A} D_X\left(a, \frac{\epsilon_a}{2}\right) \text{ and } V := \bigcup_{b \in B} D_X\left(b, \frac{\epsilon_b}{2}\right).$$

If now there exists $z \in U \cap V$, then for some $a \in A$ and $b \in B$ we have $z \in D_X\left(a, \frac{\epsilon_a}{2}\right) \bigcap D_X\left(b, \frac{\epsilon_b}{2}\right)$. Then, by the triangular inequality, $d(a, b) < \frac{\epsilon_a + \epsilon_b}{2}$. It is not restrictive to assume $\epsilon_a \leq \epsilon_b$. Then $d(a, b) < \epsilon_b$, contradicting $D_X(b, \epsilon_b) \cap A = \emptyset$. This implies that $U \cap V = \emptyset$, proving the statement. □

Theorem 1.14. *Every compact and Hausdorff space X is normal.*

Proof. First of all we prove that X is regular. Consider $x \in X$ and B closed subspace of X with $x \notin B$. Notice that B is compact. By the Hausdorff property, for any $b \in B$ there are a neighborhood $U^{(b)}$ of x and V_b of b with $U^{(b)} \cap V_b = \emptyset$. Since B is compact, it is possible to find a cover of $B \subseteq V_{b_1} \cup ... \cup V_{b_n}$ which is disjoint from $U^{(b_1)} \cap ... \cap U^{(b_n)}$, which is a neighborhood of x.

Give now any pair A and B of disjoint closed subspaces of X, by the previous part of the proof, for any $a \in A$ there exist a neighborhood U_a of a and a neighborhood $V^{(a)}$ of B so that $V^{(a)} \cap U_a = \emptyset$. It is possible to find a cover of $A \subseteq U_{a_1} \cup ... \cup U_{a_n}$ which is disjoint from $V^{(a_1)} \cap ... \cap V^{(a_n)}$, which is a neighborhood of B.

□

Theorem 1.15 (Urysohn's Lemma). *Let X be a normal space, A and B be two disjoint closed subspaces of X and $[a,b] \subset \mathbb{R}$ a compact interval. Then there exists $f \in C^0(X,[a,b])$ with $f \equiv a$ in A and $f \equiv b$ in B.*

Proof. It is enough to consider $[a,b] = [0,1]$.
Let P be the set of rational numbers in $[0,1]$. We will define a family of open sets $\{U_p\}_{p \in P}$ with

$$\overline{U}_q \subseteq U_p \text{ if } q < p, \tag{1.6}$$

with $A \subseteq U_0$ and $U_1 = X \backslash B$.
Suppose that we have defined $\{U_p\}_{p \in P}$. We can extend this to a family $\{U_p\}_{p \in \mathbb{Q}}$ setting $U_p = \emptyset$ for $p < 0$ and $U_p = X$ for $p > 1$. Notice that (1.7) continues to hold. For any $x \in X$ set now $\mathbb{Q}(x) = \{p \in \mathbb{Q} : x \in U_p\}$. Set now $f : X \to \mathbb{R}$ by

$$f(x) = \inf \mathbb{Q}(x)$$

Notice that $f \equiv 0$ in A (since $A \subseteq U_0$ and $A \cap U_p = \emptyset$ for any $p < 0$) and $f \equiv 1$ in B (since $B \subseteq U_p$ for any $p > 1$ and $B \cap U_1 = \emptyset$. Before proving the continuity of f, we prove the following two statements:

- $x \in \overline{U}_r \Rightarrow f(x) \leq r$
- $x \notin U_r \Rightarrow f(x) \geq r$

Indeed, $x \in \overline{U}_r$ by (1.7) implies $\mathbb{Q}(x) \supseteq \mathbb{Q} \cap (r, +\infty)$ and so $f(x) \leq \inf(\mathbb{Q} \cap (r, +\infty)) = r$; if $x \notin U_r$ with $f(x) < r$ then there exists $p \in \mathbb{Q}(x)$ with $p < r$, which implies $x \in U_p \subset \overline{U}_p \subsetneq U_r$, yielding a contradiction with (1.7).

Let us now prove the continuity, fixing $x_0 \in X$ and an $\epsilon > 0$. Fix two rational numbers $p < q$ with $f(x_0) - \epsilon < p < f(x_0) < q < f(x_0) + \epsilon$. We show that there exists an open neighborhood U of x_0 such that $f(U) \subseteq (f(x_0) - \epsilon, f(x_0) + \epsilon)$. We can choose the open set $U := U_q \backslash \overline{U}_p$. Notice $x \notin U_p \Rightarrow f(x) \geq p > f(x_0) - \epsilon$ and that $x \in \overline{U}_q \Rightarrow f(x) \leq q < f(x_0) + \epsilon$, so it is true that $f(U) \subseteq (f(x_0) - \epsilon, f(x_0) + \epsilon)$. Furthermore, $p < f(x_0)$ implies $x_0 \notin \overline{U}_p$ and $f(x_0) < q$ implies $x_0 \in U_q$, so $x_0 \in U = U_q \backslash \overline{U}_p$.

To complete the proof, we need to define the family of open sets $\{U_p\}_{p\in P}$. Recall that $A \subseteq U_0$ we have $U_1 = X\backslash B$ with $\overline{U}_0 \subsetneqq U_1$. We can arrange P as a sequence, which starts with 0 and 1. Let P_n the set formed by the first n terms and suppose that (1.7) holds for elements of P_n. Consider now $P_{n+1} = P_n \cup \{r\}$. Here $0 < r < 1$, and there are $p < r < q$, $p \in P_n$ the immediate predecessor and $q \in P_n$ the immediate successor of r in P_{n+1}. Consider the pair of closed sets $\widetilde{A} := \overline{U}_p$ and $\widetilde{B} = X\backslash U_q$, which are disjoint because of (1.7). Since X is normal, there exist open neighborhoods U of $\widetilde{A}$ and V of $\widetilde{B}$ with $U \cap V = \emptyset$. Let now $U_r := U$. Then $\overline{U}_p \subseteq U_r$ by definition and $\overline{U}_r \subseteq X\backslash V \subseteq X\backslash\widetilde{B} = U_q$. Hence (1.7) is true also in P_{n+1}. By induction $\{U_p\}_{p\in P}$ remains defined.

□

Corollary 1.16 (Urysohn's Lemma). *Let X be locally compact and Hausdorff and let $K \subseteq V$ with K a compact subset of X and V an open subset of X with $K \subseteq V$. Then there exists $f \in C^0(X, [0,1])$ with $f \equiv 1$ in K and $f \equiv 0$ in $X\backslash V$.*

Proof. Suppose we know that

$$\text{there exists an open set } U \text{ with } K \subseteq U \subseteq \overline{U} \subseteq V \text{ with } \overline{U} \text{ compact.} \tag{1.7}$$

Then consider $f \in C^0(\overline{U}, [0,1])$ with $f \equiv 1$ in K and $f \equiv 0$ in $\overline{U}\backslash U$ which is obtained by the previous Lemma 1.15 (after $f \rightsquigarrow 1 - f$). Then set $f = 0$ in $X\backslash\overline{U}$. In this way we obtain the desired function.

We need to prove the statement in (1.7). Notice that there exists an open set G with $\overline{G}$ compact with $K \subseteq G$. If $V = X$ we are in the previous situation with $U = G$. So assume $V \neq X$ and consider the closed set $B = X\backslash V$. Now, for any $b \in B$ there exist an open neighborhood V_b of b and a relatively compact open neighborhood $U^{(b)}$ of K with $U^{(b)} \cap V_b = \emptyset$. Notice that $U^{(b)} \subseteq X\backslash V_b$ implies $\overline{U}^{(b)} \subseteq X\backslash V_b$, because $X\backslash V_b$ is closed. So, in particular, $b \notin \overline{U}^{(b)}$. Then $\{B \cap \overline{G} \cap \overline{U}^{(b)} : b \in B\}$ is a collection of compact sets with empty intersection. It follows that there exists $\{B \cap \overline{G} \cap \overline{U}^{(b_j)} : j = 1, ..., n\}$ with empty intersection. Then set $U = G \cap U^{(b_1)} \cap ... \cap U^{(b_n)}$: it is an a relatively compact open neighborhood of K whose closure $\overline{G} \cap \overline{U}^{(b)}$ is contained in $X\backslash B = V$.

□

1.0.5 Weierstrass Approximation Theorem

Theorem 1.17 (Weierstrass Approximation Theorem). *The set of real valued polynomials is dense in $C^0([a,b], \mathbb{R})$ for any interval $[a,b]$.*

Proof. It is not restrictive to consider only case $[a,b] = [0,1]$. We recall

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{1.8}$$

Setting $r_k(x) := \binom{n}{k} x^k(1-x)^{n-k}$, we have $\sum_{k=0}^{n} r_k(x) = 1$.
Applying $x\partial_x$ to (1.8) we obtain

$$nx(x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} kx^k y^{n-k}, \tag{1.9}$$

and so

$$nx = \sum_{k=0}^{n} kr_k(x). \tag{1.10}$$

Applying $x^2\partial_x^2$ to (1.8) we obtain

$$n(n-1)x^2(x+y)^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k(k-1)x^k y^{n-k}. \tag{1.11}$$

and so

$$n(n-1)x^2 = \sum_{k=0}^{n} k(k-1)r_k(x) \tag{1.12}$$

The proof given here of this theorem is based on the following formula,

$$\begin{aligned}
&\sum_{k=0}^{n} (k-nx)^2 r_k(x) = n^2x^2 \sum_{k=0}^{n} r_k(x) - 2nx \sum_{k=0}^{n} kr_k(x) + \sum_{k=0}^{n} k^2 r_k(x) \\
&= n^2x^2 - 2nx\ nx + \sum_{k=0}^{n} k(k-1)r_k(x) + \sum_{k=0}^{n} kr_k(x) \\
&= -n^2x^2 + n(n-1)x^2 + nx = -nx^2 + nx = nx(1-x).
\end{aligned}$$

Given now $f \in C^0([0,1],\mathbb{R})$, for any given $\epsilon > 0$ we know that there exists $\delta > 0$ such that for any integral $I \subseteq [0,1]$ of length $|I| \leq \delta$ we have $\mathrm{osc}_I(f) < \epsilon$ where

$$\mathrm{osc}_I f := \sup f(I) - \inf f(I). \tag{1.13}$$

Now we write

$$\begin{aligned}
&\left| f(x) - \sum_{k=0}^{n} f\left(\frac{k}{n}\right) r_k(x) \right| = \left| \sum_{k=0}^{n} \left(f(x) - f\left(\frac{k}{n}\right)\right) r_k(x) \right| \\
&\leq \left| \sum_{|x-\frac{k}{n}|<\delta} \left(f(x) - f\left(\frac{k}{n}\right)\right) r_k(x) \right| + \left| \sum_{|x-\frac{k}{n}|\geq\delta} \left(f(x) - f\left(\frac{k}{n}\right)\right) r_k(x) \right| =: I + II.
\end{aligned}$$

The first term is bounded by

$$I \leq \left| \sum_{\left|x-\frac{k}{n}\right|<\delta} \left(f(x) - f\left(\frac{k}{n}\right)\right) r_k(x) \right| \leq \sum_{\left|x-\frac{k}{n}\right|<\delta} \left|f(x) - f\left(\frac{k}{n}\right)\right| r_k(x) < \epsilon \sum_{\left|x-\frac{k}{n}\right|\leq\delta} r_k(x) < \epsilon,$$

while, for $\operatorname{osc}_{[0,1]}(f) < M$, the 2nd term can be bounded by

$$II \leq \sum_{\left|x-\frac{k}{n}\right|\geq\delta} \left|f(x) - f\left(\frac{k}{n}\right)\right| r_k(x) \leq M \sum_{\left|x-\frac{k}{n}\right|\geq\delta} r_k(x) \leq \frac{M}{\delta^2} \sum_{\left|x-\frac{k}{n}\right|\geq\delta} \left(x - \frac{k}{n}\right)^2 r_k(x)$$

$$\leq \frac{M}{\delta^2} \sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 r_k(x) = \frac{M}{\delta^2 n^2} \sum_{k=0}^{n} (nx-k)^2 r_k(x) = \frac{M}{\delta^2 n^2} nx(1-x) \leq \frac{M}{4\delta^2 n} \xrightarrow{n\to+\infty} 0.$$

From this we derive that there exists an n such that $I + II < 2\epsilon$. □

Remark 1.18. Notice that the main steps in the above proof have probabilistic interpretation. If we consider n independent random variables X_i with $P[X_i = 1] = x$ and $P[X_i = 0] = 1 - x$, then $E[X_i] = x$, $Var[X_i] = x(1-x)$.
For $S_n = X_1 + ... + X_n$ we have $P[S_n = k] = r_k(x)$, $E[S_n] = nx$, $Var[S_n] = nx(1-x)$ and this equality corresponds to the last one in the proof of Theorem 1.17. Notice that in general, given n independent random variables X_i with $E[X_i] \equiv m$, $Var[X_i] \equiv \sigma^2$, then it is simple to prove

$$P\left[\left|\frac{X_1 + ... + X_n}{n} - m\right| \geq \delta\right] \leq \frac{\sigma^2}{n\delta^2}.$$

This inequality generalizes the inequality

$$\sum_{\left|x-\frac{k}{n}\right|\geq\delta} r_k(x) \leq \frac{x(1-x)}{n\delta^2}$$

proved above, see Varadhan Sect. 3.2 [14].

1.0.6 Ascoli Arzelá Theorem

Definition 1.19. Let X be a compact topological space and consider the set of continuous functions from X to $\mathbb{R}$, which we denote by $C^0(X, \mathbb{R})$. Notice that we can introduce in $C^0(X, \mathbb{R})$ the distance $\mathbf{d}(f, g) := \sup_{x\in X} |f(x) - g(x)|$.

Exercise 1.20. Show that in Definition 1.19 is a metric that makes $C^0(X, \mathbb{R})$ a complete metric space.

Theorem 1.21 (Ascoli Arzelá)**.** *Let X be a compact metric space and let $S \subset C^0(X, \mathbb{R})$. Then $\overline{S}$ is compact if and only if:*

1 S is bounded

2 S is equicontinuous

Proof. Suppose S is relatively compact. If S is not bounded and equicontinuous then either there is a sequence $f_n \in S$ such that (1) does not hold in the sense that $|\sup f_n(X)| + |\inf f_n(X)| \xrightarrow{n\to\infty} \infty$, or there is a sequence $f_n \in S$ such that (2) does not hold in the sense that there is an $\epsilon > 0$ such that for any n there are $x_n, y_n \in X$ such that $\mathrm{dist}(x_n, y_n) < 1/n$ and $|f_n(x_n) - f_n(y_n)| \geq \epsilon$. In either case, it is impossible to extract a subsequence of $\{f_n\}$ convergent in $C^0(X, \mathbb{R})$.

Suppose now S is bounded and equicontinuous. Since X is a compact metric space, there is a sequence $x_n \in X$ such that $\forall\, \epsilon > 0$ there is $k(\epsilon) \geq 1$ such that

$$\sup_{x\in X} \inf_{1\leq j\leq k(\epsilon)} \mathrm{dist}(x, x_j) \leq \epsilon.$$

By a diagonal process and by Bolzano Weierstrass (thanks to (1)), we obtain a subsequence f_n such that for any x_m, $\{f_n(x_m)\}_{n\in\mathbb{N}}$ converges. Let us show now that $f_n(x)$ converges for any x.
For any $\epsilon > 0$ and for any $\delta > 0$ and any $j \leq k(\delta)$, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_j)| + |f_m(x_j) - f_n(x_j)| + |f_m(x) - f_m(x_j)|.$$

By equicontinuity, for $\delta > 0$ small we have

$$|f_n(x) - f_n(x_j)| + |f_m(x) - f_m(x_j)| < 2\epsilon \text{ for all } j \leq k(\delta) \text{ with } \mathrm{dist}(x, x_j) \leq \delta \text{ and } m \text{ in } \mathbb{N}.$$

For $m, n > N(\epsilon)$, $|f_m(x_j) - f_n(x_j)| < \epsilon$ for any $j \leq k(\epsilon)$. Then we have proved that $m, n > N(\epsilon) \Rightarrow |f_n(x) - f_m(x)| < 3\epsilon$ for any $x \in X$. Hence we have proved that $\{f_m\}$ is a Cauchy sequence in $C^0(X, \mathbb{R})$. It is easy to conclude that there is an $f(x) := \lim_{n\to+\infty} f_n(x)$ pointwise well defined, and $f_n \xrightarrow{n\to+\infty} f$ uniformly. □

1.0.7 Reisz representation theorem

In this section, we will consider X, a locally compact and Hausdorff topological space. We will denote by $C_c^0(X)(= C_c^0(X, \mathbb{R}))$ to be the space of continuous maps from X to $\mathbb{R}$ which have compact support. Just for this section, if K is a compact subspace of X and $f \in C_c^0(X, [0, 1])$ is such that $f = 1$ in K, we will write $K \prec f$; if V is an open subspace s.t. $f \in C_c^0(X, [0, 1])$ is such that supp f is a compact subspace of V, we will write $f \prec V$.

Theorem 1.22. *Let X be a locally compact and Hausdorff space. Let Λ be a positive linear operator on $C_c^0(X, \mathbb{R})$. Then there exists a σ algebra $\mathcal{M}$ containing the Borelian sets, and a unique positive measure μ on $\mathcal{M}$ such that:*

1 $\Lambda f = \int_X f\, d\mu$ for any $f \in C_c^0(X, \mathbb{R})$.

2 $\mu(K) < \infty$ for any compact K.

3 $\forall E \in \mathcal{M}$ *we have* $\mu(E) = \inf\{\mu(V) : V$ *open* $V \supseteq E\}$.

4 We have $\mu(E) = \sup\{\mu(K) : K$ *compact* $K \subseteq E\}$ $\forall E \in \mathcal{M}$ *open and for all* $\forall E \in \mathcal{M}$ *s.t.* $\mu(E) < \infty$.

5 $\forall E \in \mathcal{M}$ *with* $\mu(E) = 0$ *and for any* A *with* $A \subseteq E$ *we have* $A \in \mathcal{M}$ *and* $\mu(A) = 0$.

Example 1.23. If $X = \mathbb{N}$, then any $\Lambda : C_c^0(\mathbb{N}, \mathbb{R}) \to \mathbb{R}$ like above can be identified with the sequence $\{\Lambda e_n\}_{n\in\mathbb{N}}$ in $[0, +\infty)$, where $e_n(m) = \delta_{nm}$, withe the Kronecker delta. Then

$$\Lambda\left(\{a_n\}_{n\in\mathbb{N}}\right) = \sum_{n=1}^{\infty} a_n \Lambda e_n.$$

Proof. For any open set V set

$$\mu(V) = \sup\{\Lambda f : f \prec V\}.$$

Hence for $V_1 \subseteq V_2$ we have $\mu(V_1) \le \mu(V_2)$. As a consequence, for any open subset $E \subseteq X$ the following is true

$$\mu(E) = \inf\{\mu(V) : E \subseteq V \text{ with } V \text{ open}\}. \tag{1.14}$$

Formula (1.14) makes sense for any subset $E \subseteq X$, and we use it to define $\mu(E)$ for any E. Notice that $E_1 \subseteq E_2$ implies $\mu(E_1) \le \mu(E_2)$.

Let $\mathcal{M}_F$ be the set of the E such that $\mu(E) < \infty$ and such that

$$\mu(E) = \sup\{\mu(K) : E \supseteq K \text{ compact}\}. \tag{1.15}$$

We define $\mathcal{M}$ to be the set of the $E \subseteq X$ such that $E \cap K \in \mathcal{M}_F$ for any compact K.

Claim 1.24. $\mathcal{M}_F$ contains any compact set K.

Proof. It is enough to show $\mu(K) < \infty$. Pick $K \prec f$ and let $V = \{f > 1/2\}$. Then $K \subset V$ and $g \le 2f$ for any $g \prec V$. Then

$$\mu(K) \le \mu(V) = \sup\{\Lambda g : g \prec V\} \le \Lambda(2f) < \infty.$$

□

Notice that the above claim implies also that any compact K belongs to $\mathcal{M}$.

Suppose now that $\mu(E) = 0$. Then, it follows from by monotonicity that $\mu(K) = 0$ for any compact $K \subseteq E$, so that (1.15) is true. Hence $E \in \mathcal{M}_F$. And since any subset of E, for the same reasons, belongs to $\mathcal{M}_F$, it follows that $E \in \mathcal{M}$. This proves claim 5 in the statement.

Claim 1.25. Every open set satisfies (1.15). Hence $\mathcal{M}_F$ contains every open set V with $\mu(V) < \infty$.

Proof. Let $\alpha < \mu(V)$. Then there exists $f \prec V$ such that $\alpha < \Lambda f$. Given any open set $W \supseteq \text{supp } f$, we have $f \prec W$, and hence $\alpha < \Lambda f \leq \mu(W)$. We also have $\mu(\text{supp } f) \geq \Lambda f > \alpha$. This yields formula (1.15).

□

Claim 1.26. For any sequence of arbitrary sets $E_1 ...$ in X we have

$$\mu\left(\cup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu\left(E_n\right). \tag{1.16}$$

Proof. Observe first of all that for two opens sets, we have

$$\mu\left(V_1 \cup V_2\right) \leq \mu\left(V_1\right) + \mu\left(V_2\right). \tag{1.17}$$

To show this pick $f \prec V_1 \cup V_2$. Now there are $h_j \prec V_j$ such that $h_1 + h_2 = 1$ on supp f. Then $f = fh_1 + fh_2$ with $fh_j \prec V_j$ and since, linearity $\Lambda f = \Lambda fh_1 + \Lambda fh_2$, we conclude that

$$\Lambda f = \Lambda fh_1 + \Lambda fh_2 \leq \mu\left(V_1\right) + \mu\left(V_2\right) \text{ for any } f \prec V_1 \cup V_2.$$

Hence

$$\mu\left(V_1 \cup V_2\right) = \sup\{\Lambda f : f \prec V_1 \cup V_2\} \leq \mu\left(V_1\right) + \mu\left(V_2\right).$$

Notice that (1.17) extends immediately into

$$\mu\left(V_1 \cup ... \cup V_n\right) \leq \mu\left(V_1\right) + ... + \mu\left(V_n\right) \text{ for any } n \geq 2 \text{ open sets } V_1, ..., V_n. \tag{1.18}$$

Going back to the countable subadditivity, if $\mu\left(E_n\right) = \infty$ for some n we are fine. If this is not the case, consider $E_n \subset V_n$, V_n open with $\mu\left(V_n\right) < \mu\left(E_n\right) + 2^{-n}\epsilon$. Consider the open set $V = \cup_{n=0}^{\infty} V_n$ and consider $f \prec V$. Since supp f is compact, there exists an n such that

$$\Lambda f \leq \mu\left(\cup_{i=1}^{n} V_i\right) \leq \sum_{i=1}^{n} \mu\left(V_i\right) \leq \sum_{i=1}^{\infty} \mu\left(E_i\right) + \epsilon,$$

where in the 2nd inequality we used (1.18). Since the last formula holds for any $f \prec V$ and we have $\cup_{i=1}^{\infty} E_i \subseteq V$, we we conclude that

$$\mu\left(\cup_{i=1}^{\infty} E_i\right) \leq \mu(V) \leq \sum_{i=1}^{\infty} \mu\left(E_i\right) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain (1.16). □

Claim 1.27. For any sequence $E_1 ...$ of disjoint elements in $\mathcal{M}_F$ we have for $E = \cup_{n=1}^{\infty} E_n$

$$\mu\left(E\right) = \sum_{n=1}^{\infty} \mu\left(E_n\right) \tag{1.19}$$

and, if $\mu(E) < \infty$, then $E \in \mathcal{M}_F$.

Proof. Let us first show that for disjoint compact sets we have

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2). \tag{1.20}$$

We know already that $\mu(K_1 \cup K_2) \le \mu(K_1) + \mu(K_2)$ is true
There are disjoint open sets V_1 and V_2, with $V_j \supseteq K_j$. Consider an open set $W \supseteq K_1 \cup K_2$ with $\mu(W) \le \mu(K_1 \cup K_2) + \epsilon$. There are $f_i \prec W \cap V_i$ with $\Lambda f_i > \mu(W \cap V_i) - \epsilon$. By $K_i \subset W \cap V_i$ and $f_1 + f_2 \prec W$ (where $f_1 + f_2 \in C_c^0(X, [0,1])$ because supp $f_1 \cap$ supp $f_2 \subseteq V_1 \cap V_2 = \emptyset$),

$$\mu(K_1) + \mu(K_2) \le \mu(W \cap V_1) + \mu(W \cap V_2) < \Lambda f_1 + \Lambda f_2 + 2\epsilon$$
$$\le \mu(W) + 2\epsilon < \mu(K_1 \cup K_2) + 3\epsilon.$$

By the arbitrariness of $\epsilon > 0$, we conclude $\mu(K_1 \cup K_2) \ge \mu(K_1) + \mu(K_2)$, and so (1.20). Notice that it is elementary that (1.20) extends to the case of $n \ge 2$ disjoint compact subspaces.

Going back to the countable union, if we have $\mu(E) = \infty$ both sides are equal to ∞ by the countable subadditivity. So suppose $\mu(E) < \infty$. Since $E_i \in \mathcal{M}_F$, there is a compact $H_i \subseteq E_i$ such that $\mu(E_i) \ge \mu(H_i) - 2^{-i}\epsilon$. For $K_n := H_1 \cup ... \cup H_n$,

$$\mu(E) \ge \mu(K_n) = \sum_{i=1}^{n} \mu(H_i) > \sum_{i=1}^{n} \mu(E_i) - \epsilon,$$

where the equality follows from (1.20), in the case of $n \ge 2$ disjoint compact subspaces.

This holds for any n and ϵ. So

$$\mu(E) \ge \sum_{i=1}^{\infty} \mu(E_i).$$

By the previously proved countable subadditivity we have equality, obtaining (1.19). In particular, taking N large enough,

$$\mu(E) \le \sum_{i=1}^{N} \mu(E_i) + \epsilon \le \sum_{i=1}^{N} \mu(H_i) + 2\epsilon = \mu(K_N) + 2\epsilon.$$

This proves (1.15), and so proves $E \in \mathcal{M}_F$. □

Claim 1.28. If $E \in \mathcal{M}_F$ and $\epsilon > 0$, there are $K \subseteq E \subseteq V$, K compact and V open, with $\mu(V - K) < \epsilon$.

Proof. There are K and V with $K \subseteq E \subseteq V$ and $\mu(V) - \epsilon/2 < \mu(E) < \mu(K) + \epsilon/2$. $V - K$ is open, we have $\mu(V - K) \le \mu(V) \le \mu(K) + \epsilon < \infty$, so $V - K \in \mathcal{M}_F$ by Claim 1.25. Then

$$\mu(K) + \mu(V - K) = \mu(V) < \mu(K) + \epsilon,$$

where the equality follows by Claim 1.27, yields the desired result. □

Claim 1.29. If $A, B \in \mathcal{M}_F$, then $A - B$, $A \cup B$, $A \cap B$ all belong to $\mathcal{M}_F$.

Proof. We start observing that for $\epsilon > 0$ there are $K_1 \subseteq A \subseteq V_1$ and $K_2 \subseteq B \subseteq V_2$ with $\mu(V_i \backslash K_i) < \epsilon$. It is elementary that

$$A \backslash B \subseteq V_1 \backslash K_2 \subseteq (V_1 \backslash K_1) \cup (K_1 \backslash V_2) \cup (V_2 \backslash K_2). \tag{1.21}$$

Indeed $A \subseteq V_1$ and $K_2 \subseteq B$ gives the 1st inclusion. Looking at the 2nd inclusion, elements of $V_1 \backslash K_2$ not in $V_1 \backslash K_1$ are necessarily elements of $x \in K_1 \backslash K_2$. Since $X = V_2 \cup \complement V_2$, then $K_1 = (K_1 \cap V_2) \cup (K_1 \cap \complement V_2)$. If $x \in K_1 \cap V_2$, then, since $x \notin K_2$, we have $x \in V_2 \backslash K_2$ Otherwise, $x \in K_1 \cap \complement V_2 = K_1 \backslash V_2$. So, (1.21) is proved.

By subadditivity

$$\mu(A - B) \leq 2\epsilon + \mu(K_1 - V_2).$$

Since $(K_1 - V_2)$ is a compact subset of $A - B$, we conclude $A - B \in \mathcal{M}_F$. Next, $A \cup B = (A - B) \cup B$ and $A \cup B \in \mathcal{M}_F$ by the previous step on disjoint unions. Finally, $A \cap B = A - (A - B)$. □

Claim 1.30. $\mathcal{M}$ is a σ algebra containing all Borel sets.

Proof. Let K be compact in X. If $A \in \mathcal{M}$, then $\complement A \cap K = K - (A \cap K)$ is the difference of two elements in $\mathcal{M}_F$ and so by Claim 1.29 it belongs to $\mathcal{M}_F$. So $A \in \mathcal{M}$ implies $\complement A$ in $\mathcal{M}$. Next, let $A = \cup_1^\infty A_i$ with $A_i \in \mathcal{M}$. Let K be compact, let $B_1 = A_1 \cap K$ and

$$B_n = (A_n \cap K) - (B_1 \cup ... \cup B_{n-1}).$$

Then the B_n form a disjoint sequence in $\mathcal{M}_F$ and so $A \cap K = \cup_1^\infty B_i \in \mathcal{M}_F$ by Claim 1.27. So $A \in \mathcal{M}$. Finally, let C be closed. Then, for any K, $C \cap K$ is compact so is in $\mathcal{M}_F$ and $C \in \mathcal{M}$. □

Claim 1.31. $\mathcal{M}_F$ contains exactly the $E \in \mathcal{M}$ with $\mu(E) < \infty$.

Proof. Let $E \in \mathcal{M}_F$. For any K compact, $E \cap K \in \mathcal{M}_F$ by Claims 1.24 and 1.29. By definition, this implies $E \in \mathcal{M}$.
Let us pick now $E \in \mathcal{M}$ with $\mu(E) < \infty$. Then there is $V \supseteq E$ with $\mu(V) < \infty$. By Claim 1.25, there is a compact $K \subseteq V$ with $\mu(V - K) < \epsilon$. By $E \cap K \in \mathcal{M}_F$ there is $H \subseteq E \cap K$ compact with $\mu(E \cap K) < \mu(H) + \epsilon$. We have

$$E = (E \cap K) \cup (E - K) \subseteq (E \cap K) \cup (V - K).$$

This implies

$$\mu(E) \leq \mu(E \cap K) + \mu(V - K) \leq \mu(H) + 2\epsilon,$$

so $E \in \mathcal{M}_F$. □

Claim 1.32. For any $f \in C_c^0(X)$ we have $\Lambda f = \int f d\mu$.

Proof. It is enough to prove

$$\Lambda f \leq \int f d\mu \text{ for any } f \in C_c^0(X, \mathbb{R}) \tag{1.22}$$

since by $\Lambda(-f) \leq \int(-f)d\mu$ we get $\Lambda f \geq \int f d\mu$, and thus the equality.
Let K be the support of f and $[a, b]$ the range. Pick

$$y_0 < a < y_1 < ... < y_n = b \text{ with } y_i - y_{i-1} < \epsilon \text{ for all } i = 1, ..., n.$$

Set $E_i = f^{-1}(]y_{i-1}, y_i]) \cap K$. These are Borel, disjoint with union K. There are open sets V_i with $\mu(V_i) < \mu(E_i) + \epsilon/n$ and $f < y_i + \epsilon$ in V_i. There are $h_i \prec V_i$ with $\sum h_i = 1$ on K.

$$\begin{aligned}
\Lambda f &= \sum_{i=1}^n \Lambda(h_i f) \leq \sum_{i=1}^n (y_i + \epsilon)\Lambda(h_i) \leq \sum_{i=1}^n (y_i + \epsilon)\mu(V_i) \\
&\leq \sum_{i=1}^n (y_i + \epsilon)\mu(E_i) + \sum_{i=1}^n (y_i + \epsilon)\frac{\epsilon}{n} \\
&\leq \sum_{i=1}^n (y_i - \epsilon)\mu(E_i) + 2\epsilon\mu(K) + (b + \epsilon)\epsilon \leq \sum_{i=1}^n y_{i-1}\mu(E_i) + 2\epsilon\mu(K) + (b + \epsilon)\epsilon \\
&\leq \sum_{i=1}^n \int_{E_i} f d\mu + \epsilon(2\mu(K) + b + \epsilon) = \int_X f d\mu + \epsilon(2\mu(K) + b + \epsilon).
\end{aligned}$$

□

Definition 1.33. A positive measure is regular if for any Borel set E,

$$\begin{aligned}
\mu(E) &= \sup\{\mu(K) : \ K \text{ compact set with } K \subseteq E \ \} \ (E \text{ inner regular}) \\
&= \inf\{\mu(A) : \ A \text{ open set with } A \supseteq E \ \} \ (E \text{ outer regular}).
\end{aligned}$$

Remark 1.34. In Theorem 1.22 every $E \in \mathcal{M}$ for the measure μ, while the inner regularity is proved for all $E \in \mathcal{M}_F$.

Theorem 1.35. *Let X be a locally compact, Hausdorff and σ–compact space (X is a countable union of compact sets). Let $\mathcal{M}$ and μ be like in Theorem 1.22. Then the following happens.*

1. *For any $E \in \mathcal{M}$ and $\epsilon > 0$ there exists $F \subseteq E \subseteq V$, F closed and V open, with $\mu(V \backslash K) < \epsilon$.*
2. *μ is regular.*
3. *For any $E \in \mathcal{M}$ there exist A, a countable union of closed sets, B, a countable intersection of open sets, with $A \subseteq E \subseteq B$, with $\mu(B \backslash A) = 0$.*

Proof. See Rudin [9]. □

Theorem 1.36. *Let X be a locally compact and Hausdorff space where every open set is σ–compact (a countable union of compact sets). Then any Borel measure μ such that $\mu(K) < \infty$ on any compact set, is regular.*

Proof. See Rudin [9]. $\square$

2 Topological vector spaces on $K = \mathbb{R}, \mathbb{C}$

We will consider Topological Vector Spaces, that is, vector spaces, where the algebraic and the topological structure are *compatible*.

Definition 2.1 (Topological Vector Space)**.** Consider a vector space X on the field $K = \mathbb{R}, \mathbb{C}$. A Hausdorff topological structure $(X, \mathcal{T})$ on E is said to be compatible with the vector space structure if the maps

$$X \times X \ni (x, y) \to x + y \in X \text{ and } K \times X \ni (\lambda, x) \to \lambda x \in X \text{ are continuous.} \tag{2.1}$$

Exercise 2.2. Given a topological vector space X show that $X^n = \overbrace{X \times .... \times X}^{n} \ni (x_1, ..., x_n) \to x_1 + ... + x_n \in X$ is continuous for any n, for the product topology in X^n.

Exercise 2.3. Show that in a topological vector space X a subset $U \subseteq X$ is a neighborhood of a point $x_0 \in X$ is an only if $U = x_0 + V$, where $V \subseteq X$ is a neighborhood of $0 \in X$.

Definition 2.4. Consider a vector space X on K and a subset $\Omega \subseteq X$. Then Ω is said to be

1. *balanced*, if $x \in \Omega$ and $|\lambda| \leq 1$ imply $\lambda x \in \Omega$,
2. *absorbing*, if for any $x \in X$ there exists a scalar λ such that $x \in \lambda\Omega$.

Exercise 2.5. Any neighborhood U of 0, in a topological vector space X, is absorbing.

Answer. Consider the U, neighborhood 0 and let $x \neq 0$. Since $\mathbb{R} \ni \lambda \to \mu x \in X$ is a continuous map, there exists $\delta > 0$ such that $\mu x \in U$ for $|\mu| < \delta$. Pick one such $\mu \neq 0$. Then $\mu x \in U \iff x \in \lambda U$ for $\lambda = 1/\mu$. $\square$

Lemma 2.6. *For any given neighborhood U of 0 of a topological vector space X, there exists a balanced neighborhood V of 0 such that $V \subseteq U$.*

Proof. Fix any neighborhood U of 0 in X. By continuity, there exists an open neighborhood $\widetilde{V}$ of 0 and a $\delta > 0$ such that $\lambda\widetilde{V} \subseteq U$ for any $|\lambda| \leq \delta$. Let $V = \bigcup_{|\lambda| \leq \delta} \lambda\widetilde{V}$. Then V is an open neighborhood of 0 contained in U, and it is easy to see that it is absorbing. $\square$

Remark 2.7. It is well known, and easy to check, that in a Hausdorff topological space X every subset $\{x\}$ for $x \in X$ is closed. In the context of topological vector spaces, if we subtract from the hypotheses that X is Hausdorff, but we ask that each $\{x\}$ for $x \in X$ is closed, then in fact, X is Hausdorff.

Lemma 2.8. *Assume that X is a vector space, that it has a topology for which* (2.1) *is true and that each $\{x\}$ for $x \in X$ is closed, then X is Hausdorff.*

Proof. It is enough to show that if $x \neq 0$, then there exists a neighborhoods U of 0 and V of x such that $U \cap V = \emptyset$. Since $\{x\}$ is closed, we know that there exists a neighborhood U_1 of 0 such that $x \notin U_1$. Furthermore, there exists a neighborhood U of 0 such that $U + U \subseteq U_1$. Furthermore, since Lemma 2.6 continues to hold under our hypotheses, we ca assume that U is balanced. So, in particular, $U - U \not\ni x$. It follows that $U \cap (x + U) = \emptyset$. $V := x + U$ is the desired neighborhood of x. □

Definition 2.9. Given two topological vector spaces X and Y we denote by $\mathcal{L}(X, Y)$ the set of linear operators defined in X and with values in Y which are continuous. In particular, for $Y = K$, we set $X' = \mathcal{L}(X, K)$ and we call it the dual space of X. We call the elements of X' the *linear functionals* on X. Finally, when $X = Y$ we write $\mathcal{L}(X) := \mathcal{L}(X, Y)$.

Exercise 2.10. Show that a linear map $T : X \to Y$ between two topological vector spaces is continuous if an only if it is continuous in just one point $x_0 \in X$.

Exercise 2.11. Let X be a topological vector space on $\mathbb{C}$. Show that the map $v : X \to \mathbb{R}$ in (2.2) is an $\mathbb{R}$–linear and continuous if and only if the map $f : X \to \mathbb{C}$ in (2.2) is $\mathbb{C}$–linear and continuous,

$$f(x) := v(x) - \mathrm{i}v(\mathrm{i}x). \tag{2.2}$$

Obviously any $\mathbb{C}$–vector space X is also an $\mathbb{R}$–vector space. It is easy to turn an $\mathbb{R}$–vector space into a $\mathbb{C}$–vector space. There are various possibilities, with the first indicated in the following exercise.

Exercise 2.12. Suppose that X is vector space on $\mathbb{C}$ and that $J : X \to X$ is a linear map such that $J^2 = -1$. Then show that $\mathbb{C} \times X \ni (z, x) \to (\mathrm{Re}(z) + \mathrm{Im}(z)J)x \in X$ makes X into a $\mathbb{C}$–vector space. If furthermore X is a topological vector space and $J \in \mathcal{L}(X)$, then show that the above gives X a structure of topological vector space on $\mathbb{C}$.

Another possibility is the following.

Remark 2.13 (Complexification). Suppose that X is vector space on $\mathbb{C}$ and consider the space $\mathbb{C} \otimes_{\mathbb{R}} X$. There is an obvious identification of $\mathbb{C} \otimes_{\mathbb{R}} X \supset \mathbb{R} \otimes_{\mathbb{R}} X \cong X$ and a complex structure on $\mathbb{C} \otimes_{\mathbb{R}} X$, by $\lambda_1(\lambda_2 \otimes x) = (\lambda_1\lambda_2) \otimes x$. If X has a structure as topological vector space, then so does $\mathbb{C} \otimes_{\mathbb{R}} X$. Finally, and crucially, for any $\mathbb{R}$–linear $T : X \to X$, setting $T(\lambda \otimes x) = \lambda \otimes T(x)$, a related $\mathbb{C}$–linear operator remains defined, and if the initial T is continuous, also the other T is continuous.

For time dependent PDE's, especially for Hamiltonian systems, when it is necessary to consider the spectrum of the operators, it is important to complexify.

Definition 2.14. Given a topological vector space X, a subset $B \subset X$ is called *bounded* if for any neighborhood V of 0 there exists a $\lambda > 0$ such that $\lambda V \supseteq B$.

Definition 2.15. Given two topological vector spaces X and Y, a linear operator $T : X \to Y$ is *bounded* if, for any bounded subset $B \subset X$, the image TB is bounded in Y.

Exercise 2.16. Show that if X and Y are topological vector spaces and $T : X \to Y$ is a continuous linear operator, then it is also bounded, in the sense of Definition 2.15.

The following is very important.

Lemma 2.17. *Let X be a topological vector space on K and let $T : X \to K$ be a linear map with $Tx \neq 0$ for some $x \in X$. The following statements are equivalent:*

a $T \in X'$*;*

b $\ker T$ *is closed;*

c $\ker T$ *is not dense in X;*

d T *is bounded on some neighborhood of* $0 \in X$*.*

Proof. Clearly **a** $\Rightarrow$ **b** $\Rightarrow$ **c**. Now assume **c**. It follows that there exists a point x and a neighborhood V of 0 such that $x + V \cap \ker T = \emptyset$. We can also assume by Lemma 2.6 that V is balanced. Then $TV \subseteq K$ is balanced. If TV is a bounded set, **d** follows. Otherwise, we claim that

$$TV = K. \tag{2.3}$$

Indeed, if $|Tx_n| \xrightarrow{n \to +\infty} +\infty$ for $\{x_n\}_{n \in \mathbb{N}}$ a sequence in V, for each n we have $D_K(0, |Tx_n|) \subseteq TV$ by the fact that TV is balanced, thus proving $TV = K$.
If (2.3) is true, there exists $y \in V$ such that $Ty = -Tx$ and so $x + y \in (x + V) \cap \ker T$, giving a contradiction.
Finally suppose **d**. Then $|Ty| < M$ for all y in a neighborhood V of 0 and for a fixed $M \in \mathbb{R}_+$. Then, for any $\epsilon > 0$, for $x \in \dfrac{\epsilon}{M} V$, for $\dfrac{\epsilon}{M} V \ni x = \dfrac{\epsilon}{M} y$, where $y \in V$, we have $|Tx| = \dfrac{\epsilon}{M}|Ty| < \dfrac{\epsilon}{M} M = \epsilon$, hence the continuity in 0, and so everywhere. □

Definition 2.18. Let X be a topological vector space. A subset $H \subseteq X$ is called a hyperplane if $H = f^{-1}(a)$ where $f : H \to K$ is a (bounded or unbounded) linear map.

Exercise 2.19. Let X be a topological vector space on K and let $T : X \to K$ be a linear map. Let $k_0 \in K$ be $k_0 \neq 0$. Show that the following statements are equivalent:

a $T \in X'$;

b $T^{-1}(k_0)$ is closed;

c $T^{-1}(k_0)$ is not dense in X.

Answer. Notice that **a** $\Rightarrow$ **b** $\Rightarrow$ **c**. Assuming **c** we have either $T^{-1}(k_0) = \emptyset$, which implies $T = 0$ and so is continuous, or there exists x_0 such that $T(x_0) = k_0$. Then from linearity it follows $T^{-1}(k_0) = x_0 + \ker T$ and it is easy to conclude that $T^{-1}(k_0)$ is not dense in X if and only if $\ker T$ is not dense in X. Hence **c** $\Rightarrow$ **a** by Lemma 2.17. □

Definition 2.20. A topological vector space X is metrizable if there is a metric on X which induces the topology of X.
A metric d on a vector space is translation invariant if

$$d(x,y) = d(x+z, y+z) \text{ for all } x,y,z \in X. \tag{2.4}$$

Definition 2.21. A basis of neighborhoods of a point x_0 in a topological space X is a family $\mathfrak{U}$ of neighborhoods of x_0 such that for any neighborhood V of x_0 in X there exists $U \in \mathfrak{U}$ with $U \subseteq V$.

A subbasis of neighborhoods of a point x_0 in a topological space X is a family $\mathfrak{U}$ of neighborhoods of x_0 such that the family of finite intersections of elements of $\mathfrak{U}$ is a basis of neighborhoods of x_0.

It is obvious that if a topological vector space X is metrizable, then any point of X has a countable basis of neighborhoods. The following converse is true, see Theorem 1.24 [10].

Theorem 2.22. *If X is a topological vector space such that each point of X has a countable basis of neighborhoods then X is metrizable and admits a translation invariant metric compatible to the topology such that all the balls centered in 0 are balanced. If furthermore X is locally convex*[1]*, then it is possible to find a metric compatible to the topology which, in addition to the above properties, is such that all the open balls are convex.*

Proof. We can consider a basis $\{V_n\}_{n\in\mathbb{N}}$ of neighborhoods of 0. We can assume them to be balanced and such

$$V_{n+1} + V_{n+1} \subset V_n \text{ for all } n \in \mathbb{N}. \tag{2.5}$$

Next we consider $D := \mathbb{Q} \cap [0,1)$. Now any $r \in D$ can be written as

$$r = \sum_{n=1}^{\infty} c_n(r) 2^{-n}, \text{ with } c_n(r) = 1 \text{ for finitely many } n\text{'s and with } c_n(r) = 0 \text{ otherwise.} \tag{2.6}$$

Let $A(r) := X$ for $r \geq 1$ and set

$$A(r) := \sum_{n=1}^{\infty} c_n(r) V_n, \text{ for } r \in D\text{-} \tag{2.7}$$

Now we define $f : X \to [0,\infty)$ and $d : X^2 \to [0,\infty)$ by

$$f(x) := \inf\{r : x \in A(r)\} \text{ and } d(x,y) = f(x-y). \tag{2.8}$$

Notice that if d is a metric, it is obviously translation invariant. We claim that

$$A(r) + A(s) \subseteq A(r+s) \text{ for all } r,s \in D. \tag{2.9}$$

[1]See later Sect. 4.

Let us assume (2.9). We then claim that

$$f(x+y) \leq f(x) + f(y) \text{ for all } x, y \in X. \tag{2.10}$$

This is true if the r.h.s. equals 1, so we assume we are in a case with the r.h.s. < 1. Then for any $\epsilon > 0$ there exists $r, s \in D$ with

$$f(x) < r\,, \quad f(y) < s \text{ and } r + s < f(x) + f(y) + \epsilon.$$

Then $x \in A(r)$ and $y \in A(r)$. Then (2.9) implies that $x + y \in A(r + s)$ and

$$f(x+y) \leq r + s < f(x) + f(y) + \epsilon \Longrightarrow f(x+y) \leq f(x) + f(y).$$

Hence (2.10) is proved.
Since the $\{V_n\}_{n\in\mathbb{N}}$ are balanced, from (2.7) we see that the $A(r)$ are balanced as well. Hence $f(\lambda x) = f(x)$ for all $|\lambda| = 1$. That $f(0) = 0$ follows from $0 \in A(r)$ for all r. If $x \neq 0$, then we must have $x \notin V_n = A(2^{-n})$ for some n. This implies that $f(x) \geq 2^{-n} > 0$.
We conclude that (2.8) defines a metric on X where

$$D(0,\delta) = \{x \in X : f(x) < \delta\} = \bigcup_{r<\delta} A(r)$$

is a neighborhood of 0 in X. If $\delta < 2^{-n}$, then $D(0, \delta) \subseteq V_n$. This impies that the topology induced by d, is the same of the initial one.

We now prove (2.9) by an induction argument. We consider the proposition

$$A(r) + A(s) \subseteq A(r+s) \text{ if } r + s < 1 \text{ and } c_n(r) = c_n(s) = 0 \text{ for all } n > N. \tag{P_N}$$

For $N = 1$, if $r = 0$ then $A(r) = 0$ and so the formula is obvious. So we reduce to $r = s = 1/2$, so $c_2(r) = c_2(s) = 1$ and all the others are nil,

$$A(r) + A(s) = V_2 + V_2 \subseteq V_1$$

by (2.5).
Suppose now that (P_{N-1}) is true for an $N > 1$. Consider $r, s \in D$ with $c_n(r) = c_n(s) = 0$ for all $n > N$ an let r' and s' be defined by

$$r = r' + c_N(r)2^{-N}\ , \quad s = s' + c_N(s)2^{-N}.$$

Then

$$A(r) = A(r') + c_N(r)V_N\ , \quad A(s) = A(s') + c_N(s)V_N.$$

By (P_{N-1}) we have $A(r') + A(s') \subseteq A(r' + s')$. Then

$$A(r) + A(s) \subseteq A(r' + s') + c_N(r)V_N + c_N(s)V_N.$$

If $c_N(r) = c_N(s) = 0$ from the above we get (P_N). If $c_N(r) = 0$ and $c_N(s) = 1$ we have

$$A(r) + A(s) \subseteq A(r' + s') + V_N = A(r' + s' + 2^{-N}) = A(r + s).$$

Finally, for $c_N(r) = c_N(s) = 1$ we have

$$\begin{aligned} A(r) + A(s) \subseteq A(r' + s') + V_N + V_N \subseteq A(r' + s') + V_{N-1} &= A(r' + s') + A(2^{-(N-1)}) \\ &\subseteq A(r' + s' + 2^{-(N-1)}), \end{aligned}$$

where in the last step we have used case (P_{N-1}). This completes the proof, because $r' + s' + 2^{-(N-1)} = r + s$ and we have shown that (P_{N-1}) implies (P_N).

$\square$

Definition 2.23. A sequence $\{x_n\}$ in a topological vector space X is a Cauchy sequence when for any neighborhood V of 0 in X there exists a $n(V)$ such that for $n, m > n(V)$ we have $x_n - x_m \in V$. A topological vector space X is sequentially complete if any Cauchy sequence in X is convergent in X.

Exercise 2.24. Show that if a topological vector space X is metrizable and if we consider on it a translation invariant metric, then a sequence $\{x_n\}$ in X is a Cauchy sequence in the sense of Definition 2.23 if and only if it is a Cauchy sequence in the sense in Sect. 1.0.2.

3 Norms on Vector Spaces on $K = \mathbb{R}, \mathbb{C}$

By far, the most important topological vector spaces, are the normed spaces.

The most basic notion in Functional Analysis is that of norm in a vector space X on the field $K = \mathbb{R}, \mathbb{C}$.

Definition 3.1 (Norms)**.** A map $\| \cdot \| : X \to [0, +\infty)$ is called a norm on a vector space X if it satisfies the following properties:

1. $\|x\| = 0 \iff x = 0$
2. $\|x + y\| \le \|x\| + \|y\|$ for all pairs $x, y \in X$
3. $\|\lambda x\| = |\lambda| \; \|x\|$ for all $\lambda \in K$ and $x \in X$.

   A vector space X endowed with a norm $\| \cdot \|_X$ is called a normed space.

Exercise 3.2. Check that if on a vector space X there is a norm $\|\cdot\|$, then $d(x, y) := \|x - y\|$ defines a metric on X.

Exercise 3.3. Check that if on a vector space X there is a norm $\|\cdot\|$, then for the topology associated to the corresponding metric, we have that the maps $X \times X \ni (x, y) \to x + y \in X$ and $K \times X \ni (\lambda, x) \to \lambda x \in X$ are continuous.

The important normed spaces, are the complete ones.

Definition 3.4 (Banach space)**.** A normed vector space $(X, \|\cdot\|)$ which is complete for the associated metric, is called a Banach space.

Exercise 3.5. Consider a non–complete normed vector space $(X, \|\cdot\|_X)$ on K and let $(\widehat{X}, \widehat{d})$ be its completion provided by Theorem 1.5. Show that $\widehat{X}$ is a complete normed vector space.

Example 3.6 (Lebesgue spaces)*.* Let us consider a measure space (X, μ) with a positive measure μ and let us consider the spaces $L^p(X, d\mu)$ for $p \geq 1$. Then, for any $f \in L^p(X, d\mu)$ let

$$\|f\|_{L^p(X,d\mu)} := \left(\int_X |f(x)|^p d\mu \right)^{\frac{1}{p}} \text{ for } p < \infty \text{ and} \tag{3.1}$$

$$\|f\|_{L^\infty(X,d\mu)} := \sup\{c \geq 0 : \mu\left(\{x : |f(x)| \geq c\}\right) > 0\}. \tag{3.2}$$

These, as we will see below, are norms, by the Minkowsky inequality, see below Theorem 16.2.

Example 3.7 (Spaces of Continuous functions)*.* Let Ω be an open subspace of $\mathbb{R}^d$. Interesting vector subspaces of $L^\infty(\Omega)$ are

$$C_c^0(\Omega) := \{f \in C^0(\Omega) : \ f \text{ has compact support in } \Omega\} \tag{3.3}$$

which is often denoted $C_0^0(\Omega)$,

$$BC^0(\Omega) := C^0(\Omega) \cap L^\infty(\Omega). \tag{3.4}$$

An important space is

$$C_0^0(\mathbb{R}^d) := \{f \in C^0(\mathbb{R}^d) : \lim_{x \to \infty} f(x) = 0\}. \tag{3.5}$$

Exercise 3.8. Show that for $f \in BC^0(\Omega)$

$$\|f\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |f(x)|. \tag{3.6}$$

Example 3.9 ($H^\infty(\Omega)$)*.* Let Ω be an open subspace of $\mathbb{R}^2$. We consider the vector space

$$H(\Omega) := \{f \in C^0(\Omega, \mathbb{C}) : \ f \text{ is a holomorphic function } \Omega \to \mathbb{C}\} \tag{3.7}$$

We consider the following subspace of $L^\infty(\Omega, \mathbb{C})$,

$$H^\infty(\Omega) := H(\Omega) \cap L^\infty(\Omega, \mathbb{C}). \tag{3.8}$$

Notice that if $f \in H^\infty(\mathbb{R}^2)$ then $f(z)$ is a constant function.

Example 3.10. In the notation of Example 3.7 for $n \in \mathbb{N}$

$$C^n_c(\Omega) := \{f \in C^n(\Omega) : \ f \text{ has compact support in } \Omega\} \tag{3.9}$$

which is often denoted $C^n_0(\Omega)$,

$$BC^n(\Omega) := \{f \in C^n(\Omega) : \ \partial^\alpha_x f \in BC^0(\Omega) \text{ for all } |\alpha| \le n\}. \tag{3.10}$$

Notice that the following is a norm on $BC^n(\Omega)$,

$$\|f\|_{BC^n(\Omega)} = \sum_{|\alpha|\le n} \|\partial^\alpha_x f\|_{L^\infty(\Omega)}. \tag{3.11}$$

For $\theta \in (0,1)$ and for $f \in BC^n(\Omega)$ let

$$[f]_{C^{n,\theta}(\Omega)} := \sup_{|\mu|=n} \sup_{x\neq y \text{ in } \Omega} \frac{|\partial^\mu f(x) - \partial^\mu f(y)|}{|x-y|^\theta}.$$

Then we set

$$C^{n,\theta}(\Omega) := \{f \in BC^n(\Omega) : [f]_{C^{n,\theta}(\Omega)} < +\infty\}. \tag{3.12}$$

Notice that the following is a norm on $C^{n,\theta}(\Omega)$,

$$\|f\|_{C^{n,\theta}(\Omega)} := \|f\|_{BC^n(\Omega)} + [f]_{C^{n,\theta}(\Omega)}. \tag{3.13}$$

In particular, for $n = 0$ the space of globally Hölder functions in Ω is

$$C^{0,\theta}(\Omega) := \{f \in BC(\Omega) : [f]_{C^{0,\theta}(\Omega)} < +\infty\} \text{ with norm} \tag{3.14}$$

$$\|f\|_{C^{0,\theta}(\Omega)} := \|f\|_{L^\infty(\Omega)} + [f]_{C^{0,\theta}(\Omega)}. \tag{3.15}$$

In the context of Lebesgue spaces, the analogue of the space of globally Hölder functions in Ω is the following.

Example 3.11. For $\theta \in (0,1)$ and $1 \le p < \infty$ is, for Ω open subspace in $\mathbb{R}^d$,

$$W^{\theta,p}(\Omega) := \{f \in L^p(\Omega) : [f]_{W^{\theta,p}(\Omega)} < +\infty\} \text{ where} \tag{3.16}$$

$$[f]^p_{W^{\theta,p}(\Omega)} := \int_{\Omega\times\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{\theta p + d}} dxdy \text{ with norm}$$

$$\|f\|_{W^{\theta,p}(\Omega)} := \|f\|_{L^p(\Omega)} + [f]_{W^{\theta,p}(\Omega)}. \tag{3.17}$$

Example 3.12. A special case of Example 3.6 is obtained taking $X = \mathbb{N}, \mathbb{Z}$ with $\mu(\{n\}) = 1$. Then we have the spaces

$$\|\{x_n\}_{n\in X}\|_{\ell^p(X)} := \left(\sum_{n\in X} |x_n|^p\right)^{\frac{1}{p}} \text{ for } p < \infty \text{ and} \tag{3.18}$$

$$\|\{x_n\}_{n\in X}\|_{\ell^\infty(X)} := \sup\{|x_n| : n \in X\}. \tag{3.19}$$

A special vector subspace of $\ell^\infty(X)$ is

$$c_0(X) := \{\{x_n\}_{n\in X} \in \ell^\infty(X) : \lim_{n\to\infty} x_n = 0\}. \tag{3.20}$$

Exercise 3.13. Check that for $0 < p < q \le \infty$ one has $\ell^p(X) \subset \ell^q(X)$, in particular with

$$\|\{x_n\}_{n\in X}\|_{\ell^q(X)} \le \|\{x_n\}_{n\in X}\|_{\ell^p(X)}. \tag{3.21}$$

Answer. Notice that for any $n_0 \in X$ and for $p \in (0,\infty)$,

$$|x_{n_0}|^p \le \sum_{n\in X} |x_n|^p = \|x.\|^p_{\ell^p(X)}.$$

This implies $\|x.\|_{\ell^\infty(X)} \le \|x.\|_{\ell^p(X)}$, and in particular (3.21) for $q=\infty$. Let now $0 < p < q < \infty$. Then

$$\|x.\|^q_{\ell^q(X)} = \sum_{n\in X} |x_n|^q \le \|x.\|^{q-p}_{\ell^\infty(X)} \sum_{n\in X} |x_n|^p = \|x.\|^{q-p}_{\ell^\infty(X)} \|x.\|^p_{\ell^p(X)}.$$

So we conclude

$$\|x.\|_{\ell^q(X)} \le \|x.\|^{\frac{q-p}{q}}_{\ell^\infty(X)} \|x.\|^{\frac{p}{q}}_{\ell^p(X)} \le \|x.\|^{\frac{q-p}{q}}_{\ell^p(X)} \|x.\|^{\frac{p}{q}}_{\ell^p(X)} = \|x.\|_{\ell^p(X)}.$$

$\square$

Example 3.14. The following are Banach spaces.

1. The Lebesgue spaces $L^p(X,\mu)$ for $1 \le 1 \le +\infty$.

2. $BC^0(\Omega)$. Indeed, if $\{f_n\}$ is a Cauchy sequence in $BC^0(\Omega)$, by the completeness of $L^\infty(\Omega)$ there exists $f \in L^\infty(\Omega)$ such that $f_n \xrightarrow{n\to\infty} f$ in $L^\infty(\Omega)$. Notice that by (3.6), that is by $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_{L^\infty(\Omega)}$ for all $x \in \Omega$, it follows that $\{f_n(x)\}$ is a Cauchy sequence for any $x \in \Omega$. We can assume $f(x) = \lim_{n\to+\infty} f_n(x)$ for any $x \in \Omega$. Notice that then

$$|f(x) - f_n(x)| \le \|f - f_n\|_{L^\infty(\Omega)} \text{ for any } x \in \Omega \text{ and } n \in \mathbb{N}.$$

Indeed, for any pair pair $x \in \Omega$ and $n \in \mathbb{N}$ we have

$$|f(x) - f_n(x)| = \lim_{m\to+\infty} |f_m(x) - f_n(x)| \le \lim_{m\to+\infty} \|f_m - f_n\|_{L^\infty(\Omega)} = \|f - f_n\|_{L^\infty(\Omega)}.$$

Let us show now that $f \in C^0(\Omega)$. Let $x_0 \in \Omega$ and let $\epsilon > 0$. Then

$$\text{there exists } N \text{ such that } n > N \Longrightarrow \|f - f_n\|_{L^\infty(\Omega)} < \frac{\epsilon}{3}.$$

Fix now $n > N$ and let $\delta > 0$ such that

$$|f_n(x) - f_n(x_0)| < \frac{\epsilon}{3} \text{ for all } x \in D_\Omega(x_0,\delta).$$

Then we have the following, which completes the proof of $f \in C^0(\Omega)$,

$$\begin{aligned}&|f(x) - f(x_0)| \le |f(x) - f_n(x)| + |f(x_0) - f_n(x_0)| + |f_n(x) - f_n(x_0)| \\ &< 2\frac{\epsilon}{3} + |f_n(x) - f_n(x_0)| < \epsilon \text{ for all } x \in D_\Omega(x_0,\delta).\end{aligned}$$

3. $H^\infty(\Omega)$, where Ω is an open subset of $\mathbb{R}^2$. Notice that $H^\infty(\Omega) \subset BC^0(\Omega)$. So if $\{f_n\}$ is a Cauchy sequence in $H^\infty(\Omega)$, from what see above we know that there exists an $f \in BC^0(\Omega)$ such that $f_n \xrightarrow{n\to\infty} f$ in $L^\infty(\Omega)$. Notice that by Cauchy Theorem on triangles [9, Theorem 10.13] we have (by triangle, we mean also the interior)
$$\int_{\partial T} f_n(z)dz = 0 \text{ for any triangle } T \subset \Omega \text{ and any } n \in \mathbb{N}$$
Since now $f_n \xrightarrow{n\to\infty} f$ in $BC^0(\Omega)$ implies
$$\int_{\partial T} f(z)dz = \lim_{n\to+\infty} \int_{\partial T} f_n(z)dz = 0 \text{ for any triangle } T \subset \Omega,$$
by Morera Theorem [9, Theorem 10.17] we have $f \in H(\Omega)$. Hence $H^\infty(\Omega)$ is a closed subspace in $L^\infty(\Omega)$.

4. $BC^l(\Omega)$ for any $l \in \mathbb{N}$. Let $\{f_n\}$ be a Cauchy sequence in $BC^l(\Omega)$. This is expressed equivalently saying that $\{\partial_x^\alpha f_n\}$ is a Cauchy sequence in $BC^0(\Omega)$ for any $|\alpha| \le l$. We know that for any $|\alpha| \le l$ there exists a $g_\alpha \in BC^0(\Omega)$ such that $\partial_x^\alpha f_n \xrightarrow{n\to\infty} g_\alpha$ in $BC^0(\Omega)$. We need to show that $g_\alpha = \partial_x^\alpha g_0$. It is enough to prove this for $|\alpha| = 1$. It is not restrictive to assume $\alpha = e_1 := (1, 0, ..., 0)$. We know that for any $x \in \Omega$ there exists a $\delta_x > 0$ such that for $0 < |h| < \delta_x$ we have
$$\frac{f_n(x_1+h, x') - f_n(x_1, x')}{h} = \int_{x_1}^{x_1+h} \partial_1 f_n(t, x')dt \text{ for all } n \in \mathbb{N}, \text{ where } x' = (x_2, ..., x_d).$$
Taking the limit $n \longrightarrow +\infty$ the above equalities yields
$$\frac{g_0(x_1+h, x') - g_0(x_1, x')}{h} = \frac{1}{h}\int_{x_1}^{x_1+h} g_{e_1}(t, x')dt \text{ for } 0 < |h| < \delta_x\ .$$
Hence we conclude
$$\lim_{h\to 0} \frac{g_0(x_1+h, x') - g_0(x_1, x')}{h} = \lim_{h\to 0} \frac{1}{h}\int_{x_1}^{x_1+h} g_{e_1}(t, x')dt = g_{e_1}(x) \Longrightarrow$$
$$g_{e_1}(x) = \partial_1 g_0(x) \text{ for all } x \in \Omega.$$
By symmetry, $g_{e_j}(x) = \partial_j g_0(x)$ for all $x \in \Omega$ and all $j = 1, ..., d$.

5. $C^{l,\theta}(\Omega)$ for any $l \in \mathbb{N}_0$ and any $\theta \in (0, 1)$. Let $\{f_n\}$ be a Cauchy sequence in $C^{l,\theta}(\Omega)$. We know that $f_n \xrightarrow{n\to\infty} f$ in $BC^l(\Omega)$. It is enough to focus on the case $l = 0$. For $x \neq y$ we have
$$\begin{aligned}\frac{|f(x) - f(y)|}{|x-y|^\theta} &\le \frac{|f_n(x) - f_n(y)|}{|x-y|^\theta} + \frac{|f(x) - f_n(x)|}{|x-y|^\theta} + \frac{|f(y) - f_n(y)|}{|x-y|^\theta} \\ &\le [f_n]_{C^{0,\theta}(\Omega)} + \frac{2}{|x-y|^\theta}\|f - f_n\|_{L^\infty(\Omega)}.\end{aligned}$$

Then, sending $n \longrightarrow +\infty$ we get

$$\frac{|f(x)-f(y)|}{|x-y|^\theta} \le \limsup_{n\to+\infty}[f_n]_{C^{0,\theta}(\Omega)} < +\infty \text{ for all } x \neq y \text{ in } \Omega.$$

So we conclude that $f \in C^{l,\theta}(\Omega)$. Let us now show that

$$\lim_{n\to+\infty}[f-f_n]_{C^{0,\theta}(\Omega)} = 0.$$

For any $\epsilon > 0$ we know there exists n_ϵ such that for any pair $n, m > n_\epsilon$ we have $[f_m - f_n]_{C^{0,\theta}(\Omega)} < \epsilon$. Now, for $x \neq y$ and $n > n_\epsilon$ we have

$$\frac{|f(x)-f_n(x)-(f(y)-f_n(y)|}{|x-y|^\theta} = \lim_{m\to+\infty}\frac{|f_m(x)-f_n(x)-(f_m(y)-f_n(y)|}{|x-y|^\theta} \le \epsilon.$$

This implies that $n > n_\epsilon$ we have $[f-f_n]_{C^{0,\theta}(\Omega)} \le \epsilon$ and proves $f_n \xrightarrow{n\to\infty} f$ in $C^{0,\theta}(\Omega)$.

Notice that there are very natural vector spaces, for example $C_c^\infty(\mathbb{R}^d)$, that do not have an obvious norm.

4 Locally convex spaces

Normed and Banach spaces are not the only important topological vector spaces. A very important notion is that of a convex subspace of a vector space.

Definition 4.1. 1. A subset Ω of a vector space is convex if for any $x_0, x_1 \in \Omega$ we have $x_t := (1-t)x_0 + tx_1 \in \Omega$ for all $t \in [0,1]$

2. A subset Ω of a topological vector space is strictly convex if it is convex and if $x_t \in \mathring{\Omega}$(:=interior of Ω) for all $t \in (0,1)$ for any distinct pair $x_0, x_1 \in \Omega$.

Example 4.2. 1. Let $(E, \|\cdot\|)$ be a normed space. Then $\overline{D_E(0,1)} := \{x \in E : \|x\| \le 1\}$ is convex. Indeed, given any pair $x_0, x_1 \in \overline{D_E(0,1)}$ and for any $t \in [0,1]$, we have

$$\|(1-t)x_0 + tx_1\| \le (1-t)\|x_0\| + t\|x_1\| \le (1-t) + t = 1.$$

2. Notice that in $\mathbb{R}^d$, $\|x\| := \sup\{|x_j| : j = 1, ..., d\}$ for which

$$\overline{D_{\mathbb{R}^d}(0,1)} = [-1,1]^d,$$

which is convex but not strictly convex.

3. The previous example can be generalized noticing that any $L^\infty(X, d\mu)$ is such that $\overline{D_{L^\infty(X,d\mu)}(0,1)}$ is not strictly convex (except trivial cases). Indeed, consider two disjoint measurable sets E ad F of finite positive measure, and consider $f_0 := 1_E$ and $f_1 := 1_E + 2^{-1}1_F$. We have $\|f_0\|_{L^\infty(X,d\mu)} = \|f_1\|_{L^\infty(X,d\mu)} = 1$ and

$$\|(1-t)f_0 + tf_1\|_{L^\infty(X,d\mu)} = \|1_E + 2^{-1}t1_F\|_{L^\infty(X,d\mu)} = 1.$$

Exercise 4.3. Let X be vector space and let $\{\Omega_j\}_{j\in J}$ be a family of convex subspaces. Show that $\bigcap_{j\in J}\Omega_j$ is convex.

Exercise 4.4. Let X be a topological vector space and let Ω be a convex subspace. Then, the following are true.

1. The closure $\overline{\Omega}$ is convex.

2. The interior $\mathring{\Omega}$ is convex.

Answer. Let $\overline{x}_0, \overline{x}_1 \in \overline{\Omega}$. This means that for any neighborhood V of 0, there exist elements $(\overline{x}_0 + V) \cap \Omega \ni x_0$ and $(\overline{x}_1 + V) \cap \Omega \ni x_1$. Consider $\overline{x}_t = (1-t)\overline{x}_0 + t\overline{x}_1$ and $x_t = (1-t)x_0 + tx_1$ for a fixed $t \in (0,1)$. For any given neighborhood V of 0 in X, we know that there exists a neighborhoods U and W of 0 in X such that $(1-t)U \subseteq W$, $tU \subseteq W$ and $W + W \subseteq V$. Then

$$x_t - \overline{x}_t = (1-t)(x_0 - \overline{x}_0) + t(x_1 - \overline{x}_1) \in (1-t)U + tU \subseteq V.$$

So, for any $t \in (0,1)$ and any neighborhood V of 0 in X we have $(\overline{x}_t + V) \cap \Omega \neq \emptyset$ and so $\overline{x}_t \in \overline{\Omega}$.

Let us now turn to the second part. We need to show that if $x_0, x_1 \in \mathring{\Omega}$, then for any $t \in (0,1)$ we have $x_t = (1-t)x_0 + tx_1 \in \mathring{\Omega}$. We know that there exists a neighborhood U of 0 such that $x_j + U \subseteq \mathring{\Omega}$ for both $j = 0, 1$. Then we claim that $x_t + U \subseteq \Omega$. To prove this claim notice that any element of $x_t + U \subseteq \Omega$ can be written as

$$x_t + u = (1-t)(x_0 + u) + t(x_1 + u) \in \Omega.$$

□

Definition 4.5. **a** A topological vector space X is said locally convex if, given any neighborhood U of 0, there exists a convex neighborhood V of 0 such that $V \subseteq U$.

b A topological vector space X is said a Frechét space if it is locally convex, metrizable with a translation invariant metric and complete.

Remark 4.6. Recall that it follows by Theorem 2.22 that a locally convex topological vector space is metrizable with a translation invariant metric if and only if 0 has a numberable basis of neighborhoods.

Lemma 4.7. *Given a vector space X and a subset $\Omega \subseteq X$, there exists a convex set $\mathcal{C}$ which is the smallest convex set containing Ω.*

Definition 4.8. We call the above $\mathcal{C}$ the *convex hull* of Ω in X.

Proof of Lemma 4.7. We consider $\mathfrak{C} = \{C : \Omega \subseteq C \subseteq X \text{ and } C \text{ convex}\}$. Obviously $\mathfrak{C} \ni X$. Then the intersection $\bigcap_{C\in\mathfrak{C}} C$ is the desired set. □

Exercise 4.9. Let $\Omega \subseteq X$ and $\lambda \in K$. Show that if $\mathcal{C}$ is the convex hull of Ω then $\lambda\mathcal{C}$ is the convex hull of $\lambda\Omega$.

Lemma 4.10. *For any given neighborhood U of 0 of a locally convex topological vector space X, there exists a convex neighborhood V of 0 which is balanced, absorbing and such that $V \subseteq U$.*

Proof. It is not restrictive to consider U convex. For any $\lambda \in \overline{D}_K(0,1)$ we know that there exists a neighborhood W_λ of λ in K and a convex neighborhood U_λ of 0 in X such that $W_\lambda U_\lambda \subseteq U$. If we consider now a finite cover $W_{\lambda_1} \cup ... \cup W_{\lambda_n} \supset \overline{D}_K(0,1)$ and set $\widetilde{U} := U_{\lambda_1} \cap ... \cap U_{\lambda_n}$, then we consider $\widetilde{V} := \overline{D}_K(0,1)\widetilde{U} \subseteq U$. Finally, let V be the convex hull of $\widetilde{V}$. Notice that V is a neighborhood of 0, since it contains $\widetilde{U}$, V is obviously convex, and $V \subseteq U$. We know that V is absorbing. We need to show that V is balanced, that is that $\lambda V \subseteq V$ for any $\lambda \in \overline{D}_K(0,1)$. Notice $\lambda\widetilde{V} \subseteq \widetilde{V} \subseteq V$ and, by a previous exercise, the convex hull of $\lambda\widetilde{V}$ is λV. So it follows $\lambda V \subseteq V$.

□

It is not clear yet why locally convex spaces are so important. To understand this point we need to introduce the notion of seminorm.

Definition 4.11. Let X be a vector space and let $p : X \to [0, +\infty)$ be a function with

$$p(x + y) \leq p(x) + p(y) \text{ for all } x, y \in X \tag{4.1}$$

$$p(\lambda x) = \lambda p(x) \text{ for all } x \in X \text{ and } \lambda > 0. \tag{4.2}$$

Then p is called a seminorm.

Exercise 4.12. Let X be a vector space and let $p : X \to [0, +\infty)$ be a seminorm. Let

$$C = \{x \in X : p(x) < 1\}. \tag{4.3}$$

Then show that C is convex, $0 \in C$, C is absorbing.

Partial answer. Notice the following, which proves the convexity of C,

$$p(x_t) = p((1-t)x_0 + tx_1) \leq p((1-t)x_0) + p(tx_1) = (1-t)p(x_0) + tp(x_1) < (1-t) + t = 1.$$

□

The following lemma, shows that to any open, convex, absorbing and balanced subspace $C \subseteq X$ we can associate a seminorm $p : X \to [0, +\infty)$.

Lemma 4.13. *Let X be a topological vector space and let C be an open convex set with $0 \in C$. Then there exists a seminorm $p : X \to [0, +\infty)$ satisfying* (4.1)–(4.3).

Proof. Set

$$p(x) := \inf\{a > 0 : \frac{x}{a} \in C\}. \tag{4.4}$$

First of all, it is clear that for $x \in C$ we have $1 \in \{a > 0 : \frac{x}{a} \in C\}$, and so $p(x) \leq 1$. Furthermore, since C is open, then $(1+\epsilon)x \in C$ for $\epsilon > 0$ small, so $p((1+\epsilon)x) \leq 1$ and so $p(x) \leq \frac{1}{1+\epsilon} < 1$.

If for some $x \in X$ we have $p(x) < 1$, then for some $\alpha < 1$ we have $\frac{x}{\alpha} \in C$ and so by the convexity of C and by $0 \in C$ we have $x = \alpha \frac{x}{\alpha} + (1-\alpha)0 \in C$.
So we have proved (4.3). Now let us prove first (4.2) and then (4.1).
For $\lambda > 0$, for $a = \lambda\alpha$ we obtain (4.2) from

$$p(\lambda x) = \inf\{a > 0 : \frac{\lambda x}{a} \in C\} = \inf\{\lambda\alpha > 0 : \frac{x}{\alpha} \in C\} = \lambda \inf\{\alpha > 0 : \frac{x}{\alpha} \in C\} = \lambda p(x).$$

Given $x, y \in X$, then for any $\epsilon > 0$ we have $\frac{x}{p(x)+\epsilon} \in C$, $\frac{y}{p(y)+\epsilon} \in C$. Then for $t \in [0,1]$,

$$t\frac{x}{p(x)+\epsilon} + (1-t)\frac{y}{p(y)+\epsilon} \in C. \tag{4.5}$$

For $t = \frac{p(x)+\epsilon}{p(x)+p(y)+2\epsilon}$ we get $\frac{x+y}{p(x)+p(y)+2\epsilon} \in C$, as can be seen from

$$1 - t = 1 - \frac{p(x)+\epsilon}{p(x)+p(y)+2\epsilon} = \frac{p(x)+p(y)+2\epsilon - (p(x)+\epsilon)}{p(x)+p(y)+2\epsilon} = \frac{p(y)+\epsilon}{p(x)+p(y)+2\epsilon}.$$

Hence we obtain the following which, by the arbitrariness of $\epsilon > 0$, yields (4.1),

$$p(x+y) < p(x) + p(y) + 2\epsilon.$$

□

Exercise 4.14. Consider the p of Lemma 4.13 and show that $p \in C^0(X, [0, +\infty))$.

Answer. Notice that $|p(x) - p(x_0)| \le p(x - x_0)$ for any $x, x_0 \in X$. For $\epsilon > 0$ then $\epsilon\mathcal{C}$ is an open neighborhood of 0 and coincides with the solutions of the inequality $p(y) < \epsilon$. Then if x belongs to the open neighborhood $x_0 + \epsilon\mathcal{C}$ of x_0, it follows that $|p(x) - p(x_0)| \le p(x - x_0) < \epsilon$, proving the continuity of p at the point x_0. □

Remark 4.15. Lemma 4.13 and Exercise 4.31 show that there exists a correspondence between open and convex neighborhoods of 0 and continuous seminorms.

Summing up, above we have proved the following.

Lemma 4.16. *Let X be a locally convex Hausdorff topological vector space. Then there exists a family $\{p_j\}_{j\in J}$ of continuous seminorms (this family is called a subbasis of seminorms of X) such that for any $x_0 \in X\backslash\{0\}$ there exists a $j_0 \in J$ such that $p_{j_0}(x_0) \neq 0$ and such that the family $\{p_j^{-1}([0,r)) : r > 0$ and $j \in J\}$ is a subbasis of neighborhoods of 0.*

□

Definition 4.17. A function $f : X \to K$ is homogeneous of order $\alpha \ge 0$ if $f(\lambda x) = \lambda^\alpha f(x)$ for any $x \in X$ and any $\lambda > 0$.

Remark 4.18. Notice that a seminorm $p : X \to K$ is a homogeneous function of order 1. A Linear map $f : X \to K$ is a homogeneous function of order 1.

Exercise 4.19. Consider the setup Lemma 4.16, that is X with the seminorms $\{p_j\}_{j\in J}$. Show that a homogeneous of order 1, $f : X \to K$, is continuous in 0 if and only if there exist finitely many indexes $j_1, ..., j_n \in J$ and a $\epsilon > 0$ such that

$$|f(x)| \leq 1 \text{ for all } x \text{ such that } p_{j_1}(x) < \epsilon, ..., p_{j_n}(x) < \epsilon. \tag{4.6}$$

Exercise 4.20. Consider the setup of Exercise 4.19. Show that a homogeneous function of order 1, $f : X \to K$, is continuous in 0 is and only if there exist finitely many indexes $j_1, ..., j_n \in J$ and a constant $C > 0$ such that

$$|f(x)| \leq C\,(p_{j_1}(x) + ... + p_{j_n}(x)) \text{ for all } x \in X. \tag{4.7}$$

Answer. We can assume formula (4.6) is true. We claim that (4.7) is true for $C := \frac{2}{\epsilon}$. Set $p(x) := p_{j_1}(x) + ... + p_{j_n}(x)$ and let $C := \frac{2}{\epsilon}$. Then, for $p(x) = \frac{\epsilon}{2}$ we have $p_{j_k}(x) \leq \frac{\epsilon}{2}$ for all $k = 1, ..., n$ and so $|f(x)| \leq 1$. So (4.7) is true for $p(x) = \frac{\epsilon}{2}$. By the homogeneity this yields automatically all cases where $p(x) \neq 0$. Notice that by linearity, if $p(x) = 0$ then by (4.6) we conclude $f(x) = 0$. So our claim is true. □

Example 4.21. Consider the space $L^p(0,1)$ for $0 < p < 1$. We can define a metric by setting

$$d(f,g) := \int_0^1 |f(t) - g(t)|^p dt.$$

Let us see the above is a metric. First of all, it is obviously it is symmetric and $d(f,g) = 0 \iff f = g$. By

$$(a+b)^p \leq a^p + b^p \text{ for any pair } a, b \geq 0,$$

(which follows by $\left(\dfrac{a}{a+b}\right)^p + \left(\dfrac{b}{a+b}\right)^p \geq \dfrac{a}{a+b} + \dfrac{b}{a+b} = 1$) we have

$$\begin{aligned} d(f,g) &= \int_0^1 |f(t) - g(t)|^p dt = \int_0^1 |\,(f(t) - h(t)) + (h(t) - g(t))\,|^p dt \\ &\leq \int_0^1 (|f(t) - h(t)|^p + |h(t) - g(t)|^p)\, dt = d(f,h) + d(h,g) \text{ for any } f, g, h \in L^p(0,1). \end{aligned}$$

It is easy to see that with this metric, $L^p(0,1)$ becomes a topological vector space.

We claim now that the only open convex subsets of $L^p(0,1)$ are $\emptyset$ and $L^p(0,1)$. To see this, let V be open, convex, not empty and $V \ni 0$ and let $f \in L^p(0,1)$. Since V is open, there exists $\varepsilon_0 > 0$ such that $D_{L^p(0,1)}(0, \varepsilon_0) \subseteq V$. Let $n \in \mathbb{N}$ such that $n^{p-1} \int_0^1 |f(t)|^p dt < \varepsilon_0$ and consider a decomposition $t_0 = 0 < t_1 < ... < t_n = 1$ such that $\int_{t_{j-1}}^{t_j} |f(t)|^p dt = n^{-1} \int_0^1 |f(t)|^p dt$. Then set $g_j(t) := n\chi_{[t_{j-1}, t_j]} f$. We have

$$\int_0^1 |g_j(t)|^p dt = n^p \int_{t_{j-1}}^{t_j} |f(t)|^p dt = n^{p-1} \int_0^1 |f(t)|^p dt < \varepsilon_0$$

so that we have $g_j \in D_{L^p(0,1)}(0, \varepsilon_0) \subseteq V$ for all j. Then, since

$$f = \frac{g_1 + ... + g_n}{n},$$

by the convexity of V we have also $f \in V$. So $V = L^p(0, 1)$.

Example 4.22. Consider the space $L^p(0, 1)$ for $0 < p < 1$ of Example 4.21. Then if X is a locally convex topological vector space, then the only continuous linear map $T : L^p(0, 1) \to X$ is the 0 one. Indeed, for any non–empty open convex $V \subseteq X$, $T^{-1}V$ is a convex open set in $L^p(0, 1)$, and so, from what we saw in Example 4.21, it is either the empty set or the whole $L^p(0, 1)$. So we conclude $T^{-1}V = L^p(0, 1)$ for any non–empty open convex neighborhood $V \subseteq X$ of 0 (for which $T^{-1}V \ni 0$), and so also $TL^p(0, 1) \subseteq V$. So in particular, $TL^p(0, 1) \subseteq \bigcap V$ where the intersection is done on all open convex sets containing 0. Since the intersection is 0, we conclude $TL^p(0, 1) = 0$, that is, T is 0.

Remark 4.23. As we mentioned earlier, in Functional Analysis what matters are most of all the linear or nonlinear operators. A consequence of Example 4.22, we have $(L^p(0, 1))' = 0$ for $0 < p < 1$: this makes $L^p(0, 1)$ for $0 < p < 1$ a not very useful space.

Exercise 4.24. Let $f : [0, +\infty) \to [0, +\infty)$ be a concave function with $f(0) = 0$. Show that

$$f(x + y) \leq f(x) + f(y) \text{ for any } x, y \in [0, +\infty). \tag{4.8}$$

Answer. Notice that f satisfies

$$f((1 - t)x_0 + tx_1) \geq (1 - t)f(x_0) + tf(x_1) \text{ for any } x_0, x_1 \in [0, +\infty) \text{ any } t \in [0, 1].$$

Now, notice that if we consider the triple $0, x, x + y$ we have

$$x = \frac{x}{x + y}(x + y) + \left(1 - \frac{x}{x + y}\right) 0$$

and so

$$f(x) \geq \frac{x}{x + y} f(x + y) + \left(1 - \frac{x}{x + y}\right) f(0) = \frac{x}{x + y} f(x + y)$$

and, similarly,

$$f(y) \geq \frac{y}{x + y} f(x + y).$$

So, summing up, we get (4.8) by

$$f(x) + f(y) \geq \frac{x}{x + y} f(x + y) + \frac{y}{x + y} f(x + y) = f(x + y).$$

□

Exercise 4.25. Consider the setup and the hypotheses of Lemma 4.16, and consider the topological vector space X with the structure arising from the seminorms $\{p_j\}_{j\in J}$. Suppose that $J \subseteq \mathbb{N}$. Show that

$$d(x, y) := \sum_{j\in J} 2^{-j} \frac{p_j(x-y)}{1 + p_j(x-y)} \tag{4.9}$$

is a translation invariant metric and that the topology this metric induces on X is the initial one.

Answer. First of all, let us check that d is a metric. Here we check only the triangular inequality. Here notice that

$$\begin{aligned} &p_j(x-y) \le p_j(x-z) + p_j(z-y) \Longrightarrow f(p_j(x-y)) \le f(p_j(x-z) + p_j(z-y)) \\ &\le f(p_j(x-z)) + f(p_j(z-y)) \text{ for } f(t) = \frac{t}{1+t}, \end{aligned}$$

where we use that f is concave with $f(0) = 0$, and hence we can apply (4.8). The above implies the triangular inequality. Next, let (X, τ_1) be the initial topology and (X, τ_2) the topology induced by d. It is enough to compare the neighborhoods of 0.
Let us consider the ball $D(0, \epsilon) = \{x : d(x, 0) < \epsilon\}$. Now, if $N \in \mathbb{N}$ is such that $2^{-N} < \epsilon/2$, we claim that

$$U_N(\epsilon/2) := \{x \in X : p_j(x) < \epsilon/2 \text{ for all } j = 1, ..., N\} \subseteq D(0, \epsilon). \tag{4.10}$$

To see this, notice that for any $x \in X$ we have

$$\begin{aligned} d(x, 0) &= \sum_{j=1}^{N} 2^{-j} \frac{p_j(x)}{1 + p_j(x)} + \sum_{j=N+1}^{\infty} 2^{-j} \frac{p_j(x)}{1 + p_j(x)} \\ &\le \sum_{j=1}^{N} 2^{-j} \frac{p_j(x)}{1 + p_j(x)} + \sum_{j=N+1}^{\infty} 2^{-j} \le \sum_{j=1}^{N} 2^{-j} p_j(x) + 2^{-N} < \sum_{j=1}^{N} 2^{-j} p_j(x) + \frac{\epsilon}{2}. \end{aligned}$$

So, if $x \in U_N(\epsilon/2)$ we have

$$d(x, 0) < \sum_{j=1}^{N} 2^{-j} p_j(x) + \frac{\epsilon}{2} < \frac{\epsilon}{2} \sum_{j=1}^{N} 2^{-j} + \frac{\epsilon}{2} < \frac{\epsilon}{2} \sum_{j=1}^{\infty} 2^{-j} + \frac{\epsilon}{2} = \epsilon.$$

This proves our claim and shows that the topology τ_1 is finer. Now we want to show they are equal. To this effect, consider an $U_M(\epsilon)$. It is enough to show that there exists a $\delta > 0$ such that $D(0, \delta) \subseteq U_M(\epsilon)$. Now notice that if $x \in D(0, \delta)$, that is, if

$$d(x, 0) = \sum_{j=1}^{\infty} 2^{-j} \frac{p_j(x)}{1 + p_j(x)} < \delta \Longrightarrow 2^{-j} \frac{p_j(x)}{1 + p_j(x)} < d(x, 0) < \delta \text{ for all } j \in \mathbb{N}.$$

Now let us focus on the inequalities

$$2^{-j}\frac{p_j(x)}{1+p_j(x)} < d(x,0) \text{ for all } j \leq M \iff \left(1-2^j d(x,0)\right) p_j(x) < 2^j d(x,0) \text{ for all } j \leq M \ .$$

Now, if $2^M d(x,0) < 2^M\delta < 1$ we conclude that the above inequalities are equivalent to

$$p_j(x) < \frac{2^j d(x,0)}{1-2^j d(x,0)} \text{ for all } j \leq M \ .$$

If now we choose δ so that $\frac{2^M\delta}{1-2^M\delta} < \epsilon$, it follows that $x \in D(0,\delta)$ imples $p_j(x) < \epsilon$ for all $j \leq M$, and so $D(0,\delta) \subseteq U_M(\epsilon)$.

□

Example 4.26. Given a topological vector space X and X' its dual, then the following topologies discussed later in these notes, the $\sigma(X,X')$ topology in X and the $\sigma(X',X)$ topology in X', are examples of locally convex space structures.

A very important topological vector space of *test functions*, discussed in the 2nd semester, related to the notion of tempered distribution, is the following.

Example 4.27 (Schwartz functions). Consider the set of Schwartz functions defined by

$$\mathcal{S}(\mathbb{R}^d) = \{\phi \in C^\infty(\mathbb{R}^d) : p_{\alpha\beta}(\phi) := \sup_{x\in\mathbb{R}^d} |x^\beta \partial_x^\alpha \phi(x)| < +\infty \text{ for all multi–indexes } \alpha \text{ and } \beta\}.$$

Notice that the $p_{\alpha,\beta}(\phi)$ are seminorms on $\mathcal{S}(\mathbb{R}^d)$ and, as α and β vary in all possible ways among the multi–indexes, they provide $\mathcal{S}(\mathbb{R}^d)$ with a structure of Hausdorff and locally convex and complete topological vector space.

Exercise 4.28. Prove the completeness of $\mathcal{S}(\mathbb{R}^d)$.

Exercise 4.29. Show that $\mathcal{S}(\mathbb{R}^d)$ with the above topology is metrizable.

Exercise 4.30. Show that the above topology of $\mathcal{S}(\mathbb{R}^d)$ does not come from a norm.

Answer. If it did and we had a norm $\|f\|$, then by the statement in Exercise 4.36 there would be pairs (α_1,β_1),..., (α_n,β_n) a constant $C_0 > 0$ such that

$$\|f\| \leq C_0\left(p_{\alpha_1\beta_1}(x) + ... + p_{\alpha_n\beta_n}(f)\right) \text{ for all } f \in \mathcal{S}(\mathbb{R}^d). \tag{4.11}$$

Furthermore, since all the seminorms $p_{\alpha\beta}$ are continuous, for the same reason, for any of them there would exist a a constant $C_{\alpha\beta} > 0$ such that

$$p_{\alpha\beta}(f) \leq C_{\alpha\beta}\|f\| \text{ for all } f \in \mathcal{S}(\mathbb{R}^d). \tag{4.12}$$

Hence for any of the seminorms $p_{\alpha\beta}$ we would conclude

$$p_{\alpha\beta}(f) \leq C_0 C_{\alpha\beta}\left(p_{\alpha_1\beta_1}(x) + ... + p_{\alpha_n\beta_n}(f)\right) \text{ for all } f \in \mathcal{S}(\mathbb{R}^d). \tag{4.13}$$

It is easy to conclude, taking for example (α,β) in $\mathbb{N}_0^d \times \mathbb{N}_0^d$ sufficiently "large", that this is false. □

4.1 Inductive limits

The following is a *supremely* important space in Mathematics, treated in some depth next semester.

Example 4.31 (Test functions). Consider an open set $\Omega \subseteq \mathbb{R}^d$ and denote $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$. For any compact $K \subset \Omega$ let

$$\mathcal{D}_K(\Omega) = \{\phi \in C_c^\infty(\Omega) : \text{supp } \phi \subseteq K\}.$$

In $\mathcal{D}_K(\Omega)$, for any $\phi \in \mathcal{D}_K(\Omega)$ let

$$p_{n,K}(\phi) := \sup\{|\partial_x^\alpha \phi(x)| : |\alpha| \leq n \text{ and } x \in K\}. \tag{4.14}$$

Then the $\{p_{n,K}\}_{n\in\mathbb{N}}$ are a basis of seminorms for a Hausdorff and locally convex topological vector space structure on $\mathcal{D}_K(\Omega)$.

Exercise 4.32. Show that each $\mathcal{D}_K(\Omega)$ with the above topology is metrizable and complete.

Lemma 4.33 (Inductive limit)**.** *Consider a vector space X and let $\{X_n\}_{n\in\mathbb{N}}$ be a growing sequence of subspaces of X, such that $\bigcup_{n\in\mathbb{N}} X_n = X$. Suppose that each X_n has a structure of locally convex topological vector space and that the topology on each X_n coincides with the topology induced on X_n by the topology of X_{n+1}, for all n. Let $\mathcal{O}$ be the collection of all convex subsets of X containing 0 for which each $\mathbf{O} \in \mathcal{O}$ is such that the set $\mathbf{O} \cap X_n$ is an open neighborhood $0 \in X_n$ for any $n \in \mathbb{N}$. Then:*

1. *$\mathcal{O}$ is a basis of neighborhoods of 0 for a locally convex topology in X;*
2. *the topology generated by $\mathcal{O}$ is the strongest locally convex topology such that all the immersions $X_n \hookrightarrow X$ are continuous;*
3. *the restriction of the topology of X on X_n yields the topology of X_n for any $n \in \mathbb{N}$;*
4. *if each X_n is complete, so is X;*
5. *if each X_n is Hausdorff, so is is X.*

Proof. This is discussed in Treves [13] . □

Example 4.34 (Topology on $\mathcal{D}(\Omega)$). We consider

$$\text{a sequence } K_n \text{ of compact subsets of } \Omega \text{ with } K_n \subset \overset{\circ}{K}_{n+1}\ \forall\ n \text{ and } \bigcup_{n=1}^{\infty} K_n = \Omega. \tag{4.15}$$

Then we consider on $\mathcal{D}(\Omega)$ the topology from the direct limit of the sequence of spaces $\{D_{K_n}(\Omega)\}$. Notice that it is easy to show that for any n the topology induced on $D_{K_n}(\Omega)$ by $D_{K_{n+1}}(\Omega)$ coincides with the topology of $D_{K_n}(\Omega)$.
It can also be shown that the topology on $\mathcal{D}(\Omega)$ thus defined does not depend on the specific sequence K_n in (4.15).

Exercise 4.35. Let X and Y be two locally convex topological vector spaces, where X is the inductive limit of a sequence $\{X_n\}_{n\in\mathbb{N}}$ locally convex topological vector spaces. Let $T : X \to Y$ be a linear map. Show that the following statements are equivalent.

a $T \in \mathcal{L}(X, Y)$.

b The restriction $T|_{X_n}$ is in $\mathcal{L}(X_n, Y)$ for any $n \in \mathbb{N}$.

Answer. That **a**$\Longrightarrow$**b** is true, follows from the fact that the inclusion $X_n \subseteq X$ is continuous. Let now $V \subseteq Y$ be open, convex and containing $0\in Y$. Then $X_n \cap T^{-1}(V)$ is an open, convex neighborhood of $0\in X_n$. Then $T^{-1}(V) \in \mathcal{O}$. □

Exercise 4.36. Consider the setup of Exercise 4.19. Show that a homogeneous of order 1, $f : X \to K$, is continuous in 0 is and only if there exist finitely many indexes $j_1, ..., j_n \in J$ and a constant $C > 0$ such that

$$|f(x)| \leq C\,(p_{j_1}(x) + ... + p_{j_n}(x)) \text{ for all } x \in X. \tag{4.16}$$

Remark 4.37. One of the most important modern notions in Mathematics is that of *distribution*. The distributions on an open set Ω are the elements T of the dual $\mathcal{D}'(\Omega)$. This means that $T : \mathcal{D}(\Omega) \to \mathbb{R}$ is linear and for any K compact subspace of Ω we have $T : \mathcal{D}_K(\Omega) \to \mathbb{R}$ is continuous, which, by Exercise 4.36 means that there exists an $n \in \mathbb{N}_0$ and constant $C_{nK} > 0$ such that

$$|f(x)| \leq C_{nK}\; p_{nK}(x) \text{ for all } f \in \mathcal{D}_K(\Omega). \tag{4.17}$$

You will see this in detail the next semester.

Remark 4.38. It can be shown that $\mathcal{D}(\Omega)$ is not metrizable, see later Exercise 7.7 .

5 Continuous linear operators between normed spaces

For linear maps between normed spaces we have the following.

Lemma 5.1. *Suppose that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two normed spaces. Let $T : X \to Y$ be a linear map. Then the following two statements are equivalent.*

1. *T is a continuous map in X.*

2. *T is a* ***bounded operator****, that is*

$$\|T\|_{\mathcal{L}(X,Y)} := \sup_{x\in D_X(0,1)\setminus\{0\}} \frac{\|Tx\|_Y}{\|x\|_X} < \infty. \tag{5.1}$$

Proof. First of all, it is easy to check that T is a continuous map in X if and only if T is continuous in the point $0 \in X$. Suppose now that T is continuous in $0 \in X$. So since

$T0 = 0 \in Y$, for any $\epsilon > 0$ there is $\delta > 0$ such that $\|x\|_X = \delta$ implies $\|Tx\|_Y \le \epsilon$. Given any $x \neq 0$, then $\widetilde{x} = \frac{\delta}{\|x\|_X} x$ is $\|\widetilde{x}\|_X = \delta$ and $T\widetilde{x} = T\left(\frac{\delta}{\|x\|_X} x\right) = \frac{\delta}{\|x\|_X} Tx$ so

$$\frac{\|Tx\|_Y}{\|x\|_X} = \frac{\|T\widetilde{x}\|_Y}{\|\widetilde{x}\|_X} \le \frac{\epsilon}{\delta}$$

and hence we conclude T is bounded with $\|T\|_{\mathcal{L}(X,Y)} \le \frac{\epsilon}{\delta}$.
Viceversa suppose T is bounded. Then, for some constant $\|T\|_{\mathcal{L}(X,Y)}$ we have $\|Tx\|_Y \le \|T\|_{\mathcal{L}(X,Y)} \|x\|_X$ for any $x \in X$, and so we conclude that T is continuous in 0 because for any $\epsilon > 0$ if we set $\delta = \frac{\epsilon}{\|T\|_{\mathcal{L}(X,Y)}}$ we have $\|x\|_X < \delta$ implies $\|Tx\|_Y \le \|T\|_{\mathcal{L}(X,Y)} \|x\|_X < \|T\|_{\mathcal{L}(X,Y)} \delta = \epsilon$.

$\square$

Exercise 5.2. Check that given two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ then $\mathcal{L}(X,Y)$ with the $\|\cdot\|_{\mathcal{L}(X,Y)}$ in (5.1) is a normed space.
Check that if $(Y, \|\cdot\|_Y)$ is a Banach space, so is $\mathcal{L}(X,Y)$ with the above norm.

Show that

$$\|T\|_{\mathcal{L}(X,Y)} := \sup_{x \in D_X(0,1)} \|Tx\|_Y. \tag{5.2}$$

In particular the dual X' of $(X, \|\cdot\|_X)$ has a natural norm given by

$$\|f\|_{X'} = \sup_{x \in D_X(0,1)\setminus\{0\}} \frac{|\langle f, x\rangle_{X' \times X}|}{\|x\|_X} = \sup_{x \in D_X(0,1)} |\langle f, x\rangle_{X' \times X}|, \tag{5.3}$$

and X' with this norm is a Banach space.

Example 5.3. We have $(L^p(X, d\mu))' = L^{p'}(X, d\mu)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \le p < \infty$. We will discuss this later.

Definition 5.4. Given a sequence $\{T_n\}_{n \in \mathbb{N}}$ in $\mathcal{L}(X,Y)$ we say that the sequence converges uniformly to a $T \in \mathcal{L}(X,Y)$ if $\|T_n - T\|_{\mathcal{L}(X,Y)} \xrightarrow{n \to +\infty} 0$. We say that the sequence converges strongly to an operator T, a standard notation is $s - \lim_{n \to +\infty} T_n = T$, if $T_n x \xrightarrow{n \to +\infty} Tx$ for any $x \in X$.

Example 5.5. Consider $\chi_{D_{\mathbb{R}^d}(0,\lambda)}$ thinking them as the operators $f \to \chi_{D_{\mathbb{R}^d}(0,\lambda)} f$. Then, for $1 \le p < \infty$ we have that $s - \lim_{\lambda \to +\infty} \chi_{D_{\mathbb{R}^d}(0,\lambda)} = 1$ in $L^p(\mathbb{R}^d)$ (see later Exercise 16.13), while it is not true, in general, that $\lim_{\lambda \to +\infty} \chi_{D_{\mathbb{R}^d}(0,\lambda)} = 1$ in $\mathcal{L}(L^p(\mathbb{R}^d))$.

Example 5.6. Consider $\varphi \in BC^0(\mathbb{R}^d, \mathbb{R})$ with $\varphi(0) = 1$. Then, for $1 \le p < \infty$, for the operators $f \to \varphi\left(\frac{\cdot}{\lambda}\right) f$ we have that $s - \lim_{\lambda \to +\infty} \varphi\left(\frac{\cdot}{\lambda}\right) = 1$ in $L^p(\mathbb{R}^d)$, while it is not true, in general, that $\lim_{\lambda \to +\infty} \varphi\left(\frac{\cdot}{\lambda}\right) = 1$ in $\mathcal{L}(L^p(\mathbb{R}^d))$.

Example 5.7. Other important examples are obtained with groups or semigroups of operators, like the semigroup $e^{t\triangle}$, see Remark 7.27, which $\mathcal{L}(L^p(\mathbb{R}^d))$ is strongly continuous in $t \in [0, +\infty)$, but is not uniformly continuous.

Example 5.8. For $f \in C^0([0,1])$ let $B_n(f)(x) := \sum_{k=0}^n f\left(\frac{k}{n}\right)\begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}$, see Sect. 1.0.5. The B_n are called the Bernstein operators, and in Sect. 1.0.5 we showed that $s - \lim_{n\to+\infty} B_n$ =identity. On the other hand, it is not true that $B_n \xrightarrow{n\to+\infty}$identity uniformly in $\mathcal{L}(C^0([0,1]))$.

Exercise 5.9. Show that in a normed space X the sets bounded in terms of the metric, are exactly the sets bounded in the sense of Definition 2.14.

Exercise 5.10. Show that if X and Y in Definition 2.15 are normed spaces, then Definition 2.15 is equivalent to the definition inside Lemma 5.1.

Exercise 5.11. Let E, F be two normed spaces, G a dense vector subspace of E and $T : G \to F$ a bounded linear map and F a Banach space. Show that T extends in a unique way in a bounded linear map $\overline{T} : E \to F$ and that T and $\overline{T}$ have the same operator norms.

Exercise 5.12. Let $E_1, ..., E_n, F$ be normed spaces, with $n > 1$. Then a map $T : E_1 \times .... \times E_n \to F$ an n–th linear map is bounded if

$$\|T\| := \sup\{\|T(x_1, ..., x_n)\|_F : \|x_1\|_E = ... = \|x_n\|_E = 1\} < \infty.$$

Show that $T : E_1 \times .... \times E_n \to F$ is continuous if an only is bounded,

Exercise 5.13. Let $E_1, ..., E_n, F$ be normed spaces, with $n > 1$ and F a Banach space. Let $G_1 \subseteq E_1$, ... ,$G_n \subseteq E_n$ be dense vector subspaces, and let $T : G_1 \times .... \times G_n \to F$ be a bounded n–th linear map. Show that T extends in a unique way in a bounded n–the linear map $\overline{T} : E_1 \times .... \times E_n \to F$ and that T and $\overline{T}$ have the same operator norms.

We only consider Functional Analysis because we are interested to linear and non–linear operators.

Example 5.14. Consider for $z \in \mathbb{C}\backslash[0, \infty)$ the equation in $L^2(\mathbb{R}, \mathbb{C})$

$$\left(-\frac{d^2}{dx^2} - z\right) u = f \text{ where } f \in L^2(\mathbb{R}, \mathbb{C}). \tag{5.4}$$

It turns out that for $z \in \mathbb{C}\backslash[0, +\infty)$ we have

$$u = R_{-\frac{d^2}{dx}}(z) = \int_{\mathbb{R}} R_0(x-y, z) f(y) dy \text{ where } R_0(x, z) := \frac{\mathrm{i}}{2\sqrt{z}} e^{\mathrm{i}\sqrt{z}|x|} \text{ where } \operatorname{Im}\sqrt{z} > 0. \tag{5.5}$$

Notice that the operator in (5.5) is the resolvent in the language of Sect. 5.1, where we introduce the resolvent only for bounded operators (so, not for the operator $-\frac{d^2}{dx^2}$).

To see how it comes about, consider the homogeneous equation

$$\left(-\frac{d^2}{dx^2} - z\right) u = 0. \tag{5.6}$$

It has solutions

$$\psi_\pm(x, \sqrt{z}) = e^{\pm \mathrm{i}\sqrt{z}x} \text{ , where } \arg \sqrt{z} > 0 \text{ and where } \psi_\pm(x, \sqrt{z}) \xrightarrow{x\to\pm\infty} 0. \tag{5.7}$$

Notice that, for the Wronskian $w(f, g) = f'g - fg'$, we have

$$w\left(\psi_+(x, \sqrt{z}), \psi_-(x, \sqrt{z})\right) = 2\mathrm{i}\sqrt{z}. \tag{5.8}$$

Then set

$$R_0(x, y, z) = \begin{cases} -\frac{\psi_+(x,\sqrt{z})\psi_-(y,\sqrt{z})}{w\left(\psi_+(y,\sqrt{z}),\psi_-(y,\sqrt{z})\right)} & \text{if } x > y \\ -\frac{\psi_+(y,\sqrt{z})\psi_-(x,\sqrt{z})}{w\left(\psi_+(y,\sqrt{z}),\psi_-(y,\sqrt{z})\right)} & \text{if } x < y, \end{cases} \tag{5.9}$$

If we consider now,

$$\begin{aligned} R_0(z)f &:= \int_{\mathbb{R}} R_0(x, y, z) f(y) dy \qquad (5.10) \\ = -\int_{-\infty}^{x} &\frac{\psi_+(x, \sqrt{z})\psi_-(y, \sqrt{z})}{w\left(\psi_+(y, \sqrt{z}), \psi_-(y, \sqrt{z})\right)} f(y)dy - \int_{x}^{+\infty} \frac{\psi_-(x, \sqrt{z})\psi_+(y, \sqrt{z})}{w\left(\psi_+(y, \sqrt{z}), \psi_-(y, \sqrt{z})\right)} f(y)dy, \end{aligned}$$

it is elementary to see that $\left(-\frac{d^2}{dx^2} - z\right) R_0(z)f = f$ and that $R_0(x, y, z) = \frac{\mathrm{i}}{2\sqrt{z}} e^{\mathrm{i}\sqrt{z}|x-y|}$.

Example 5.15. Consider for $z \in \mathbb{C}\backslash[0, +\infty)$ the equation in $L^2(\mathbb{R}, \mathbb{C})$

$$\left(-\frac{d^2}{dx^2} + V - z\right) u = f \text{ where } f \in L^2(\mathbb{R}, \mathbb{C}), \tag{5.11}$$

where $V \in C_c^0(\mathbb{R}, \mathbb{R})$.

In this case, if we consider the homogeneous equation

$$\left(-\frac{d^2}{dx^2} + V - z\right) u = 0, \tag{5.12}$$

it is easy to see that there are solutions

$$\psi_{V\pm}(x, \sqrt{z}) = \psi_\pm(x, \sqrt{z}) \text{ for } x \notin \operatorname{supp} V. \tag{5.13}$$

Notice that, for the Wronskian $w(f, g) = f'g - fg'$, we consider the

$$w\left(\psi_{V+}(x, \sqrt{z}), \psi_{V-}(x, \sqrt{z})\right). \tag{5.14}$$

Notice that we can have $w\left(\psi_{V+}(x,\sqrt{z}),\psi_{V-}(x,\sqrt{z})\right)=0$ for some $z\in\mathbb{R}_-$. In this case $z\in\mathbb{R}_-$ is an eigenvalue of the (unbounded) operator $-\frac{d^2}{dx^2}+V$. However, here we consider only values of z so that $w\left(\psi_{V+}(x,\sqrt{z}),\psi_{V-}(x,\sqrt{z})\right)\neq 0$.

Then set, for $w\left(\psi_{V+}(x,\sqrt{z}),\psi_{V-}(x,\sqrt{z})\right)\neq 0$, if for

$$R_V(x,y,z)=\begin{cases} -\frac{\psi_{V+}(x,\sqrt{z})\psi_{V-}(y,\sqrt{z})}{w\left(\psi_{+}(y,\sqrt{z}),\psi_{V-}(y,\sqrt{z})\right)} & \text{if } x>y \\ -\frac{\psi_{V+}(y,\sqrt{z})\psi_{V-}(x,\sqrt{z})}{w\left(\psi_{V+}(y,\sqrt{z}),\psi_{V-}(y,\sqrt{z})\right)} & \text{if } x<y, \end{cases} \tag{5.15}$$

we consider now,

$$\begin{aligned} R_V(z)f &:= \int_{\mathbb{R}} R_V(x,y,z)f(y)dy \qquad (5.16) \\ &= -\int_{-\infty}^{x}\frac{\psi_{V+}(x,\sqrt{z})\psi_{V-}(y,\sqrt{z})}{w\left(\psi_{V+}(y,\sqrt{z}),\psi_{V-}(y,\sqrt{z})\right)}f(y)dy-\int_{x}^{+\infty}\frac{\psi_{V-}(x,\sqrt{z})\psi_{V+}(y,\sqrt{z})}{w\left(\psi_{V+}(y,\sqrt{z}),\psi_{V-}(y,\sqrt{z})\right)}f(y)dy, \end{aligned}$$

it is elementary to see that $\left(-\frac{d^2}{dx^2}+V-z\right)R_V(z)f=f$.
In other words, the operators $R_V(z)$ defined in (5.10) or, more generally (5.16), are resolvents of $-\frac{d^2}{dx^2}$ or, more generally, $-\frac{d^2}{dx^2}+V$, see Sect. 5.1, where, however, we consider only resolvents of bounded operators.

5.1 Spectrum and exponential of a bounded operator

Definition 5.16. Let X be a Banach space on $\mathbb{C}$, and let $T\in\mathcal{L}(X)$. Then the resolvent set of T is

$$\rho(T)=\{z\in\mathbb{C}:(T-z)\text{ is invertible and }(T-z)^{-1}\in\mathcal{L}(X)\}. \tag{5.17}$$

If $z\in\rho(T)$ we will denote $R_T(z):=(T-z)^{-1}$.

The spectrum of T is

$$\sigma(T)=\mathbb{C}\backslash\rho(T). \tag{5.18}$$

$R_T(z)$ is the resolvent of T.

Exercise 5.17. Show that if $\lambda\in\mathbb{C}$ is an eigenvalue of T, that is there exists $0\neq x\in X$ with $Tx=\lambda x$, then $\lambda\in\sigma(T)$.

The set of eigenvalues is called also the point spectrum, denoted with $\sigma_p(T)$

Exercise 5.18. Consider the space $L^p((0,1),\mathbb{C})$ and the bounded operator $Tf:=xf$ in $L^p((0,1),\mathbb{C})$. Show that $\sigma(T)=[0,1]$. Show that T does not have eigenvalues.

Exercise 5.19. More generally, consider the space $L^p((0,1),\mathbb{C})$, a function $m\in C^0([0,1],\mathbb{C})$ and the bounded map $T_mf:=mf$. Show that $\sigma(T_m)=m([0,1])$.

Exercise 5.20. In the framework of of the above exercise, in $L^p((0,1),\mathbb{C})$ and with $m \in C^0([0,1],\mathbb{C})$, show that $\|T_m\|_{\mathcal{L}(L^p((0,1),\mathbb{C}))} = \|m\|_{L^\infty(0,1)}$.

Exercise 5.21. More generally, for $m \in L^\infty([0,1],\mathbb{C})$ show that $\|T_m\|_{\mathcal{L}(L^p((0,1),\mathbb{C}))} = \|m\|_{L^\infty(0,1)}$.

Exercise 5.22. Consider the space $L^p((0,1),\mathbb{C})$ and let $m(x) = \sum_{j=1}^{n} \lambda_j \chi_{I_j}(x)$, where I_1,..., I_n are pairwise disjoint intervals contained in $(0,1)$. Show that each of the coefficients λ_j is an eigenvalue of the bounded operator $T_m f := mf$. Find whether or not $\dim(T_m - \lambda_j)$ is finite.

Lemma 5.23. *$\rho(T)$ is an open subset of $\mathbb{C}$, $\sigma(T)$ is an closed subset of $\mathbb{C}$ and*

$$\sigma(T) \subseteq \overline{D_{\mathbb{C}}(0, \|T\|_{\mathcal{L}(X)})}. \tag{5.19}$$

Proof. Let us start with $|z|_{\mathbb{C}} > \|T\|_{\mathcal{L}(X)}$. Then consider

$$z - T = z\left(1 - \frac{T}{z}\right).$$

Obviously the invertibility of $T - z$ is equivalent to the invertibility of $\left(1 - \frac{T}{z}\right)$. Not consider the series

$$\sum_{n=0}^{\infty} \frac{T^n}{z^n}.$$

Notice that this series is convergent, because the tails converge to 0:

$$\left\| \sum_{n=m}^{\infty} \frac{T^n}{z^n} \right\|_{\mathcal{L}(X)} \le \sum_{n=m}^{\infty} \left\| \frac{T^n}{z^n} \right\|_{\mathcal{L}(X)} \le \sum_{n=m}^{\infty} \frac{\|T\|^n_{\mathcal{L}(X)}}{|z|^n} = \frac{\frac{\|T\|^m_{\mathcal{L}(X)}}{|z|^m}}{1 - \frac{\|T\|_{\mathcal{L}(X)}}{|z|}} \xrightarrow{m \to +\infty} 0.$$

Notice also that

$$\left(1 - \frac{T}{z}\right) \sum_{n=0}^{m} \frac{T^n}{z^n} = \left(\sum_{n=0}^{m} \frac{T^n}{z^n}\right)\left(1 - \frac{T}{z}\right) = 1 - \frac{T^{m+1}}{z^{m+1}} \xrightarrow{m \to +\infty} \text{identity, in } \mathcal{L}(X).$$

So $\rho(T) \supseteq \mathbb{C} \backslash \overline{D_{\mathbb{C}}(0, \|T\|_{\mathcal{L}(X)})}$ or, what is the same, $\sigma(T) \subseteq \overline{D_{\mathbb{C}}(0, \|T\|_{\mathcal{L}(X)})}$.

One can prove similarly that $\rho(T)$ is an open subset of $\mathbb{C}$. Suppose that $z \in \rho(T)$. Then, for some other $\zeta \in \mathbb{C}$ we can write

$$T - \zeta = T - z + z - \zeta = (T - z)(1 + (T - z)^{-1}(z - \zeta)).$$

Picking $|z - \zeta| < \frac{1}{\|R_T(z)\|}$ we have $\|R_T(z)(z - \zeta)\|_{\mathcal{L}(X)} < 1$, so again

$$R_T(\zeta) = (1 + R_T(z)(z - \zeta))^{-1} R_T(z) = \sum_{n=0}^{\infty} (-1)^n (R_T(z))^n (z - \zeta)^n \quad R_T(z),$$

where the above series converges absolutely. $\square$

Remark 5.24. Notice that if λ is an eigenvalue from $|\lambda| \le \|T\|$ we can derive also $|\lambda| \le \|T^n\|^{\frac{1}{n}}$ for all $n \in \mathbb{N}$. So, in particular, if $\|T^n\|^{\frac{1}{n}} \xrightarrow{n\to+\infty} 0$, we get $\lambda = 0$.

Remark 5.25. It is rather elementary to show that the map $\rho(T) \ni z \to R_T(z) \in \mathcal{L}(X)$ is a holomorphic map from $\rho(T)$ to $\mathcal{L}(X)$.

Remark 5.26. Notice that $\rho(T)$ is non empty. Indeed, otherwise, the holomorphic function $\mathbb{C} \ni z \to R_T(z) \in \mathcal{L}(X)$ is bounded with $R_T(z) \xrightarrow{z\to\infty} 0$ and one can prove by Liouville Theorem that $R_T(z) \equiv 0$ for all $z \in \mathbb{C}$, which is impossible.

Exercise 5.27. Prove rigorously the statement in Remark 5.26.

Definition 5.28 (geometric dimension)**.** If $\lambda \in \sigma_p(T)$ and $n := \dim(T-\lambda) < \infty$, n is the geometric dimension of λ.

Remark 5.29. It is elementary to check that the sequence of vector spaces $\ker(T-\lambda)^n$ is non decreasing.

Definition 5.30 (algebraic dimension)**.** If $\lambda \in \sigma_p(T)$ and if the space

$$N_g(T-\lambda) := \bigcup_{n=1}^{\infty} \ker(T-\lambda)^n \tag{5.20}$$

has dimension $m := \dim N_g(T-\lambda) < \infty$, m is the algebraic dimension of λ.

Exercise 5.31. Check that the usual definition of geometric and algebraic dimension in the context of $\dim X < +\infty$ coincide with the above ones (Hint: use the canonical Jordan bloc decomposition).

Definition 5.32 (Exponential of an operator)**.** X a Banach space and for $A \in \mathcal{L}(X)$ the exponential of A is the operator

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}. \tag{5.21}$$

Exercise 5.33. a Check that the series in (5.21) is convergent in $\mathcal{L}(X)$.

b Check that if $A, B \in \mathcal{L}(X)$ commute, that is $[A, B] := AB - BA = 0$, then $e^{A+B} = e^A e^B = e^B e^A$.

c Check that $Ue^A U^{-1} = e^{UAU^{-1}}$.

Example 5.34. Obviously, the exponentials are important because if we have for X a Banach space and for $T \in \mathcal{L}(X)$ and $f \in C^0(\mathbb{R}, X)$ the simple ODE

$$\begin{cases} \dot{x} = Tx + f \\ x(0) = x_0, \end{cases} \tag{5.22}$$

then the solution to (5.22) is

$$x(t) = e^{tT}x_0 + \int_0^t e^{(t-s)T} f(s)ds. \tag{5.23}$$

In fact these formulas are true also for appropriate unbounded operators, like for example the Laplacian $\triangle = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_d^2}$ in $X = L^2(\mathbb{R}^d)$.

Obviously, a very important topic is the study of e^{tT} as $t \to +\infty$.

Remark 5.35. Notice that if, for X a Banach space, for $f \in C^0(\mathbb{R}, X)$ and $T(\cdot) \in C^0(\mathbb{R}, \mathcal{L}(X))$, we consider the ODE

$$\begin{cases} \dot{x} = T(t)x + f \\ x(0) = x_0, \end{cases} \tag{5.24}$$

then the following formula

$$x(t) = e^{\int_0^t T(s)ds}x_0 + \int_0^t e^{\int_s^t T(s')ds'} f(s)ds, \tag{5.25}$$

which is valid for scalar equations, that is when $X = \mathbb{R}$, is in general false.

Exercise 5.36. Show that (5.25) is correct in a Banach space X if additionally we assume $[T(t), T(s)] = 0$ for all pairs $t, s \in \mathbb{R}$.

Example 5.37. It is worth computing the exponential of some matrix. For

$$A = \begin{bmatrix} A_1 & 0 & 0 & \\ 0 & A_2 & 0 & \\ 0 & 0 & A_3 & \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

we have

$$e^{tA} = \begin{bmatrix} e^{tA_1} & 0 & 0 & \\ 0 & e^{tA_2} & 0 & \\ 0 & 0 & e^{tA_3} & \\ 0 & 0 & 0 & \ddots \end{bmatrix}.$$

So, also using the conclusions of Exercise 5.33, it can be shown that in finite dimension, it is sufficient to understand case

$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 & \dots & & & \\ 0 & \lambda & 1 & 0 & \dots & & & \\ 0 & 0 & \lambda & 1 & 0 & \dots & & \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & & \\ & & & \dots & 0 & \lambda & 1 \\ & & & \dots & 0 & 0 & \lambda \end{bmatrix}.$$

We have $A = \lambda I + N$ where

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & & \\ 0 & 0 & 1 & 0 & \dots & & \\ 0 & 0 & 0 & 1 & 0 & \dots & \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & \\ & & & \dots & 0 & 0 & 1 \\ & & & \dots & 0 & 0 & 0 \end{bmatrix}.$$

Since $\lambda I N = N \lambda I$, risulta

$$e^{t(\lambda I+N)} = e^{t\lambda I} e^{tN}.$$

Obviously $e^{t\lambda I} = e^{t\lambda} I$. Notice that if N is an $n \times n$ matrix, then we have $N^n = 0$ and

$$N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & & \\ 0 & 0 & 0 & 1 & \dots & & \\ 0 & 0 & 0 & 0 & 1 & \dots & \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & \\ & & & \dots & 0 & 0 & 1 \\ & & & \dots & 0 & 0 & 0 \\ & & & \dots & 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & \dots & & \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \\ & & & & & & \\ & & & \dots & 0 & 0 & 1 \\ & & & \dots & 0 & 0 & 0 \\ & & & \dots & 0 & 0 & 0 \\ & & & \dots & 0 & 0 & 0 \end{bmatrix}$$

$$N^{n-1} = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ & & & \\ \vdots & \vdots & \vdots & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

Since

$$e^{tN} = I + tN + \frac{1}{2!} t^2 N^2 + ... + \frac{1}{(n-1)!} t^{n-1} N^{n-1}$$

we get

$$e^{t(\lambda I+N)} = \begin{bmatrix} e^{t\lambda} & te^{t\lambda} & \frac{t^2}{2!}e^{t\lambda} & \frac{t^3}{3!}e^{t\lambda} & \dots & \frac{t^{n-2}}{(n-2)!}e^{t\lambda} & \frac{t^{n-1}}{(n-1)!}e^{t\lambda} \\ 0 & e^{t\lambda} & te^{t\lambda} & \frac{t^2}{2!}e^{t\lambda} & \dots & & \frac{t^{n-2}}{(n-2)!}e^{t\lambda} \\ 0 & 0 & e^{t\lambda} & te^{t\lambda} & \frac{t^2}{2!}e^{t\lambda} & \dots & \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & \\ & & & \dots & 0 & e^{t\lambda} & te^{t\lambda} \\ & & & \dots & 0 & 0 & e^{t\lambda} \end{bmatrix}.$$

Example 5.38. Notice that there is a deep connection, between resolvent and exponential, related to the Laplace transform. Indeed, if X is a Banach space and if $A \in \mathcal{L}(X)$, then

$$R_A(z) = \int_0^{+\infty} e^{tA}e^{-tz}dt \tag{5.26}$$

is absolutely convergent for $\operatorname{Re} z > \|A\|_{\mathcal{L}(X)}$, and can be extended in a larger region in $\rho(A)$. Notice that (5.26) is the Laplace transform of e^{tA}.
Obviously, it is possible to express e^{tA} in terms of $R_A(z)$ in terms of the Inverse Laplace transform, which for $A \in \mathcal{L}(X)$ can be written as

$$e^{tA} = -\frac{1}{2\pi \mathrm{i}} \int_\gamma e^{tz} R_A(z) dz \tag{5.27}$$

with γ a counter clockwise oriented closed path containing in the interior a topological disk containing $\sigma(A)$. In many important examples, it is possible to study e^{tA} only by studying $R_A(z)$. See for example, the classical paper by Jensen and Kato [5].

Exercise 5.39. Show that $\|e^A\|_{\mathcal{L}(X)} \leq e^{\|A\|_{\mathcal{L}(X)}}$, where $A \in \mathcal{L}(X)$ for a Banach space X. Then use this inequality to prove that for $\operatorname{Re} z > \|A\|_{\mathcal{L}(X)}$ we have $\|e^{tA}e^{-tz}\|_{\mathcal{L}(X)} \leq e^{t\left(\|A\|_{\mathcal{L}(X)} - \operatorname{Re} z\right)}$, which decays exponentially to 0 for $t \longrightarrow +\infty$.

Answer. Since by the triangular inequality

$$\left\| \sum_{n=0}^N \frac{A^n}{n!} \right\|_{\mathcal{L}(X)} \leq \sum_{n=0}^N \frac{\|A^n\|_{\mathcal{L}(X)}}{n!} \leq \sum_{n=0}^N \frac{\|A\|^n_{\mathcal{L}(X)}}{n!},$$

we get $\|e^A\|_{\mathcal{L}(X)} \leq e^{\|A\|_{\mathcal{L}(X)}}$. Next,

$$\|e^{tA}e^{-tz}\|_{\mathcal{L}(X)} = \|e^{tA}e^{-t\operatorname{Re} z}\|_{\mathcal{L}(X)} = e^{-t\operatorname{Re} z}\|e^{tA}\|_{\mathcal{L}(X)} \leq e^{-t\operatorname{Re} z}e^{\|A\|_{\mathcal{L}(X)}}.$$

$\square$

Example 5.40. Notice that if X is a Banach space, and if $A \in \mathcal{L}(X)$ and if $f \in H(\mathbb{C})$, then it is possible to define the function of the operator $f(A)$.

$$f(A) = -\frac{1}{2\pi \mathrm{i}} \int_\gamma f(z) R_A(z) dz \text{ , with } \gamma \text{ as in (5.27).} \tag{5.28}$$

A version of this, extends to unbounded operators. Obviously, for bounded operators, one could use power series.

Let us check for example that for $f \equiv 1$, then the right hand side of (5.28) is the identity operator. We notice that the integral coincides for $R \gg 1$ with

$$\begin{aligned} &- \frac{R}{2\pi} \int_0^{2\pi} \frac{1}{A - Re^{\mathrm{i}\vartheta}} e^{\mathrm{i}\vartheta} d\vartheta = -\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\frac{A}{R} e^{-\mathrm{i}\vartheta} - 1} d\vartheta = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1 - \frac{A}{R} e^{-\mathrm{i}\vartheta}} d\vartheta \\ &= 1 + \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=1}^\infty \frac{A^n}{R^n} e^{-\mathrm{i}n\vartheta} d\vartheta = 1 + \frac{A}{2\pi R} \int_0^{2\pi} e^{-\mathrm{i}\vartheta} \sum_{n=0}^\infty \frac{A^n}{R^n} e^{-\mathrm{i}n\vartheta} d\vartheta. \end{aligned} \tag{5.29}$$

Now notice that

$$\begin{aligned} &\left\| \frac{A}{2\pi R} \int_0^{2\pi} e^{-\mathrm{i}\vartheta} \sum_{n=0}^\infty \frac{A^n}{R^n} e^{-\mathrm{i}n\vartheta} d\vartheta \right\|_{\mathcal{L}(X)} \le \frac{\|A\|_{\mathcal{L}(X)}}{2\pi R} \int_0^{2\pi} \sum_{n=0}^\infty \frac{\|A\|^n_{\mathcal{L}(X)}}{R^n} d\vartheta \\ &= \frac{\|A\|_{\mathcal{L}(X)}}{R} \frac{1}{1 - \frac{\|A\|_{\mathcal{L}(X)}}{R}} \xrightarrow{R \to +\infty} 0. \end{aligned}$$

This implies that for the operator in (5.30)$\xrightarrow{R \to +\infty}$ 1 uniformly.

More generally, if we define $f(A)$ using the power series, we claim that then equality (5.28) is true. In fact, using the special case $f \equiv 1$ just shown,

$$f(A) + \frac{1}{2\pi \mathrm{i}} \int_\gamma f(z) R_A(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \left(f(z) - f(A) \right) R_A(z) dz.$$

Now, if $f(z) = \sum_{n=0}^\infty a_n z^n$ is the power series expansion of f

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma \sum_{n=0}^\infty \left(z^n - A^n \right)) \frac{1}{A - z} dz = \frac{1}{2\pi \mathrm{i}} \sum_{n=1}^\infty \int_\gamma \left(z^n - A^n \right) \frac{1}{A - z} dz. \tag{5.30}$$

We have the elementary factorization formula

$$z^n - A^n = (z - A) \sum_{j=1}^n z^{n-j} A^{j-1}.$$

Hence

$$\frac{1}{2\pi \mathrm{i}} \sum_{n=0}^\infty \int_\gamma \left(z^n - A^n \right) \frac{1}{A - z} dz = -\frac{1}{2\pi \mathrm{i}} \sum_{n=1}^\infty \sum_{j=1}^n A^{j-1} \int_\gamma z^{n-j} dz = 0,$$

which proves our claim that (5.28) is true.

Exercise 5.41. Check that if $f \in H(D_{\mathbb{C}}(0,r))$ and $\|A\|_{\mathcal{L}(X)} < r$ where X is a Banach space, then for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ the power series of f, we have that

$$f(A) := \sum_{n=0}^{\infty} a_n A^n$$

is a well defined element in $\mathcal{L}(X)$, with the series convergent uniformly in $\mathcal{L}(X)$.

Exercise 5.42. Check that if $A \in \mathcal{L}(X)$ where X is a Banach space, then $\sigma(A) \neq \emptyset$.

Answer. If we have $\sigma(A) = \emptyset$, then $z \to R_A(z)$ is holomorphic over the entire $\mathbb{C}$ with values in $\mathcal{L}(X)$. We claim that $R_A(z) \xrightarrow{z\to\infty} 0$ uniformly in $\mathcal{L}(X)$. Assuming for a moment this claim, it follows that $R_A(z)$ is also bounded in $\mathbb{C}$ and hence, by Liouville Theorem (which is true like for scalar holomorphic functions) it follows $R_A(z)$ is constant, and so necessarily identically equal to 0, which is absurd. Now, by a previous computation, for $|z| > \|A\|_{\mathcal{L}(X)}$ we have

$$\|R_A(z)\|_{\mathcal{L}(X)} \leq |z|^{-1} \| \left(1 - z^{-1}A\right)^{-1} \|_{\mathcal{L}(X)} \leq |z|^{-1} \frac{1}{1 - |z|^{-1}\|A\|_{\mathcal{L}(X)}} \xrightarrow{z\to\infty} 0.$$

6 The Theorem of Hahn–Banach

Theorem 6.1 (Hahn–Banach, Analytic form)**.** *Let X be a vector space on $\mathbb{R}$. Let $p : X \to \mathbb{R}$ be a seminorm, Y a linear subspace of X and $g : Y \to \mathbb{R}$ a linear map such that*

$$g(y) \leq p(y) \text{ for all } y \in Y. \tag{6.1}$$

Then there is a linear map $f : X \to \mathbb{R}$ such that $f|_Y = g$ and such that

$$f(x) \leq p(x) \text{ for all } x \in X. \tag{6.2}$$

Proof. If $x_0 \notin Y$, then the elements of the vector space $\mathbb{R}x_0 + Y$ can be written in a unique way as $x = tx_0 + y$ for $y \in Y$ and $t \in \mathbb{R}$. Then define $f : \mathbb{R}x_0 + Y \to \mathbb{R}$ by $f(tx_0 + y) = t\alpha + g(y)$ for $\alpha \in \mathbb{R}$ to be chosen. We want

$$t\alpha + g(y) \leq p(tx_0 + y) \text{ for all } t \in \mathbb{R} \text{ and } y \in Y. \tag{6.3}$$

Notice that (6.3), considering only the case $t > 0$, is equivalent to

$$\alpha + g(y) \leq p(x_0 + y) \text{ for all } y \in Y. \tag{6.4}$$

To see this, just observe that

$$t\alpha + g(y) \leq p(tx_0 + y) \text{ for all } t > 0 \text{ and } y \in Y \iff \alpha + g\left(\frac{y}{t}\right) \leq p\left(x_0 + \frac{y}{t}\right) \text{ for all } t > 0 \text{ and } y \in Y$$
$$\iff \alpha + g(y) \leq p(x_0 + y) \text{ for all } y \in Y.$$

Similarly

$t\alpha + g(y) \le p(tx_0 + y)$ for all $t < 0$ and $y \in Y \iff -\alpha + g(y) \le p(-x_0 + y)$ for all $y \in Y$.

So we are reduced to searching an $\alpha \in \mathbb{R}$ satisfying (6.4) and

$$-\alpha + g(y) \le p(-x_0 + y) \text{ for all } y \in Y. \tag{6.5}$$

In other words, we need to have

$$\sup_{y\in Y} \left(-p(-x_0 + y) + g(y)\right) \le \alpha \le \inf_{y\in Y} \left(p(x_0 + y) - g(y)\right). \tag{6.6}$$

Notice that

$$-p(-x_0 + y_1) + g(y_1) \le p(x_0 + y_2) - g(y_2) \iff g(y_1) + g(y_2) \le p(x_0 + y_2) + p(-x_0 + y_1).$$

The latter is true for all $y_1, y_2 \in Y$. Indeed we have

$$g(y_1) + g(y_2) = g(y_1 + y_2) \le p(y_1 + y_2) = p(x_0 + y_2 - x_0 + y_1) \le p(x_0 + y_2) + p(-x_0 + y_1).$$

This implies that there exists an $\alpha \in \mathbb{R}$ such that (6.6) holds true.

We now define

$P :=\{(h, D)$ s.t. D is a linear subspace of X with $Y \subseteq D$, $h : D \to \mathbb{R}$ is a linear extension of g with $h(x) \le p(x)$ for all $x \in D$ $\}$.

Notice that in P there is a partial ordering

$$(h_1, D_1) \preceq (h_2, D_2) \iff D_1 \subseteq D_2 \text{ and } h_2|_{D_1} = h_1. \tag{6.7}$$

P is *inductive*, that is, any totally ordered subset Q of P has an upper bound. Just take for $Q = \{(h_q, D_q)\}_{q\in Q}$, then set $\widehat{D} = \cup_{q\in Q} D_q$, which is a linear subspace of X, and for any $x \in \widehat{D}$ set $h(x) = h_q(x)$ if $x \in D_q$. Then we applying Zorn's Lemma 1.1 we conclude that P has a maximal element (D, h). If $D \subsetneqq X$, then by the above argument, if we pick $x_0 \notin D$, we can extend $h : D \to \mathbb{R}$ into a linear map $h : \{tx_0 + y : t \in \mathbb{R} \text{ and } y \in D\} \to \mathbb{R}$ and conclude that (D, h) is not a maximal element in P for the order relation (6.7). Hence $D = X$ and h is the desired linear functional.

□

Let us see some corollaries of the Hahn–Banach Theorem.

Corollary 6.2. *Let $(X, \|\cdot\|_X)$ be a normed space and let $Y \subset X$ be a vector subspace, with respect to the field K. If $g : Y \to K$ is a linear functional, there exists a $f \in X'$ which extends g and such that*

$$\|f\|_{X'} = \sup_{y\in D_Y(0,1)} |g(y)| =: \|g\|_{Y'}. \tag{6.8}$$

Proof. Let us start considering the case $K = \mathbb{R}$. Apply Theorem 6.1 using $p(x) := \|g\|_{Y'}\|x\|_X$. Notice that $x \to p(x)$ satisfies (4.1)–(4.2) and that by the definition of $\|g\|_{Y'}$ we have that (6.1) is true. Then Theorem 6.1 yields $f : X \to \mathbb{R}$ such that $f(x) \leq \|g\|_{Y'}\|x\|_X$ for all $x \in X$. Notice that this implies $|f(x)| \leq \|g\|_{Y'}\|x\|_X$ for all $x \in X$ and in particular yields $\|f\|_{X'} \leq \|g\|_{Y'}$. We must have $\|f\|_{X'} = \|g\|_{Y'}$ since obviously

$$\|f\|_{X'} = \sup_{x \in D_X(0,1)} |f(x)| \geq \sup_{y \in D_Y(0,1)} |f(y)| = \|g\|_{Y'}.$$

The statement has been proven in the case $K = \mathbb{R}$. Let us consider now the case $K = \mathbb{C}$. So Y is a complex subspace of X and g is linear with respect to $\mathbb{C}$. Then $u = \operatorname{Re} g$ is a linear operator with respect to $\mathbb{R}$. Apply the first part of the theorem, and let $v \in X'$ the extension of u. Then, using formula (2.2),

$$f(x) := v(x) - \mathrm{i}v(\mathrm{i}x),$$

It is elementary to check that f is an extension of g and that it is linear with respect to $\mathbb{C}$. Next, since $|v(x)| \leq |f(x)|$, obviously $\|v\|_{X'} \leq \|f\|_{X'}$. On the other hand, for any x there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $f(\lambda x) = |f(x)|$. So $|f(x)| = |v(\lambda x)| \leq \|v\|_{X'}\|x\|_X$, which implies $\|v\|_{X'} \geq \|f\|_{X'}$ and, so, the equality.

□

Exercise 6.3. Given a normed space X and $Y \subsetneqq X$ be a closed vector subspace, show that there exists a continuous functional $\phi : X \to K$ such that $Y \subseteq \ker \phi$.

Definition 6.4. Given a topological vector space X and Y a closed vector subspace, we say that Y is complementary in X if there is a closed subspace W of X such that $X = Y + W$ and $Y \cap G = 0$ (and so $X = Y \oplus W$).

Remark 6.5. Notice that Corollary 6.2 becomes trivial if Y has a closed complementary space such that $\|y + w\|_X \geq \|y\|_X$ for any $y \in Y$ and $w \in W$. This happens always in a Hilbert or in a pre–Hilbert space, but in general Banach spaces it is not true, see Remark 10.9.

Corollary 6.6. *Let $(X, \|\cdot\|_X)$ be a normed space. For any $x_0 \in X$ there exists a $f \in X'$ such that*

$$\|f\|_{X'} = \|x_0\|_X \text{ and } f(x_0) = \|x_0\|_X^2. \tag{6.9}$$

Proof. Let $Y = \mathbb{R}x_0 = \{\lambda x_0 : \lambda \in \mathbb{R}\}$ and let $g \in Y'$ defined by $g(\lambda x_0) = \lambda\|x_0\|_X^2$. Then $\|g\|_{Y'} = \left|g\left(\frac{1}{\|x_0\|_X}x_0\right)\right| = \frac{1}{\|x_0\|_X}\|x_0\|_X^2 = \|x_0\|_X$. Applying Corollary 6.2 we obtain the desired result.
We can define a $\mathbb{C}$–linear functional using an analogue of formula (2.2).

□

Remark 6.7. Recall that in Example 4.22 we have $(L^p(0,1))' = 0$ for $0 < p < 1$. So Corollary 6.6 shows a completely different behavior of X' for X normed.

Example 6.8. Let $T : X \to Y$ be a continuous linear operator between two normed spaces. Then the for any $y' \in Y'$, that is a bounded linear map $y' : Y \to \mathbb{R}$, it is elementary that $y' \circ T$ defines an element in X'. This defines a linear map

$$Y' \ni y' \xrightarrow{T^*} y' \circ T \in X' \tag{6.10}$$

which is called the dual map of T. T^* is a bounded map and in particular we have

$$\|T\|_{\mathcal{L}(X,Y)} = \|T^*\|_{\mathcal{L}(Y',X')}. \tag{6.11}$$

To see this, notice that for $\|y'\|_{Y'} = 1$, by the definition of the norm in X' we have

$$\begin{aligned}
&\|T^*y'\|_{X'} = \sup\{\langle T^*y', x\rangle_{X'\times X} : \|x\|_X = 1\} = \sup\{\langle y' \circ T, x\rangle_{X'\times X} : \|x\|_X = 1\}\\
&= \sup\{\langle y', Tx\rangle_{Y'\times Y} : \|x\|_X = 1\} \le \|y'\|_{Y'} \sup_{\|x\|_X=1} \|Tx\|_Y \le \|y'\|_{Y'}\|T\|_{\mathcal{L}(X,Y)} = \|T\|_{\mathcal{L}(X,Y)},
\end{aligned}$$

which yields $\|T^*\|_{\mathcal{L}(Y',X')} \le \|T\|_{\mathcal{L}(X,Y)}$. Similarly, for $\|x\|_X = 1$, by Hahn–Banach we have

$$\begin{aligned}
&\|Tx\|_Y = \sup\{\langle Tx, y'\rangle_{Y'\times Y} : \|y'\|_{Y'} = 1\} \text{ (by Corollary 6.6)}\\
&= \sup\{\langle T^*y', x\rangle_{X'\times X} : \|y'\|_{Y'} = 1\}\\
&\le \|x\|_X \sup_{\|y'\|_{Y'}=1} \|T^*y'\|_{X'} \le \|x\|_X\|T^*\|_{\mathcal{L}(Y',X')} = \|T^*\|_{\mathcal{L}(Y',X')},
\end{aligned}$$

which yields $\|T^*\|_{\mathcal{L}(Y',X')} \ge \|T\|_{\mathcal{L}(X,Y)}$.

Corollary 6.9. *Let $U := D_{\mathbb{C}}(0,1)$ and $\mathbb{T} = \partial U$. Let $A \subseteq C^0(\overline{U}, \mathbb{C})$ be a vector space. Suppose that A contains the set $\mathbb{C}[z]$ of polynomials $p_n(z) = a_n z^n + ... + a_0$ and that*

$$\|f\|_{L^\infty(\overline{U})} = \|f\|_{L^\infty(\mathbb{T})} \text{ for any } f \in A, \tag{6.12}$$

(notice that any element $f \in \mathbb{C}[z]$ satisfies (6.12) by the Maximum Modulus Theorem). Then we have

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|z|^2}{|z-e^{it}|^2} f(e^{it})dt \text{ for any } f \in A \text{ and any } z \in U. \tag{6.13}$$

Proof. Let Y be the subspace of $C^0(\mathbb{T}, \mathbb{C})$ formed by the restrictions on $\mathbb{T}$ of the functions in A. We fix $z \in U$ and we consider the linear map $Y \ni f_{|\mathbb{T}} \to \Lambda f_{|\mathbb{T}} := f(z) \in \mathbb{C}$. By (6.12) it follows

$$|f(z)| \le \|f_{|\mathbb{T}}\|_{L^\infty(\mathbb{T})} \text{ for any } f_{|\mathbb{T}} \in Y.$$

So the norm of this operator is ≤ 1. In fact, since $1(z) = 1$, the norm is exactly 1. By Hahn–Banach there exists an extension $\Lambda : C^0(\mathbb{T}, \mathbb{C}) \to \mathbb{C}$ with norm

$$\Lambda 1 = 1 \text{ and } \|\Lambda\| = 1. \tag{6.14}$$

We claim that

$$\text{for any } C^0(\mathbb{T},\mathbb{C}) \ni f \geq 0 \text{ we have } \Lambda f \geq 0. \tag{6.15}$$

Assuming (6.15), we can conclude that there exists a positive Borel measure $d\mu_z$ in $\mathbb{T}$, such that by Theorem 1.22 we have

$$\Lambda f = \int_{\mathbb{T}} f d\mu_z \text{for any } f \in C^0(\mathbb{T},\mathbb{C}). \tag{6.16}$$

We have

$$z^n = \int_{\mathbb{T}} w^n d\mu_z(w) \text{ for any } n \in \mathbb{N} \tag{6.17}$$

and taking complex conjugation, for $z = re^{\mathrm{i}\theta}$ we have

$$r^{|n|}e^{in\theta} = \int_{\mathbb{T}} w^n d\mu_z(w) \text{ for any } n \in \mathbb{Z} \tag{6.18}$$

Now notice that for

$$\begin{aligned} P_r(\theta-t) &:= \sum_{n=-\infty}^{+\infty} r^{|n|}e^{\mathrm{i}n(\theta-t)} = \mathrm{Re}\left\{1+2\sum_{n=1}^{\infty}\left(ze^{-\mathrm{i}t}\right)^n\right\} = \mathrm{Re}\left\{-1+2\sum_{n=0}^{\infty}\left(ze^{-\mathrm{i}t}\right)^n\right\} \\ &= \mathrm{Re}\left\{\frac{2}{1-ze^{-\mathrm{i}t}}-1\right\} = \mathrm{Re}\left\{\frac{1+ze^{-\mathrm{i}t}}{1-ze^{-\mathrm{i}t}}\right\} = \frac{\mathrm{Re}\{(1+ze^{-\mathrm{i}t})(1-\overline{z}e^{\mathrm{i}t})\}}{|z-e^{\mathrm{i}t}|^2} \\ &= \frac{1-|z|^2+\mathrm{Re}\{ze^{-\mathrm{i}t}-\overline{z}e^{\mathrm{i}t}\}}{|z-e^{\mathrm{i}t}|^2} = \frac{1-|z|^2}{|z-e^{\mathrm{i}t}|^2}, \end{aligned}$$

we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)e^{\mathrm{i}nt}dt = r^{|n|}e^{\mathrm{i}n\theta} \text{ for any } n \in \mathbb{Z}. \tag{6.19}$$

Comparing (6.18) and (6.20) we conclude that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)f(e^{\mathrm{i}t})dt = \int_{\mathbb{T}} f(w)d\mu_z(w) \tag{6.20}$$

for any trigonometric polynomial

$$f(w) = \sum_{j=-n}^{n} a_n w^j \text{ for } w \in \mathbb{T}. \tag{6.21}$$

We will see in Corollary 7.25 that the trigonometric polynomials form a dense set in $C^0(\mathbb{T},\mathbb{C})$. We conclude therefore that the equality (6.20) is true for all $f \in C^0(\mathbb{T},\mathbb{C})$. Hence (6.17) and (6.20) yield (6.13).

To complete the proof of Corollary 6.9 we need to prove (6.15). It is enough to assume $0 \leq f(z) \leq 1$, since any $C^0(\mathbb{T}, \mathbb{C}) \ni f \geq 0$ is of this type, up to multiplication by a sufficiently small constant $0 < c$, and if $0 \leq \Lambda c f = c \Lambda f$, obviously also $\Lambda f \geq 0$. Set $g = 2f - 1$. Then $-1 \leq g \leq 1$. Let $\Lambda g = \alpha + \mathrm{i}\beta$. Notice that for any $r \in \mathbb{R}$,

$$|g + \mathrm{i}r|^2 = g^2 + r^2 \leq 1 + r^2.$$

Then,

$$(\beta + r)^2 \leq |\alpha + \mathrm{i}\beta + \mathrm{i}r|^2 = |\Lambda(g + \mathrm{i}r)|^2 \leq 1 + r^2,$$

where we used $\Lambda \mathrm{i}r = \mathrm{i}r\Lambda 1 = \mathrm{i}r$ and $|\Lambda(g + \mathrm{i}r)|^2 \leq \|g + \mathrm{i}r\|^2_{L^\infty(\mathbb{T})} \leq 1 + r^2$. Then $\beta^2 + 2\beta r \leq 1$ for any $r \in \mathbb{R}$. This implies $\beta = 0$. We have $|\alpha| = |\Lambda g| \leq \|g\|_{L^\infty(\mathbb{T})} \leq 1$. Then we obtain the desired result:

$$\Lambda f = \Lambda \frac{1 + g}{2} = \frac{1 + \alpha}{2} \geq 0.$$

□

Example 6.10. If we set

$$A := \{f \in C^0(\overline{U}, \mathbb{C}) \cap C^2(U, \mathbb{C}) : \triangle f = 0 \text{ in } U\}$$

then $A \supset \mathbb{C}[z]$ and, by the Maximum Modulus Theorem for harmonic functions, Corollary 6.9 applies.

Remark 6.11. Notice that for any $z_0 \in \mathbb{T}$ the function $\dfrac{1 - |z|^2}{|z - z_0|^2}$ is harmonic in U, as can be checked by direct inspection. Indeed, $\triangle = 4\partial_z \partial_{\overline{z}}$. Then

$$\begin{aligned}
\partial_{\overline{z}} \frac{1 - |z|^2}{|z - z_0|^2} &= \partial_{\overline{z}} \frac{1 - z\overline{z}}{(z - z_0)(\overline{z} - \overline{z}_0)} \\
&= \frac{-z}{(z - z_0)(\overline{z} - \overline{z}_0)} - \frac{1 - z\overline{z}}{(z - z_0)(\overline{z} - \overline{z}_0)^2}
\end{aligned}$$

and

$$\begin{aligned}
&\partial_z \partial_{\overline{z}} \frac{1 - |z|^2}{|z - z_0|^2} = \partial_z \left[\frac{-z}{(z - z_0)(\overline{z} - \overline{z}_0)} - \frac{1 - z\overline{z}}{(z - z_0)(\overline{z} - \overline{z}_0)^2} \right] \\
&= \frac{1}{|z - z_0|^2} \left[-1 + \frac{z}{(z - z_0)} + \frac{1 - z\overline{z}}{(z - z_0)(\overline{z} - \overline{z}_0)} + \frac{\overline{z}}{(\overline{z} - \overline{z}_0)} \right] \\
&= \frac{1}{|z - z_0|^4} \left[-(z - z_0)(\overline{z} - \overline{z}_0) + z(\overline{z} - \overline{z}_0) + 1 - |z|^2 + \overline{z}(z - z_0) \right] \\
&= \frac{1}{|z - z_0|^4} \left[(z - z_0)\overline{z}_0 + z(\overline{z} - \overline{z}_0) + 1 - |z|^2 \right] = \frac{1 - |z_0|^2}{|z - z_0|^4} = 0
\end{aligned}$$

So all functions in the space A in Corollary 6.9 are harmonic inside U. This means that $A \supseteq \mathbb{C}[z]$, the fact that A is a vector space and (6.12), taken together are a powerful *rigidity* condition.

6.1 Geometric form of the Theorem of Hahn–Banach

Definition 6.12. Let A and B nonempty subsets of a topological vector space X on $\mathbb{R}$. Let $H = f^{-1}(a)$ be a hyperplane with $f : X \to \mathbb{R}$ a linear map.

1. H separates A and B if $f(A) \subseteq (-\infty, a]$ and $f(B) \subseteq [a, +\infty)$ (or viceversa $f(B) \subseteq (-\infty, a]$ and $f(A) \subseteq [a, +\infty)$)

2. H separates strictly A and B if there exists an $\epsilon > 0$ such that $f(A) \subseteq (-\infty, a - \epsilon]$ and $f(B) \subseteq [a + \epsilon, +\infty)$ (or viceversa).

A special case of the geometric form of the Hahn–Banach theorem is the following, which states that if $A = C$ is an open convex set and $B = \{x_0\}$ with $x_0 \notin C$, then there exists a closed hyperplane separating them.

Lemma 6.13. *Let X be a topological vector space on $\mathbb{R}$, let C be an open and convex nonempty subspace and let $x_0 \notin C$. Then there exists a bounded linear map $f : X \to \mathbb{R}$ such that $f(x) < f(x_0)$ for all $x \in C$.*

Proof. It is not restrictive to assume $0 \in C$. Let us consider the seminorm p defined in (4.4). Then $p(x_0) \geq 1$ by $x_0 \notin C$. Set $Y := \mathbb{R}x_0$ and on Y define the linear map $g(tx_0) = t$ for all $t \in \mathbb{R}$. We claim have $g(y) \leq p(y)$ for all $y \in Y$. Indeed, in the special case $x = x_0$, $g(x_0) = 1$ and $p(x_0) \geq 1$ and so $1 = g(x_0) \leq p(x_0)$. This inequality continues to hold if we multiply the above inequality by $t \geq 0$, getting $g(tx_0) \leq p(tx_0)$ for $t \geq 0$. Finally we have

$$t = g(tx_0) \leq p(tx_0) \text{ for all } t \in \mathbb{R}$$

since, for $t < 0$, the l.h.s. is negative while the r.h.s. is non negative.
We can now apply Theorem 6.1 and conclude that there is a linear operator $f : X \to \mathbb{R}$ which extends g and is such that $f(x) \leq p(x)$ for all $x \in X$. Then $f(x) \leq p(x) < 1 = f(x_0)$ for all $x \in C$. Notice that since $f^{-1}(1) \cap C = \emptyset$, the hyperplane $f^{-1}(1)$ is not dense in X, and so, c.f.r. Exercise 2.19, is closed and f is bounded. □

Theorem 6.14 (Hahn–Banach, Geometric form)**.** *Consider a topological vector space X. Let A and B be nonempty and disjoint convex sets, with A open. Then there is a closed hyperplane H separating them.*

Proof. Set $C := A - B = \{a - b | a \in A \text{ and } b \in B\}$. Then C is convex since if $a_0, a_1 \in A$, $b_0, b_1 \in B$, $a_t \in A$, $b_t \in B$, we have $a_t - b_t = (a - b)_t \in C$.
We notice now that $C = \cup_{b \in B}(A - b)$, as a union of open sets, is open and $C \not\ni 0$. We apply Lemma 6.13 to the pair C and $x_0 = 0$. Then there exists a continuous linear map $f : X \to \mathbb{R}$ such that $f(c) < f(0) = 0$ for all $c \in C$.

This is the same as having $f(a) < f(b)$ for any $a \in A$ and $b \in B$. Then picking

$$\sup_{x \in A} f(x) \leq \alpha \leq \inf_{x \in B} f(x),$$

we have that $H = f^{-1}(\alpha)$ separates A and B. □

Remark 6.15. Notice that in Brezis [3, Exercise 1.9] it is stated, and later in the solutions it is discussed, the fact that in finite dimension the statement holds just under the hypothesis that A and B are nonempty and disjoint convex sets, without further specifications. This would be false in an infinite dimensional topological vector space X. Take for example a not continuous linear map $f : X \to \mathbb{R}$ and let $A = f^{-1}(\mathbb{R}_-)$ and $B = f^{-1}(\mathbb{R}_+)$. Then A and B are nonempty and disjoint convex sets, but there is no closed hyperplane H separating them. If fact, if it existed, it would be of the form $H = g^{-1}(\alpha)$ for a nonzero $g \in X'$ and with $g \leq \alpha$ in A and $g \geq \alpha$ in B. But the set where $g < \alpha$ is non empty and open in X, while B is dense in X, by Exercise 2.19. So it is impossible that $g \geq \alpha$ in B, and we get a contradiction.

Theorem 6.16 (Hahn–Banach, Geometric form, 2nd version)**.** *Consider a locally convex space X. Let A and B be nonempty and disjoint convex sets, with A closed and B compact. Then there is a closed hyperplane H separating strictly them.*

Proof. We claim that

$$\exists \text{ a convex balanced open neighborhood } U \text{ of } 0 \text{ such that } (A+U) \cap (B+U) = \emptyset. \tag{6.22}$$

Let us assume (6.22). Then, since $A+U$ and $B+U$ are easily shown to be open convex sets, by Theorem 6.14 we know that there exists a closed hyperplane $H = f^{-1}(\alpha)$ separating $A+U$ and $B+U$, that is

$$f(a) + f(z_1) \leq \alpha \leq f(b) + f(z_2) \text{ for all } a \in A,\, b \in B \text{ and all } z_1, z_2 \in U. \tag{6.23}$$

Notice that there exists an $\epsilon > 0$ such that $f(U) \supseteq [-\epsilon, \epsilon]$. Indeed there exists $x_0 \in X$ such that $f(x_0) = 1$ and there exists $\epsilon > 0$ such that $\epsilon x_0 \in U$. Then $\lambda x_0 \in U$ for any $|\lambda| \leq \epsilon$ because U is balanced. Then $f(U) \supseteq f(\{\lambda x_0 : |\lambda| \leq \epsilon\}) = [-\epsilon, \epsilon]$. Then from (6.27) we derive

$$f(a) \leq \alpha - \epsilon < \alpha + \epsilon \leq f(b) \text{ for all } a \in A \text{ and } b \in B.$$

Now we turn to the proof of (6.22). Let us consider the family $\mathfrak{V}$ of the open convex and balanced neighborhoods of 0 in X. For any $V \in \mathfrak{V}$, the complement $\complement(\overline{A+V})$ of the closure $\overline{A+V}$ is obviously open. We claim

$$\bigcup_{V \in \mathfrak{V}} \complement(\overline{A+V}) \supseteq B. \tag{6.24}$$

To see (6.28), we consider any $x \in B$. Since $\complement A$ is an open neighborhood of x, there exists a balanced open and convex neighborhood of 0, V, such that $V + V + x \subseteq \complement A$. Since the latter inclusion implies

$$v_1 - v_2 + x \neq a \text{ for any } a \in A \text{ and } v_1 \text{ and } v_2 \text{ in } V, \tag{6.25}$$

it implies $V + x \subseteq \complement(A + V)$. Since $V + x$ is open, $V + x \subseteq \mathring{\complement}(A + V) = \complement\left(\overline{A + V}\right)$. This proves (6.28).
Since B is compact, there is a finite cover

$$\complement(\overline{A + V_1}) \cup ... \cup \complement(\overline{A + V_n}) \supseteq B.$$

Taking $W = V_1 \cap ... \cap V_n$, we conclude $\complement(\overline{A + W}) \supseteq B$. Let us consider now a convex balanced open neighborhood U of 0 such that $U + U \subseteq W$. Then, like in (6.25),

$$B \cap \subseteq \complement\left(\overline{A + U + U}\right) \Longleftrightarrow B + U \subseteq \complement\left(A + U\right) \Longleftrightarrow (B + U) \cap (A + U) = \emptyset,$$

thus proving (6.22) and completing the proof of the theorem. □

Remark 6.17. When X is a normed space, then the proof of (6.22) is simpler. In fact it is easy to prove that

$$\text{there is an } \epsilon > 0 \text{ such that } (A + D_X(0, \epsilon)) \cap (B + D_X(0, \epsilon)) = \emptyset. \tag{6.26}$$

Indeed, if this is false, for any sequence $\epsilon_n \xrightarrow{n \to +\infty} 0^+$, there exists

$$z_n \in (A + D_X(0, \epsilon_n)) \cap (B + D_X(0, \epsilon_n)).$$

Then there are sequences $\{a_n\}$ in A and $\{b_n\}$ in B with $\|a_n - z_n\|_X < \epsilon_n$ and $\|b_n - z_n\|_X < \epsilon_n$. So we have $\|a_n - b_n\|_X < 2\epsilon_n$. On the other hand, since B is compact, there is a subsequence of $\{b_n\}$ in B convergent in B. It is not restrictive to assume that $b_n \xrightarrow{n \to +\infty} b$ in B. We have also $a_n \xrightarrow{n \to +\infty} b$. But then $b \in A \cap B$, which contradicts $A \cap B = \emptyset$.

Remark 6.18. It is obvious that, if we replace in the hypothesis in Theorem 6.16, B compact with B closed, then there is no hope, even in finite dimension, that in general these can be separated strictly with a closed hyperplane H. In Brezis [3, Exercise 1.14], there is an example of two closed convex sets inside $\ell^1(\mathbb{N})$, which cannot be separated by a closed hyperplane H. The idea is to consider the two vector spaces

$$X := \{(x_n) : x_{2n} = 0 \text{ for all } n \in \mathbb{N}\} \text{ and}$$
$$Y := \{(y_n) : y_{2n} = \frac{1}{2^n} y_{2n-1} \text{ for all } n \in \mathbb{N}\}.$$

These are closed vector spaces. The key point now is that $\overline{X + Y} = \ell^1(\mathbb{N})$ but $X + Y \subsetneqq \ell^1(\mathbb{N})$. Let $c \notin X + Y$. Then there is no closed hyperplane separating c and $X + Y$ (since the latter is dense). Now let $Z = X - c$. Then Y and Z are closed disjoint convex sets, and the claim is that they are not separated by a closed hyperplane H. Otherwise there would be $f \in (\ell^1(\mathbb{N}))'$ separating them, that is there would exist $\alpha \in \mathbb{R}$ and $f \leq \alpha$ in Y and $f \geq \alpha$ in Z. Necessarily $\alpha \geq 0$ and $f = 0$ in Y, by linearity and, similarly, $f = 0$ in X. Then $0 \leq \alpha \leq -f(c)$. So $f(c) \leq -\alpha \leq 0$ and $f \equiv 0$ in $X + Y$ would give us $f^{-1}(\alpha)$, a closed hyperplane separating c and $X + Y$, which in fact cannot exist.

Corollary 6.19. *Let Y be a nonempty subspace of a locally convex space X on $\mathbb{R}$. Assume $\overline{Y} \subsetneqq X$. Then there exists $f \in X'$ nonzero and such that*

$$f(y) = 0 \text{ for all } y \in Y. \tag{6.27}$$

Proof. Take $x_0 \notin \overline{Y}$. Then $A := \overline{Y}$ and $B := \{x_0\}$ satisfy the hypotheses of Theorem 6.16. So in particular there exists $f \in X'$ such that for some $\alpha \in \mathbb{R}$

$$f(y) < \alpha < f(x_0) \text{ for all } y \in Y. \tag{6.28}$$

But then we must have (6.27) for, if we had $f(y_0) \neq 0$ for an $y_0 \in Y$, then

$$\sup_{\lambda \in \mathbb{R}} f(\lambda y_0) = \sup_{\lambda \in \mathbb{R}} (\lambda f(y_0)) = +\infty,$$

contradicting (6.28).

□

Remark 6.20. Notice that in the spaces $L^p(0,1)$ with $0 < p < 1$, where $(L^p(0,1))' = 0$, then Corollary 6.19 is false. Take just any line $Y = \text{sp}\{f\}$, which is a closed subspace of $L^p(0,1)$, with $0 \neq f \in L^p(0,1)$. The reason why $Y = \text{sp}\{f\}$, is a closed subspace of $L^p(0,1)$, it that it is isomorphic to $\mathbb{R}$. Indeed, $\mathbb{R} \ni t \to tf \in Y$ is continuous and viceversa, the map $tf \to t$ is continuous since its kernel, equal to $\{0\}$, is closed in $L^p(0,1)$, and so also in Y, and hence is not dense in Y.

Example 6.21. An interesting result is the Müntz-Szasz Theorem which states the following:

Let $I = [0,1]$ and let $0 < \lambda_1 < \lambda_2 < ...$ be a strictly increasing sequence with $\lambda_n \xrightarrow{n \to +\infty} +\infty$. Then the closure Y of the subspace in $C^0(I)$ generated by $1, t^{\lambda_1}, t^{\lambda_2}, ...$ is such that

(1) *If* $\displaystyle\sum_{n=1}^{\infty} 1/\lambda_n = +\infty$ *then* $Y = C^0(I)$.

(2) *If* $\displaystyle\sum_{n=1}^{\infty} 1/\lambda_n < +\infty$ *and if* $\lambda \neq 0$ *is* $\lambda \notin \{\lambda_n\}_1^{\infty}$, *then* $t^{\lambda} \notin Y$.

Notice that, in the particular case $\lambda_n \equiv n$, we reobtain the Weierstrass Approximation Theorem 1.17. The Müntz-Szasz Theorem, in particular, implies that if we eliminate any number N of elements $\lambda_{n_1} < ... < \lambda_{n_N}$ from the sequence in case (1), the set Y remains the same and continues to coincide with $C^0(I)$.

For the proof we refer to Rudin [9]. We sketch the proof of statement (1), which is a beautiful application of Corollary 6.19, of the fact that $\left(C^0(I)\right)'$ is the space of Borel measures on I, see Theorem 16.19 later, and some basic fact on bounded holomorphic functions in in the unit disk U.

To prove (1) it enough to prove that for any complex Borel measure μ we have

$$\int_I t^{\lambda_n} d\mu(t) = \int_I d\mu(t) = 0 \,\forall\, n \Rightarrow \int_I t^n d\mu(t) = 0 \,\forall\, n = 0, 1, ... \tag{6.29}$$

But then, since $\mathrm{Sp}\{1, t, t^2, ...\}$ is dense in $C^0(I)$, we conclude $\mu = 0$. Hence, since all the elements of $(C^0(I))'$ which are null in the closed set Y are also null on $C^0(I)$, by Corollary 6.19 we conclude that we cannot have $Y \subsetneqq C^0(I)$.

To prove (6.29), define

$$f(z) := \int_I t^z d\mu(t). \tag{6.30}$$

Then $f \in H(\{z : \mathrm{Re}\, z > 0\})$. Indeed, $f(z)$ is bounded and continuous in $\{z : \mathrm{Re}\, z \geq 0\}$ and, applying Morera Theorem, is holomorphic in $\{z : \mathrm{Re}\, z > 0\}$. Next, set

$$g(z) = f\left(\frac{1+z}{1-z}\right).$$

Then $g \in H^\infty(U)$, where $U = D_{\mathbb{C}}(0,1)$, with $g(\alpha_n) = 0$ for $\alpha_n = \dfrac{\lambda_n - 1}{1 + \lambda_n}$. Using $\left(\frac{t-1}{t+1}\right)' = \frac{2}{(t+1)^2} >$ we conclude that α_n is strictly increasing. So we have

$$\sum_{n=1}^{\infty} 1/\lambda_n = +\infty \Longrightarrow \sum_{n=1}^{\infty} 1/\lambda_n = \sum_{n=1}^{\infty} \frac{1-\alpha_n}{1+\alpha_n} = \frac{1}{1+\alpha_1} \sum_{n=1}^{\infty} (1 - |\alpha_n|) = +\infty. \tag{6.31}$$

Hence $\sum\limits_{n=1}^{\infty} (1 - |\alpha_n|) = +\infty$. But there is a theorem which says that then $g \equiv 0$ (it is a refinement of the theorem which guarantees that if $g \not\equiv 0$, the set of the zeros of g has no accumulation points inside U). So $f \equiv 0$ and, in particular, $f(n) = 0$ for all $n = 0, 1, 2...$ proving (6.29). We refer to Rudin [9] for the proof of (2), which is based on some beautiful theory of holomorphic functions on a half–plane, and for further details.

We will see later further applications of the Hahn–Banach Theorem.

6.2 The bidual and orthogonality

Let X be a normed space and consider the Banach space X'. Then the bidual of X is $X'' := (X')'$.

Lemma 6.22. *Consider the map $J : X \to X''$ given by $\langle Jx, x' \rangle_{X'' \times X'} := \langle x, x' \rangle_{X \times X'}$. Then J is an isometric immersion of X inside X''.*

Proof. For $\|x'\|_{X'} = 1$, by the definition of the norm in X'', we have

$$|\left\langle Jx, x' \right\rangle_{X'' \times X'}| = |\left\langle x, x' \right\rangle_{X \times X'}| \leq \|x'\|_{X'} \|x\|_X = \|x\|_X,$$

from which we derive that $\|Jx\|_{X''} \leq \|x\|_X$ for any $x \in X$. On the other hand, for any $x \in X$, by Hahn–Banach we know that there exists $x' \in X'$ with $\|x'\|_{X'} = 1$ such that

$$\|x\|_X = |\left\langle x, x' \right\rangle_{X \times X'}| = |\left\langle Jx, x' \right\rangle_{X'' \times X'}| \leq \|x'\|_{X'} \|Jx\|_{X''} = \|Jx\|_{X''},$$

so that we conclude $\|Jx\|_{X''} \geq \|x\|_X$ for any $x \in X$.

□

Definition 6.23. Given a topological vector space X and $M \subseteq X$, we set

$$M^{\perp} := \{f \in X' : \langle f, x \rangle_{X' \times X} = 0 \text{ for all } x \in M\}. \tag{6.32}$$

Similarly, for $N \subseteq X'$ we set

$$N^{\perp} := \{x \in X : \langle f, x \rangle_{X' \times X} = 0 \text{ for all } f \in N\}. \tag{6.33}$$

Let X by a normed space and consider the related norm in X'.

Lemma 6.24. *Given a normed space and a linear subspace $M \subseteq X$. Then*

$$(M^{\perp})^{\perp} = \overline{M}. \tag{6.34}$$

Given a linear subspace $N \subseteq X'$, then

$$(N^{\perp})^{\perp} \supseteq \overline{N}. \tag{6.35}$$

Proof. By definition of $M^{\perp}$ in (6.32), we have $\langle f, x \rangle_{X' \times X} = 0$ for all $x \in M$ and $f \in M^{\perp}$. So $(M^{\perp})^{\perp} \supseteq M$ by the definition of $(M^{\perp})^{\perp}$, in (6.33) for $N = M^{\perp}$. Furthermore, since the orthogonals are closed spaces, we have also $(M^{\perp})^{\perp} \supseteq \overline{M}$. This in particular proves also (6.35). Now let us prove the equality (6.34). Proceeding by contradiction, suppose that there exists $x_0 \in (M^{\perp})^{\perp} \backslash \overline{M}$. Then there is a closed hyperplane H separating strictly x_0 and $\overline{M}$. In particular, is not restrictive to assume the existence of a continuous linear functional f and an $\alpha \in \mathbb{R}$ such that

$$\langle f, x_0 \rangle_{X' \times X} = f(x_0) < \alpha < f(x) = \langle f, x \rangle_{X' \times X} \text{ for all } x \in \overline{M}.$$

By linearity we need to have $f(x) = 0$ for all $x \in \overline{M}$. This means that $f \in M^{\perp}$. On the other hand, $f(x_0) < 0$ implies that $x_0 \notin (M^{\perp})^{\perp}$, which contradicts our hypothesis that $x_0 \in (M^{\perp})^{\perp} \backslash \overline{M}$. So we have proved that $(M^{\perp})^{\perp} \backslash \overline{M} = \emptyset$. ☐

Example 6.25. We will see later that $(\ell^1(\mathbb{N}))' = \ell^{\infty}(\mathbb{N})$ and that $(c_0(\mathbb{N}))' = \ell^1(\mathbb{N})$, where $c_0(\mathbb{N})$ is a closed subspace of $\ell^1(\mathbb{N})$. Now, $c_0(\mathbb{N})^{\perp} = 0 \subseteq \ell^1(\mathbb{N})$ and $0^{\perp} = \ell^{\infty}(\mathbb{N})$, so $\left(c_0(\mathbb{N})^{\perp}\right)^{\perp} = \ell^{\infty}(\mathbb{N}) \supsetneqq c_0(\mathbb{N})$.

Lemma 6.26. *Consider a bounded operator $T : X \to Y$ between two Banach spaces and the adjoint $T : Y' \to X'$. Let $R(T) = TX$ and $R(T^*) = T^*Y'$. Then we have the following:*

$$\ker T = R(T^*)^{\perp}; \tag{6.36}$$

$$\ker T^* = R(T)^{\perp}; \tag{6.37}$$

$$(\ker T)^{\perp} \supseteq \overline{R(T^*)}; \tag{6.38}$$

$$(\ker T^*)^{\perp} = \overline{R(T)}. \tag{6.39}$$

Proof. Formula (6.36) follows from $\langle Tx, y' \rangle_{Y \times Y'} = \langle x, T^* y' \rangle_{X \times X'}$. Indeed, if $x \in \ker T$ then the formula yields $\langle x, T^* y' \rangle_{X \times X'} = 0$ for any $y' \in F'$, and so $x \in R(T^*)^\perp$. If, viceversa, $x \in R(T^*)^\perp$, then $\langle Tx, y' \rangle_{Y \times Y'} = 0$ for all $y' \in Y'$, and this implies $Tx = 0$ by Corollary 6.6, that is, $x \in \ker T$.
A similar discussion, in fact, simpler since it does not rely on a deep theorem like Hahn–Banach but stems directly from the definitions, is valid for (6.37). Indeed, if $y' \in \ker T^*$ then from $0 = \langle x, T^* y' \rangle_{X \times X'} = \langle Tx, y' \rangle_{Y \times Y'}$ for all $x \in X$ we have $y' \in R(T)^\perp$. Viceversa, if $y' \in R(T)^\perp$, then $0 = \langle Tx, y' \rangle_{Y \times Y'} = \langle x, T^* y' \rangle_{X \times X'}$ for all $x \in X$ implies, by definition of $T^* y' \in X'$ (and therefor not by any deep theorem), that $T^* y' = 0$ and, so, $y' \in \ker T^*$.
Turning to (6.38), we know from (6.35) that $(R(T^*)^\perp)^\perp \supseteq \overline{R(T^*)}$, which gives the desired result. Similarly, we know from (6.34) that $(R(T)^\perp)^\perp = \overline{R(T)}$, which gives 6.39. $\square$

Exercise 6.27. Let $T \in \mathcal{L}(X, Y)$ and consider $T^* \in \mathcal{L}(Y\,', X')$, $T^{**} \in \mathcal{L}(X'', Y'')$ and the maps $J_X : X \to X''$ and $J_Y : Y \to Y''$ in Lemma 6.22. Then for the following two maps $X \to Y''$, show that we have $T^{**} J_X = J_Y T$ or, otherwise stated, that the following diagram is commutative,

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow{\scriptstyle J_X} & & \downarrow{\scriptstyle J_Y} \\ X'' & \xrightarrow{T^{**}} & Y'' \end{array}$$

Answer. We have

$$\begin{aligned} &\left\langle Tx, y' \right\rangle_{Y \times Y'} = \left\langle J_Y Tx, y' \right\rangle_{Y'' \times Y'} \text{ and} \\ &\left\langle Tx, y' \right\rangle_{Y \times Y'} = \left\langle x, T^* y' \right\rangle_{X \times X'} = \left\langle J_X x, T^* y' \right\rangle_{X'' \times X'} = \left\langle T^{**} J_X x, y' \right\rangle_{Y'' \times Y'}. \end{aligned}$$

Then

$$\left\langle J_Y Tx - T^{**} J_X x, y' \right\rangle_{Y'' \times Y'} = 0 \text{ for all } x \in X \text{ and } y' \in Y'.$$

This by the definition of Y'' implies $T^{**} J_X x = J_Y T x$ for all $x \in X$ and so our statement. $\square$

Exercise 6.28. Consider the set up of Exercise 6.27 with $T \in \mathcal{L}(X, Y)$ and consider $T^* \in \mathcal{L}(Y', X')$, $T^{**} \in \mathcal{L}(X'', Y'')$ and the maps $J_X : X \to X''$ and $J_Y : Y \to Y''$ in Lemma 6.22. Suppose now that both X and Y are reflexive, that is, see later in Sect. 13, J_X and J_Y are isomorphisms. Then show that instead of (6.38) we have

$$(\ker T)^\perp = \overline{R(T^*)}. \tag{6.40}$$

Answer. We want to show that

$$x' \in (\ker T)^\perp \Longrightarrow x' \in (\ker T^{**})^\perp. \tag{6.41}$$

Assuming this, by Lemma 6.26 with T (resp. T^*) replaced by T^* (resp. T^{**}), we have $(\ker T^{**})^\perp = \overline{R(T^*)}$. This allows to conclude that (6.40) is true.

So let us prove 6.41 and let $x' \in (\ker T)^{\perp}$. Observe that, since $J_Y^{-1} T^{**} J_X = T$ where J_X is an isomorphism, we have $\ker T^{**} = J_X \ker T$. Since

$$0 = \langle x, x' \rangle_{X \times X'} = \langle J_X x, x' \rangle_{X'' \times X'} \text{ for all } x \in \ker T$$

and since, by our observation, as $J_X x$ spans all $\ker T^{**}$ as x varies in $\ker T$, we conclude that

$$\langle x'', x' \rangle_{X'' \times X'} = 0 \text{ for all } x'' \in \ker T^{**}.$$

Hence $x' \in (\ker T^{**})^{\perp}$ and (6.41) is proved.

□

7 Theorem by Banach and Steinhaus

Definition 7.1 (Baire Spaces)**.** A topological space X is said to be a Baire space if either of the following two equivalent statements holds:

1. for any sequence A_n of dense open subspaces, then $\cap_{n=1}^{\infty} A_n$ is dense;
2. for any sequence C_n of closed subspaces without interior points, then $\cup_{n=1}^{\infty} C_n$ has empty interior.

A subspace of a topological space X which contains the intersection of a sequence of open dense subspaces of X is called a G_δ subspace of X.

Remark 7.2. A G_δ subspace of a Baire space X is dense in X.

Exercise 7.3. Consider a compact space K and a decreasing sequence of compact sets with $\cap_{n \in \mathbb{N}} K_n = \emptyset$. Then only finitely many of them are nonempty.

There are two important classes of Baire spaces: locally compact spaces and complete metric spaces.

Theorem 7.4. *Every locally compact Hausdorff space X is a Baire space.*

Proof. Take a sequence A_n of dense open sets. Take any open set G_1. We can take a sequence of decreasing nonempty relatively compact open sets, with for $n \geq 2$, $G_n \subseteq \overline{G}_n \subseteq A_n \cap G_{n-1}$, $\overline{G}_n$ compact. Then it is easy to see that we have $\cap_{n \in \mathbb{N}} G_n = \cap_{n \in \mathbb{N}} \overline{G}_n$. By Exercise 7.3 we also know that the above infinite intersection is non empty, since otherwise there would be N such that $G_n = \emptyset$ for $n \geq N$, which, by construction, is not true.
Notice also that

$$\cap_{n=1}^{\infty} G_n \subseteq \cap_{n=2}^{\infty} G_n \subseteq \cap_{n=2}^{\infty} (A_n \cap G_{n-1}) \subseteq \cap_{n=2}^{\infty} A_n.$$

Then $G_1 \cap (\cap_{n=2}^{\infty} A_n) \neq \emptyset$. This implies also $G_1 \cap (\cap_{n=1}^{\infty} A_n) \neq \emptyset$. This implies that any sequence A_n of dense open subspaces in X is such that $\cap_{n=1}^{\infty} A_n$ is dense

□

Example 7.5. Let X be any non empty set and let us consider the topology where any $Y \subseteq X$ is open. Notice that X is locally compact (for any $x \in X$ the set $\{x\}$ is a compact neighborhood of x). If A is a dense set in X, then necessarily $A = X$. So for any sequence A_n of dense open sets we actually have $A_n = X$ for all n and trivially $\cap_{n=1}^{\infty} A_n = X$, which is obviously dense.

Theorem 7.6. *Every complete metric space (X, d) is a Baire space.*

Proof. Take a sequence A_n of dense open sets in X. Take any open set G_1. We can take a sequence of decreasing nonempty open balls, with for $n \geq 2$, $D(x_n, r_n) \subseteq \overline{D(x_n, r_n)} \subseteq D(x_{n-1}, r_{n-1}) \subset A_n \cap G_1$ with $r_n \searrow 0$ (r_n a strictly decreasing sequence, convergent to 0). Then it is easy to see that we have $\cap_{n=1}^{\infty} D(x_n, r_n) = \cap_{n=1}^{\infty} \overline{D(x_n, r_n)}$. Furthermore, since X is a complete metric space, this intersection is non empty and, in fact, is of the type $\{\overline{x}\}$ for some $\overline{x} \in X$. Here $\overline{x} \in \overline{D(x_n, r_n)}$ for all n, and so $\overline{x} \in A_n \cap G_1$ for all n. So, in particular $\overline{x} \in (\cap_{n=1}^{\infty} A_n) \cap G_1$.
This shows that $\cap_{n=1}^{\infty} A_n$ is dense.

□

Exercise 7.7. Show that $\mathcal{D}(\Omega)$ is not metrizable.

Answer. As we know, $\mathcal{D}(\Omega)$ is the direct limit of a strictly increasing sequence $\{D_{K_n}(\Omega)\}$, where $\{K_n\}$ is a sequence of compact subsets of Ω with $K_n \subset \mathring{K}_{n+1}$ $\forall$ n and $\bigcup_{n=1}^{\infty} K_n = \Omega$. Notice that since each $D_{K_n}(\Omega)$ is complete, by Theorem 4.33 also $\mathcal{D}(\Omega)$ is complete.
Each $D_{K_n}(\Omega)$ is a closed subspace of $\mathcal{D}(\Omega)$ (since it is complete) and it has empty interior, since there are elements $\varphi \in D_{K_{n+1}}(\Omega)$ arbitrarily close to 0 in $D_{K_{n+1}}(\Omega)$ and with supp $\varphi \supsetneq K_n$. Furthermore, $\mathcal{D}(\Omega) = \bigcup_{n=1}^{\infty} D_{K_n}(\Omega)$. All this implies that $\mathcal{D}(\Omega)$ is not a Baire space. If it were metrizable, it would be a complete metric space and, by Theorem 7.6, it would be Baire.

Definition 7.8. Let $\{\Lambda_j\}_{j \in J}$ be a family in $\mathcal{L}(X, Y)$ with X and Y two topological vector spaces. We say that $\{\Lambda_j\}_{j \in J}$ is equicontinuous if for any neighborhood V of 0 in Y there exists a neighborhood U of 0 in X such that $\Lambda_j U \subseteq V$ for all $j \in J$.

Exercise 7.9. Show that a family $\{\Lambda_j\}_{j \in J}$ in $\mathcal{L}(X, Y)$ with X and Y two normed spaces is equicontinuous if and only if there exists an $M \in \mathbb{R}_+$ such that $\|T_j\|_{\mathcal{L}(E,F)} \leq M < \infty$ for all $j \in J$.

Lemma 7.10. *Let $\{\Lambda_j\}_{j \in J}$ be an equicontinuous family in $\mathcal{L}(X, Y)$. Then, for any bounded set E in X there exists a bounded set F in Y such that $\bigcup_{j \in J} \Lambda_j E \subseteq F$.*

Proof. Set $F := \bigcup_{j \in J} \Lambda_j E$ and let V be any neighborhood of 0 in Y. By equicontinuity, we know that there exists a neighborhood U of 0 in X such that $\Lambda_j U \subseteq V$ for all $j \in J$. Since E is bounded, we know that there exists $t \in \mathbb{R}_+$ such that $E \subseteq tU$. It follows that for any $j \in J$ we have $\Lambda_j E \subseteq t\Lambda_j U \subseteq tV$. So also $F = \bigcup_{j \in J} \Lambda_j E \subseteq tV$, which proves that F is bonded in Y.

□

Theorem 7.11 (Banach–Steinhaus). *Consider a family $\{\Lambda_j\}_{j\in J}$ in $\mathcal{L}(X,Y)$ with X and Y two topological vector spaces. Consider the orbits*

$$\Gamma(x) := \{\Lambda_j x : j \in J\}$$

and set

$$B = \{x \in X : \Gamma(x) \textit{ is bounded in } Y\}.$$

Suppose that the complement of B in X does not contain a G_δ set. Then the family $\{\Lambda_j\}_{j\in J}$ is equicontinuous.

Proof. Since by hypothesis $\complement B := X\backslash B$ does not contain the intersection of a sequence of open dense sets in X, it follows that B is not contained in the union of a sequence of closed sets, each with empty interior.
Consider an arbitrary balanced neighborhood W of 0 in Y and let V be another balanced neighborhood of 0 in Y with $\overline{V}+\overline{V} \subseteq W$ (notice that $\overline{V} \subseteq V+V$, since if $x \in \overline{V}$ then $(x+V)\cap V \neq \emptyset$ which implies $x \in V - V = V + V$, and so we can just take $\overline{V}+\overline{V} \subseteq V+V+V+V \subseteq W$).
Set $E := \bigcap_{j\in J} \Lambda_j^{-1}\overline{V}$. Then we claim that

$$B \subseteq \bigcup_{n\in\mathbb{N}} nE.$$

Indeed, for any $x \in B$, the fact that $\Gamma(x)$ is bounded in Y implies that there exists an $n \in \mathbb{N}$ such that $\Gamma(x) \subseteq nV$. So $\Lambda_j x \in n\overline{V}$ or, equivalently $x \in n\Lambda_j^{-1}\overline{V}$, for all $j \in J$. Hence $x \in \bigcap_{j\in J} n\Lambda_j^{-1}\overline{V} = n\bigcap_{j\in J}\Lambda_j^{-1}\overline{V} = nE$.
The nE are closed sets. Each of them has non–empty interior exactly if E has a non–empty interior. So there is an interior point $x \in \mathring{E}$ and a neighborhood U of 0 in X with $x+U \subseteq E$. By the definition of E, this implies that $\Lambda_j x + \Lambda_j U \subseteq \overline{V}$ for all $j \in J$. Then

$$\Lambda_j U \subseteq \overline{V} - \Lambda_j x \subseteq \overline{V} - \overline{V} = \overline{V} + \overline{V} \subseteq W \text{ for all } j \in J.$$

So we have proved that for any neighborhood W of 0 in Y there exists a neighborhood U of 0 in X such that $\Lambda_j U \subseteq W$ for all $j \in J$ and so, that $\{\Lambda_j\}_{j\in J}$ is equicontinuous.

□

An immediate corollary is hence the following .

Corollary 7.12. *Let X and Y normed spaces and consider a sequence $T_n \in \mathcal{L}(X,Y)$. Suppose $\sup_n \|T_n x\|_Y < \infty$ for any $x \in X$. Then $\|T_n\|_{\mathcal{L}(X,Y)} \leq M < \infty$ for some M.*
If it is not true that $\sup_n \|T_n\|_{\mathcal{L}(X,Y)} \leq M < \infty$ for some M, then $\sup_n \|T_n x\|_Y = \infty$ for all the x in a G_δ set.

7.1 Some application of the Theorem by Banach and Steinhaus

A function $f(x) = P(\cos x, \sin x)$ with $P(z_1, z_2)$ a polinomial is called a trigonometric polinomial. Using repeatedly the prostaferese formulas

$$\begin{aligned}
\sin(nx)\sin(mx) &= \frac{\cos((n-m)x) - \cos((n+m)x)}{2}, \\
\cos(nx)\sin(mx) &= \frac{\sin((n+m)x) - \sin((n-m)x)}{2} \text{ and} \\
\cos(nx)\cos(mx) &= \frac{\cos((n-m)x) + \cos((n+m)x)}{2},
\end{aligned} \tag{7.1}$$

it is easy to see that any trigonometric polinomial can be written in the form

$$f(x) = \frac{a_0}{2} + \sum_{\ell=1}^{n} (a_\ell \cos(\ell x) + b_\ell \sin(\ell x)). \tag{7.2}$$

Lemma 7.13. *Given the trigonometric polinomial* (7.2)*, the following formulas are true,*

$$\begin{aligned}
a_\ell &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(\ell x)dx = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos(\ell x)dx,\ \ell = 0, 1, \cdots \\
b_\ell &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(\ell x)dx = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin(\ell x)dx\,,\ \ \ell = 1, \cdots .
\end{aligned} \tag{7.3}$$

Proof. For $\delta_{n,m}$ the Kronecker delta, we have

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \delta_{n,m} \\
&\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \delta_{n,m} \\
&\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)\sin(mx)dx = 0.
\end{aligned} \tag{7.4}$$

So, if we multiply (7.2) by $\cos(mx)$, obtaining

$$f(x)\cos(mx) = \frac{a_0}{2}\cos(mx) + \sum_{\ell=1}^{n}(a_\ell\cos(\ell x)\cos(mx) + b_\ell\sin(\ell x)\cos(mx))$$

and if we integrate, we get

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\cos(mx)dx = \\
&= \frac{a_0}{2}\cos(mx) + \sum_{\ell=1}^{n}(a_\ell\int_{-\pi}^{\pi}\cos(\ell x)\cos(mx)dx + b_\ell\int_{-\pi}^{\pi}\sin(\ell x)\cos(mx)dx)^{.}
\end{aligned}$$

By (7.4) at most one term on the left is non–zero, and we have

$$\int_{-\pi}^{\pi} f(x)\cos(mx)dx = a_m\pi.$$

In this way we obtai deln the 1st line in (7.3). The 2nd line is obtained similarly, multiplying (7.2) by $\sin(mx)$ and integrating. $\square$

Definition 7.14 (Fourier Series)**.** For any $f \in L^1(-\pi,\pi)$ its Fourier series is the series

$$\frac{a_0}{2}\cos(mx) + \sum_{\ell=1}^{\infty}(a_\ell\cos(\ell x) + b_\ell\sin(\ell x)) \tag{7.5}$$

where the coefficients a_n and b_n are defined by (7.3). Alternatively, we can define the Fourier series of f as the series

$$\begin{aligned} &\sum_{\ell\in\mathbb{Z}}^{\infty}\widehat{f}(\ell)e^{\mathrm{i}\ell x} \text{ where} \\ &\widehat{f}(\ell) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}\ell x}f(x)dx. \end{aligned} \tag{7.6}$$

The expansions can be obtained one from the other, using $e^{\mathrm{i}\ell x} = \cos(\ell x) + \mathrm{i}\sin(\ell x)$.

Example 7.15. For $[a,b] \subseteq [-\pi,\pi]$ and for the characteristic function $\chi_{[a,b]}$, we have

$$\begin{aligned} \widehat{\chi_{[a,b]}}(0) &= \frac{1}{2\pi}\int_a^b dx = \frac{b-a}{2\pi} \text{ and, for } n \neq 0, \\ \widehat{\chi_{[a,b]}}(n) &= \frac{1}{2\pi}\int_a^b e^{-\mathrm{i}nx}dx = \mathrm{i}\frac{e^{-\mathrm{i}nb} - e^{-\mathrm{i}na}}{2\pi n}. \end{aligned}$$

Definition 7.16 (Tori)**.** For any $d \in \mathbb{N}$, we set $\mathbb{T}^d := \mathbb{R}^d/2\pi\mathbb{Z}^d$ which we call the d–dimensional Torus. There is a natural identification of $L^p((-\pi,\pi)^d)$ with $L^p(\mathbb{T}^d)$. For any $f \in L^1(\mathbb{T}^d)$ its Fourier coefficients are given by

$$\widehat{f}(\mathbf{n}) := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{-\mathrm{i}\mathbf{n}\cdot x}f(x)dx. \tag{7.7}$$

Notice that if $f \in C^k(\mathbb{T}^d)$ for any $|\alpha| \leq k$ we have

$$\begin{aligned} \widehat{\partial_x^\alpha f}(\mathbf{n}) &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{-\mathrm{i}\mathbf{n}\cdot x}\partial_x^\alpha f(x)dx = (-1)^{|\alpha|}\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \partial_x^\alpha e^{-\mathrm{i}\mathbf{n}\cdot x}f(x)dx \\ &= \mathrm{i}^{|\alpha|}n_1^{\alpha_1}...n_d^{\alpha_d}\widehat{f}(\mathbf{n}). \end{aligned} \tag{7.8}$$

In other words, transformation (7.7) diagonalizes all the operators ∂_x^α. So in particular, for the Laplacian $\triangle := \sum_{j=1}^{d} \partial_j^2$ we have

$$\widehat{\triangle f}(\mathbf{n}) = \mathrm{i}^2(n_1^2 + \ldots + n_d^2)\widehat{f}(\mathbf{n}) = -\|\mathbf{n}\|^2 \widehat{f}(\mathbf{n}). \tag{7.9}$$

The following lemma expresses the fact that a bounded operator remains defined by $L^1(\mathbb{T}^d) \ni \widehat{f} \longrightarrow \widehat{f} \in c_0(\mathbb{Z}^d)$, where the latter is defined in (3.20)

Lemma 7.17. *For any $f \in L^1(\mathbb{T}^d)$ we have*

$$|\widehat{f}(\mathbf{n})| \le \frac{\|f\|_{L^1(\mathbb{T}^d)}}{(2\pi)^d} \text{ for all } \mathbf{n} \in \mathbb{Z}^d, \text{ and} \tag{7.10}$$

$$\lim_{\mathbf{n}\to\infty} \widehat{f}(\mathbf{n}) = 0. \tag{7.11}$$

Proof. Inequality (7.10) is straightforward. Let us turn to the proof of (7.11), which we know is true for $f = \chi_{[a_1,b_1]\times\ldots\times[a_d,b_d]}$ or for f a linear combination of functions $\chi_{[a_1,b_1]\times\ldots\times[a_d,b_d]}$. Such linear combinations form a dense set in $L^1((-\pi,\pi)^d)$. So, for any $\epsilon > 0$ there exists

$$g = \sum_{j=1}^{N} \lambda_j \chi_{[a_1^{(j)},b_1^{(j)}]\times\ldots\times[a_d^{(j)},b_d^{(j)}]}$$

with $\|f-g\|_{L^1(\mathbb{T}^d)} < \epsilon$. By Example 7.15 we have

$$\lim_{\mathbf{n}\to\infty} \widehat{g}(\mathbf{n}) = \lim_{\mathbf{n}\to\infty} \frac{\mathrm{i}^d}{(2\pi)^d} \sum_{j=1}^{N} \lambda_j \prod_{k=1}^{d} \frac{e^{-\mathrm{i}n_k b_k^{(j)}} - e^{-\mathrm{i}n_k a_k^{(j)}}}{n_k} = 0,$$

which implies that there exists a N_ϵ such that for $|\mathbf{n}| > N_\epsilon$ we have $|\widehat{g}(\mathbf{n})| < \epsilon$. This implies that for $|\mathbf{n}| > N_\epsilon$ we have

$$|\widehat{f}(\mathbf{n})| \le |\widehat{g}(\mathbf{n})| + |\widehat{f}(\mathbf{n}) - \widehat{g}(\mathbf{n})| \le \epsilon + (2\pi)^{-d}\|f-g\|_{L^1(\mathbb{T}^d)} < \left(1 + (2\pi)^{-d}\right)\epsilon.$$

We conclude that we have shown (7.11).

□

Let us focus in dimension $d = 1$. Obviously, it is interesting to get information on the convergence of the Fourier Series, that is on the limit

$$\lim_{n\to+\infty} S_n f(x) \text{ , with the partial sums } S_n f(x) := \frac{a_0}{2} + \sum_{\ell=1}^{n} (a_\ell \cos(\ell x) + b_\ell \sin(\ell x))$$

Definition 7.18. For any $n \ge 1$ the Dirichlet kernel is the function

$$D_n(x) = \frac{1}{2} + \sum_{\ell=1}^{n} \cos(\ell x) = \frac{\sin((n+\frac{1}{2})x)}{2\sin\frac{x}{2}}. \tag{7.12}$$

Notice that the 2nd equality follows from using the telescopic sum

$$\sin\left(\left(n+\frac{1}{2}\right)x\right) = \sin\left(\frac{x}{2}\right) + \sum_{\ell=1}^{n}\left(\sin\left(\left(\ell+\frac{1}{2}\right)x\right) - \sin\left(\left(\ell-\frac{1}{2}\right)x\right)\right)$$
$$= \sin\left(\frac{x}{2}\right) + \sum_{\ell=1}^{n} 2\sin\left(\frac{x}{2}\right)\cos(\ell x).$$

Lemma 7.19. *For any $f \in L^1(\mathbb{T})$ we have*

$$S_n f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) D_n(x-t)dt. \tag{7.13}$$

Proof. Follows from

$$S_n f(x) = \frac{1}{2}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{\ell=1}^{n}\left(\cos(\ell x)\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(\ell t)dt + \sin(\ell x)\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(\ell t)dt\right)$$
$$= \frac{1}{2}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{\ell=1}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)(\cos(\ell x)\cos(\ell t) + \sin(\ell x)\sin(\ell t))dt$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2} + \sum_{\ell=1}^{n}\cos(\ell(x-t))\right)dt.$$

□

We will denote by $C^0(\mathbb{T})$ the set of the functions $f \in C^0(\mathbb{R})$ and 2π–periodic.

Let us apply now the Theorem by Banach and Steinhaus to the Fourier series.

Theorem 7.20. *For any $x \in \mathbb{T}$ there exists $f \in C^0(\mathbb{T})$ whose Fourier series does not converge in x.*

Proof. First of all, it is not restrictive to consider $x = 0$. Recall that the partial sums are given by

$$S_n f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) D_n(t-x)dt\,,\ D_n(x) = \frac{\sin((n+\frac{1}{2})x)}{2\sin(x/2)}.$$

Notice now that

$$\|D_n\|_{L^1(\mathbb{T})} > 2\int_0^{\pi} \left|\sin((n+\frac{1}{2})t)\right|\frac{dt}{t} = 2\int_0^{(n+\frac{1}{2})\pi}\left|\sin(t)\right|\frac{dt}{t} > 2\int_0^{n\pi}\left|\sin(t)\right|\frac{dt}{t}$$
$$> 2\sum_{k=1}^{n}\frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi}\left|\sin(t)\right|dt = 4\sum_{k=1}^{n}\frac{1}{k\pi} \xrightarrow{n\to+\infty} \infty.$$

If we set $g(t) = \text{sign}(D_n(t))$ with $\text{sign}(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{cases}$, then it is easy to understand that $\text{sign}(D_n)$ is Riemann integrable.

Notice that

$$S_n g(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)D_n(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|dt = \frac{1}{2\pi}\|D_n\|_{L^1(\mathbb{T})}$$

By Lusin Theorem, for any $1/j > 0$ there exists a function $f_j \in C^0(\mathbb{T})$ with $\|f_j\|_{L^\infty(\mathbb{T})} \leq \|g\|_{L^\infty(\mathbb{T})} = 1$ such that

$$|\{x : f_j(x) \neq g(x)\}| < 1/j.$$

Hence $f_j \xrightarrow{j\to+\infty} g$ in $L^1(\mathbb{T})$ and $S_n f_j(0) \xrightarrow{j\to+\infty} S_n g(0) = \frac{1}{2\pi}\|D_n\|_{L^1(\mathbb{T})}$.
If $\{S_n f(0)\}_{n\in\mathbb{N}}$ for any $f \in C^0(\mathbb{T})$ were convergent, then for any $f \in C^0(\mathbb{T})$ we would have $\sup_n |S_n f(0)| < \infty$. Now, the operators $\mathrm{ev}_0 S_n : f \to S_n f(0)$ are bounded operators $C^0(\mathbb{T}) \to \mathbb{C}$ for any n. By Banach Steinhaus, $\sup_n |S_n f(0)| < \infty$ for any $f \in C^0(\mathbb{T})$ would imply a uniform $\|\mathrm{ev}_0 S_n\|_{(C^0(\mathbb{T}))'} \leq C$ for all $n \in \mathbb{N}$. Then $|S_n f(0)| \leq \|\mathrm{ev}_0 S_n\|_{(C^0(\mathbb{T}))'}\|f\|_{C^0(\mathbb{T})} \leq C\|f\|_{C^0(\mathbb{T})}$ for all $n \in \mathbb{N}$ and $f \in C^0(\mathbb{T})$ and so $2\pi C \geq \|\mathrm{ev}_0 S_n\|_{(C^0(\mathbb{T}))'} 2\pi \geq \|D_n\|_{L^1(\mathbb{T})} \xrightarrow{n\to+\infty} \infty$, which gives a contradiction.

□

A direct consequence of Banach–Steinhaus and Theorem 7.20, is the following.

Corollary 7.21. *For any $x \in \mathbb{T}$ the subset E_x formed by the $f \in C^0(\mathbb{T})$ whose Fourier series does not converge in x contains a G_δ set, that is it contains a countable intersection of open dense sets.*

□

The fact that $f \in C^0(\mathbb{T})$ is not the pointwise limit of its Fourier series, does not prevent f from being the pointwise limit of another sequence of trigonometric polynomials. What follows is related to the notion of Cesaro means. Recall that, given a sequence of numbers x_n, then

$$\lim_{n\to+\infty} x_n = A \Rightarrow \lim_{n\to+\infty} \frac{x_1 + x_2 + .. + x_n}{n} = A.$$

Obviously $\Leftarrow$ is not true (take $x_n = (-1)^n$). Turning to $f \in C^0(\mathbb{T})$, instead of considering the limit $\lim\limits_{n\to\infty} S_n f(x)$ we will show instead that

$$\lim_{n\to\infty} \sigma_n f(x) = f(x) \text{ for any } x \text{ and for } \sigma_n f(x) = \frac{S_0 f(x) + S_1 f(x) + .. + S_n f(x)}{n+1}. \quad (7.14)$$

Definition 7.22 (Fejer Kernel)**.** The Fejer Kernel is given by

$$K_N(t) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(t) = \frac{1}{N+1}\sum_{n=0}^{N} \frac{\sin((n+\frac{1}{2})t)}{2\sin\frac{t}{2}}. \quad (7.15)$$

Lemma 7.23. *We have the following facts.*

1. $K_N(t) \geq 0$ *for all* t.

2. *We have* $\frac{1}{\pi}\int_{-\pi}^{\pi} K_N(t)dt = 1.$

3. *For* $\mu_N(\delta) := \max\{K_N(t) : \delta \le t \le \pi\}$*, we have* $\mu_N(\delta) \le 1/(2(N+1)\sin^2\frac{\delta}{2}) \xrightarrow{N\to+\infty} 0.$

Proof. We have the following formula, which shows the 1st and 3rd claim,

$$\begin{aligned}
K_N(t) &= \frac{1}{N+1}\sum_{n=0}^{N}\frac{\sin((n+\frac{1}{2})t)\sin\frac{t}{2}}{2\sin^2\frac{t}{2}} \\
&= \frac{1}{N+1}\sum_{n=0}^{N}\frac{\cos(nt)-\cos((n+1)t)}{4\sin^2\frac{t}{2}} \\
&= \frac{1}{N+1}\frac{1-\cos((N+1)t)}{4\sin^2\frac{t}{2}} = \frac{2}{N+1}\left(\frac{\sin\left(\frac{(N+1)}{2}t\right)}{4\sin\frac{t}{2}}\right)^2.
\end{aligned}$$

Finally, the 2nd claim follows, using (7.12), from

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} K_N(t)dt &= \frac{1}{\pi}\frac{1}{N+1}\sum_{n=0}^{N}\int_{-\pi}^{\pi} D_n(t)dt \\
&= \frac{1}{\pi}\frac{1}{N+1}\sum_{n=0}^{N}\int_{-\pi}^{\pi}\left(\frac{1}{2}+\sum_{\ell=1}^{n}\cos(\ell t)\right)dt = \frac{1}{\pi}\frac{1}{N+1}\sum_{n=0}^{N}\int_{-\pi}^{\pi}\frac{1}{2}dt = 1.
\end{aligned}$$

□

Here notice that for

$$S_N f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{N}(\alpha_n\cos(nx) + \beta_n\sin(nx)) \tag{7.16}$$

then

$$\begin{aligned}
\sigma_0 f(x) &= \frac{\alpha_0}{2}\,,\ \sigma_1 f(x) = \frac{\alpha_0 + \alpha_1\cos(x) + \beta_1\sin(x)}{2} \\
\sigma_2 f(x) &= \frac{\frac{3}{2}\alpha_0 + 2\alpha_1\cos(x) + 2\beta_1\sin(x) + \alpha_2\cos(2x) + \beta_2\sin(2x)}{3} \\
&... \\
\sigma_N f(x) &= \frac{\frac{N+1}{2}\alpha_0 + N\alpha_1\cos(x) + N\beta_1\sin(x) + ... + \alpha_N\cos(Nx) + \beta_N\sin(Nx)}{N+1}
\end{aligned}$$

In particular

$$\sigma_N f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{N}\left(1 - \frac{n}{N+1}\right)(\alpha_n\cos(nx) + \beta_n\sin(nx)). \tag{7.17}$$

Lemma 7.24. *Given $f \in L^1(\mathbb{T})$ and a point x_0 where the two limits $f(x_0^{\pm})$ exist and are finite, then the sequence*

$$\sigma_N f(x) = \frac{2}{\pi(N+1)} \int_{-\pi}^{\pi} f(x+t) \left(\frac{\sin\left(\frac{(N+1)}{2}t\right)}{4\sin\frac{t}{2}} \right)^2 dt$$

converges in $x = x_0$ with

$$\lim_{n\to\infty} \sigma_n f(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2}. \tag{7.18}$$

Proof. We write

$$\begin{aligned}
&\sigma_n f(x_0) - \frac{f(x_0^+) + f(x_0^-)}{2} = \\
&= \frac{1}{2\pi(n+1)} \int_0^{\pi} (f(x_0+t) - f(x_0^+)) \left(\frac{\sin\left(\frac{(n+1)}{2}t\right)}{\sin\frac{t}{2}} \right)^2 dt \\
&+ \frac{1}{2\pi(n+1)} \int_0^{\pi} (f(x_0-t) - f(x_0^-)) \left(\frac{\sin\left(\frac{(n+1)}{2}t\right)}{\sin\frac{t}{2}} \right)^2 dt.
\end{aligned} \tag{7.19}$$

Now, considering for instance the 1st term in the r.h.s., we have

$$\begin{aligned}
&\frac{1}{2\pi(n+1)} \int_0^{\pi} (f(x_0+t) - f(x_0^+)) \left(\frac{\sin\left(\frac{(n+1)}{2}t\right)}{\sin\frac{t}{2}} \right)^2 dt \\
&= \frac{1}{2\pi(n+1)} \int_0^{\delta} (f(x_0+t) - f(x_0^+)) \left(\frac{\sin\left(\frac{(n+1)}{2}t\right)}{\sin\frac{t}{2}} \right)^2 dt \\
&+ \frac{1}{2\pi(n+1)} \int_{\delta}^{\pi} (f(x_0+t) - f(x_0^+)) \left(\frac{\sin\left(\frac{(n+1)}{2}t\right)}{\sin\frac{t}{2}} \right)^2 dt,
\end{aligned}$$

where the last line is in absolute value bounded above by

$$\frac{1}{\pi} \int_{\delta}^{\pi} \left(|f(x_0+t)| + |f(x_0^+)| \right) \mu_n(\delta) dt \le \frac{1}{\pi} \left(\|f\|_{L^1(\mathbb{T})} + \pi |f(x_0^+)| \right) \mu_n(\delta) \xrightarrow{n\to+\infty} 0.$$

So for $\delta > 0$ arbitrarily small and to be chosen, we look at the limit for $n \to \infty$ of

$$\begin{aligned}
&\frac{1}{2\pi(n+1)} \int_0^{\delta} (f(x_0+t) - f(x_0^+)) \left(\frac{\sin\left(\frac{(n+1)}{2}t\right)}{\sin\frac{t}{2}} \right)^2 dt \\
&+ \frac{1}{2\pi(n+1)} \int_0^{\delta} (f(x_0-t) - f(x_0^-)) \left(\frac{\sin\left(\frac{(n+1)}{2}t\right)}{\sin\frac{t}{2}} \right)^2 dt.
\end{aligned}$$

For $\delta > 0$ sufficiently small, $|f(x_0 \pm t) - f(x_0^{\pm})| < \epsilon$ for any preassigned $\epsilon > 0$ and using

$$8K_n(t) = \frac{1}{n+1}\left(\frac{\sin\left(\frac{(n+1)}{2}t\right)}{\sin\frac{t}{2}}\right)^2,$$

the absolute value of the previous formula is less than

$$\frac{\epsilon}{\pi(n+1)}\int_0^\delta \left(\frac{\sin\left(\frac{(n+1)}{2}t\right)}{\sin\frac{t}{2}}\right)^2 dt \le \frac{8\epsilon}{\pi}\int_{-\pi}^{\pi} K_n(t)dt = 8\epsilon.$$

So

$$\limsup_{n\to\infty}\left|\sigma_n f(x_0) - \frac{f(x_0^+) + f(x_0^-)}{2}\right| \le 8\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude with (7.18). □

Corollary 7.25. *For any $f \in C^0(\mathbb{T})$ and any point x_0 we have*

$$\lim_{n\to\infty} \sigma_n f(x_0) = f(x_0). \tag{7.20}$$

Furthermore, we have

$$\sigma_n f \xrightarrow{n\to+\infty} f \text{ in } C^0(\mathbb{T}). \tag{7.21}$$

Proof. The limit (7.20) follows immediately from Lemma 7.24. Let us now prove the uniform limit in (7.21). In analogy to (7.19)

$$\begin{aligned}
&\sigma_n f(x) - f(x) = \\
&= \frac{1}{2\pi(n+1)}\int_{|t|\le\delta}(f(x+t) - f(x))\left(\frac{\sin\left(\frac{(n+1)}{2}t\right)}{\sin\frac{t}{2}}\right)^2 dt \\
&+ \frac{1}{2\pi(n+1)}\int_{|t|\ge\delta}(f(x+t) - f(x))\left(\frac{\sin\left(\frac{(n+1)}{2}t\right)}{\sin\frac{t}{2}}\right)^2 dt =: I + II.
\end{aligned}$$

we have

$$\|II\|_{L^\infty(\mathbb{T})} \le \sup_{x\in\mathbb{T}} \frac{1}{\pi}\int_{|t|\ge\delta}(|f(x+t)| + |f(x)|)\,\mu_n(\delta)dt \le \frac{1}{\pi}\left(\|f\|_{L^1(\mathbb{T})} + 2\pi\|f\|_{L^\infty(\mathbb{T})}\right)\mu_n(\delta) \xrightarrow{n\to+\infty} 0.$$

On the other hand $f \in C^0(\mathbb{T})$ implies that $f : \mathbb{T} \to \mathbb{C}$ is uniformly continuous. So, for any $\epsilon > 0$ there exists $\delta > 0$ such that on any $I \subseteq \mathbb{T}$ with $\mathrm{diam}(I) \le \delta$ we have $\mathrm{osc}_I f < \epsilon$. Hence for such $\delta > 0$ we have

$$\|I\|_{L^\infty(\mathbb{T})} = \frac{1}{2\pi(n+1)} \sup_{x\in\mathbb{T}} \int_{|t|\le\delta} |f(x+t) - f(x)| \left(\frac{\sin\left(\frac{(n+1)}{2}t\right)}{\sin\frac{t}{2}} \right)^2 dt$$

$$\le \epsilon \frac{1}{2\pi(n+1)} \int_{|t|\le\delta} \left(\frac{\sin\left(\frac{(n+1)}{2}t\right)}{\sin\frac{t}{2}} \right)^2 dt \le \frac{4\epsilon}{\pi} \int_{-\pi}^{\pi} K_n(t)dt = 4\epsilon$$

like in the proof in Lemma 7.24. This completes the proof of (7.21).

□

Exercise 7.26. Show that it is false that

$$\sigma_n f \xrightarrow{n\to+\infty} f \text{ in } L^\infty(\mathbb{T}) \text{ for any } f \in L^\infty(\mathbb{T}). \tag{7.22}$$

Remark 7.27. We will return later to the phenomenon in Exercise 7.26. Notice that the operator defined by

$$e^{t\triangle} f(x) := (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)dy \text{ for } f \in L^p(\mathbb{R}^d) \tag{7.23}$$

solves the initial value problem

$$u_t - \triangle u = 0\,, \quad u|_{t=0} = f \tag{7.24}$$

for $1 \le p < \infty$ but not for $p = \infty$. We will prove

$$\lim_{t\searrow 0} e^{t\triangle} f = f \text{ in } L^p(\mathbb{R}^d) \tag{7.25}$$

for all $f \in L^p(\mathbb{R}^d)$ for $1 \le p < \infty$ and

$$\lim_{t\searrow 0} e^{t\triangle} f = f \text{ in } L^\infty(\mathbb{R}^d) \text{ for } f \in C_0^0(\mathbb{R}^d), \tag{7.26}$$

where

$$C_0^0(\mathbb{R}^d) := \{g \in C^0(\mathbb{R}^d) : \lim_{x\to\infty} g(x) = 0\}. \tag{7.27}$$

While we will discuss (7.25) and (7.26), we will not discuss the above PDE (this would require the Fourier Transform). Notice that it is not true, and we will discuss this, that

$$\lim_{t\searrow 0} e^{t\triangle} f = f \text{ in } L^\infty(\mathbb{R}^d) \text{ for any } f \in L^\infty(\mathbb{R}^d). \tag{7.28}$$

Remark 7.28. The operator defined by

$$e^{\mathrm{i}\triangle t} u_0(x) = (4\pi \mathrm{i} t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{\mathrm{i}|x-y|^2}{4t}} u_0(y)dy \text{ for } f \in L^p(\mathbb{R}^d) \text{ for } 1 \le p \le 2. \tag{7.29}$$

solves the linear Schrödinger equation

$$\mathrm{i}u_t + \triangle u = 0\,, u(0,x) = u_0(x). \tag{7.30}$$

8 Curiosities

8.1 Unbounded linear operators

Lemma 8.1. *Let X and Y be two normed spaces. There are linear maps $T : X \to Y$ which are unbounded.*

Before proving Lemma 8.1 let us recall the following notion.

Definition 8.2 (Hamel bases)**.** Let V be a vector space. A set $\{v_i\}_{i\in I}$ of elements of V is called a (Hamel) basis of V if for any $v \in V$ there is a unique finite subset $J \subseteq I$ and a unique family $\{\lambda_j\}_{j\in J}$ in $\mathbb{R}$ such that

$$v = \sum_{j\in J} \lambda_j v_j. \tag{8.1}$$

We say that a set $\{v_i\}_{i\in I}$ of elements of V is linearly independent if for any finite subset $J \subseteq I$

$$0 = \sum_{j\in J} \lambda_j v_j \Longrightarrow \lambda_j \equiv 0. \tag{8.2}$$

The span $\mathrm{Sp}\,(\{v_i\}_{i\in I})$ of a set $\{v_i\}_{i\in I}$ of elements of V is the set (it is a vector space) formed by the vectors v which can be expressed in the form (8.1) for finite $J \subseteq I$ and $\{\lambda_j\}_{j\in J}$ in $\mathbb{R}$.

We have the following.

Theorem 8.3. *Any vector space V has a Hamel basis.*

Proof. Let us denote by P the set of linearly independent subsets of V. It is endowed with the $\subseteq$ order relation. We claim thay P is *inductive*, that is, any totally ordered subset Q of P has an upper bound. Just take for $\mathrm{Q} = \{\mathcal{S}_q\}_{q\in \mathrm{Q}}$, the set $\widehat{\mathcal{S}} = \cup_{q\in\mathrm{Q}}\mathcal{S}_q$. If $\{v_j\}_{i\in J}$ is any finite subset of $\widehat{\mathcal{S}}$, by the total order, there must be a $\mathcal{S}_q$ containing $\{v_j\}_{i\in J}$. Then, since $\mathcal{S}_q$ is linearly independent, (8.2) is true. Since we have obtained that (8.2) is true for any finite subset $\{v_j\}_{i\in J}$ of $\widehat{\mathcal{S}}$, then $\widehat{\mathcal{S}}$ is linearly independent. Furthermore, any $\mathcal{R} \in \mathrm{P}$ with $\mathcal{S}_q \subseteq \mathcal{R}$ for all q, must satisfy $\widehat{\mathcal{S}} \subseteq \mathcal{R}$. So $\widehat{\mathcal{S}}$ is an upper bound. By Zorn's Lemma, there exists in P a maximal element $\{v_i\}_{i\in I}$. We claim this gives a basis. First of all, $\{v_i\}_{i\in I}$ is linearly independent.
If $\mathrm{Sp}\,(\{v_i\}_{i\in I}) = V$, then $\{v_i\}_{i\in I}$ is a Hamel basis. Suppose now that $\mathrm{Sp}\,(\{v_i\}_{i\in I}) = U \subsetneqq V$. Then there exists $v \in V\backslash U$ for which (8.1) is not true for all choices of J and $\{\lambda_j\}_{j\in J}$. This implies that $\{v_i\}_{i\in I} \cup \{v\} \supsetneqq \{v_i\}_{i\in I}$ is linear independent. This implies that $\{v_i\}_{i\in I}$ is not a maximal element of P, and we get a contradiction. So we must have $\mathrm{Sp}\,(\{v_i\}_{i\in I}) = V$. $\square$

Proof of Lemma 8.1. Let $\{x_i\}_{i\in I}$ be a Hamel basis of X and consider a family $\{y_i\}_{i\in I}$ in Y. There exists a linear unique map $T : X \to Y$ such that $f(x_i) = y_i$ for all $i \in I$. Let us take y_i so that $\sup\{\frac{\|y_i\|_Y}{\|x_i\|_X}\} = +\infty$. Then

$$\sup\{\frac{\|Tx\|_Y}{\|x\|_X}|x \in X\backslash\{0\}\} \geq \sup\{\frac{\|y_i\|_Y}{\|x_i\|_X}|i \in I\} = +\infty.$$

□

9 Open Mapping Theorem and Closed Graph Theorem

Theorem 9.1. *Let E and F be Banach spaces and consider a bounded linear map $T : E \to F$ which is onto. Then there is $c > 0$ such that*

$$T(D_E(0,1)) \supset D_F(0,c). \tag{9.1}$$

Proof. First we show there is $c > 0$ such that $\overline{T(D_E(0,1))} \supset D_F(0,2c)$. If this is not the case, then consider $X_n = n\,\overline{T(D_E(0,1))} = \overline{nT(D_E(0,1))} = \overline{T(nD_E(0,1))}$. Since $F = \cup X_n$, since by Theorem 7.6 the space F is a Baire space, some of the X_n must have nonempty interior. Since $X_1 = \frac{1}{n}X_n$, we conclude that X_1 has non empty interior. Then , there exists a disk

$$D_F(y_0,4c) \subset \overline{T(D_E(0,1))}. \tag{9.2}$$

Hence $y_0 \in \overline{T(D_E(0,1))}$ and by symmetry

$$-y_0 \in \overline{T(D_E(0,1))}. \tag{9.3}$$

Summing up (9.2)–(9.3), we obtain

$$D_F(0,4c) \subset \overline{T(D_E(0,1))} + \overline{T(D_E(0,1))} \subseteq 2\overline{T(D_E(0,1))}, \tag{9.4}$$

the latter inclusion by convexity of $\overline{T(D_E(0,1))}$.
Next we want to show (9.1).
Let $\|y\|_F < c$. We claim that

$$\text{there is } z_1 \in D_E(0,1/2) \text{ such that } \|y - Tz_1\|_F < c2^{-1}. \tag{9.5}$$

Indeed, $\overline{T(D_E(0,1))} \supset D_F(0,2c)$, which follows from (9.4), implies $\overline{T(D_E(0,1/2))} \supset D_F(0,c)$. This inclusion proves the claim, because either $y \in T(D_E(0,1/2))$ and so there is a $\|z_1\|_E < 2^{-1}$ with $y = Tz_1$, or y is in the closure of $T(D_E(0,1/2))$, and in this case for any $\epsilon > 0$ there is $z_1 \in D_E(0,1/2)$ with $\|y - Tz_1\|_F < \epsilon$. So we get the desired claim (9.5).
Suppose we have found $z_j \in D_E(0,2^{-j})$ for $j = 1,...,n$ so that $\|y - \sum_{j=1}^n Tz_j\|_F < c2^{-n}$. Then we claim that there exists $z_{n+1} \in D_E(0,2^{-n-1})$ so that $\|y - \sum_{j=1}^{n+1} Tz_j\|_F < c2^{-n-1}$. Indeed on one hand we have $(y - \sum_{j=1}^n Tz_j) \in D_F(0,2^{-n}c)$ and on the other $\overline{T(D_E(0,1))} \supset D_F(0,2c)$ implies $\overline{T(D_E(0,2^{-n-1}))} \supset D_F(0,2^{-n}c)$. We conclude like in the proof of (9.5).

Consider now $x = \sum_{j=1}^{\infty} z_j$. We have $\|x\|_E \le \sum_{j=1}^{\infty} \|z_j\|_E < \sum_{j=1}^{\infty} 2^{-j} = 1$, so $x \in D_E(0,1)$

On the other hand $\|Tx - y\|_F = \lim_{n\to+\infty} \|y - \sum_{j=1}^{n} Tz_j\|_F = 0$, so $y = Tx$. So we have proved (9.1). □

Corollary 9.2. *Let E and F Banach spaces and consider a bounded linear map $T : E \to F$ which is onto. Then T is open, that is, it sends open sets into open sets. Furthermore, if T is also one to one, then also T^{-1} is bounded.*

Proof. First we need to show that if $U \subseteq E$ is open, then $TU \subseteq F$ is open. Let $y_0 \in TU$ and $x_0 \in U$ with $y_0 = Tx_0$. Since U is open, there exists $r > 0$ such that $D_E(x_0, r) \subseteq U$. It then follows that $y_0 + TD_E(0, r) \subseteq TU$. By (9.1), we know that $TD_E(0, r) \supset D_F(0, c\, r)$. Then $y_0 + D_F(0, c\, r) = D_F(y_0, c\, r) \subseteq TU$. This proves that $TU \subseteq F$ is open.
We prove now the last sentence in the statement. We know that $T^{-1}F \to E$ exists and we have to prove that it is bounded. Form (9.1) we know that if $y = Tx \in D_F(0, c)$, then necessarily $x = T^{-1}y \in D_E(0, 1)$. So we conclude $\|T^{-1}y\|_E \leq \frac{1}{c}\|y\|_F$ for every $y \in F$ and T^{-1} is bounded. ☐

Example 9.3. In Sect. 9.1 we discuss the fact the map $\mathfrak{F} : f \to \widehat{f}(n)$ which sends $L^1(\mathbb{T})$ in $c_0(\mathbb{Z})$ is one to one but is not onto. Notice that the operator $L^1(\mathbb{T}) \xrightarrow{\mathfrak{F}} R(\mathfrak{F}) \subsetneqq c_0(\mathbb{Z})$ is one to one and onto on the image. Yet the inverse $R(\mathfrak{F}) \xrightarrow{\mathfrak{F}^{-1}} L^1(\mathbb{T})$ is unbounded. Indeed, if it were bounded, then $L^1(\mathbb{T}) \xrightarrow{\mathfrak{F}} R(\mathfrak{F})$ would be an isomorphism, and $R(\mathfrak{F})$ would be complete, but in fact it is not, since $\overline{R(\mathfrak{F})} = c_0(\mathbb{Z})$ inside $\ell^\infty(\mathbb{Z})$. So in other words, in Corollary 9.2 the hypothesis that F is a Banach space is essential, since otherwise the statement is false.

Example 9.4. Another example is the operator $L^p(\mathbb{R}^d) \ni f \xrightarrow{T} \langle x \rangle^{-1} f \in L^p(\mathbb{R}^d)$ where $\langle x \rangle = \sqrt{1 + |x|^2}$ is the *Japanese bracket*. This is obviously a bounded operator. Notice that the spectrum is $[0, 1]$ and is an obviously not invertible, since otherwise 0 would not be in the spectrum, or, more directly, since the inverse would be $f \to \langle x \rangle f$, which is clearly not a bounded operator in $L^p(\mathbb{R}^d)$. On the other hand $R(T) \supset C_c^\infty(\mathbb{R}^d)$, which is dense in $L^p(\mathbb{R}^d)$ for $p < \infty$. So again, $T^{-1} \notin \mathcal{L}(R(T), L^p(\mathbb{R}^d))$ since otherwise $R(T)$ would be a closed subspace of $L^p(\mathbb{R}^d)$.
Obviously this example can be replicated using any $\varphi \in C^0(\mathbb{R}^d, \mathbb{R}_+)$ with $\lim\limits_{x\to\infty} \varphi(x) = +\infty$ and $L^p(\mathbb{R}^d) \ni f \xrightarrow{T} (\varphi(x))^{-1} f \in L^p(\mathbb{R}^d)$.

9.1 An application of the Open Mapping Theorem to Fourier series

Theorem 9.5. *The map $f \to \widehat{f}(n)$ which sends $L^1(\mathbb{T})$ in $c_0(\mathbb{Z})$ is one to one but is not onto.*

Proof. Let us proceed by contradiction and let us suppose that the map is not one to one. Then there exists a nonzero $f \in L^1(-\pi, \pi)$ with $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$. In particular this implies that

$$\int_{-\pi}^{\pi} f(t)P(t)dt = 0 \tag{9.6}$$

for any trigonometric polynomial $P(t)$. Then we claim that (9.6) extends replacing P by any $g \in C^0(\mathbb{T})$. Indeed, we have

$$\int_{-\pi}^{\pi} f(t)\sigma_N g(t)dt = 0.$$

On the other hand, $f\sigma_N g \xrightarrow{n\to+\infty} fg$ in $L^1(\mathbb{T})$ since

$$\|fg - f\sigma_N g\|_{L^1(\mathbb{T})} \le \|f\|_{L^1(\mathbb{T})}\|g - \sigma_N g\|_{L^\infty(\mathbb{T})} \xrightarrow{n\to+\infty} 0,$$

by Corollary 7.25. This implies our claim:

$$0 = \lim_{N\to+\infty}\int_{-\pi}^{\pi} f(t)\sigma_N g(t)dt = \int_{-\pi}^{\pi} f(t)g(t)dt \text{ for any } g \in C^0(\mathbb{T}).$$

Now, for any interval $I \subseteq (-\pi,\pi)$ it is elementary to find a sequence $g_n \in C^0(\mathbb{T})$ with $|g_n(x)| \le 1$ everywhere and $\lim\limits_{n\to+\infty} g_n(x) = \chi_I(x)$ for any x. Indeed, if say $I = [a,b] \subset (-\pi,\pi)$, we can take

$$g_n(x) = \begin{cases} 1 \text{ if } x \in [a,b] \\ 0 \text{ if } x \notin \left[a-\frac{1}{n}, b+\frac{1}{n}\right] \\ n\left(x-a+\frac{1}{n}\right) \text{ if } x \in \left[a-\frac{1}{n}, a\right] \\ -n\left(x-b-\frac{1}{n}\right) \text{ if } x \in \left[b, b+\frac{1}{n}, a\right], \end{cases}$$

and we can deal similarly with other cases.
Then, by Dominated Convergence, we obtain

$$0 = \lim_{n\to+\infty}\int_{-\pi}^{\pi} f(t)g_n(t)dt = \int_{-\pi}^{\pi} f(t)\chi_I(t)dt \text{ for any interval } I \subseteq (-\pi,\pi).$$

This implies $f(t) = 0$ for a.a. t.

Having proved that our map is one to one, we show that it is not onto. Indeed, since it is a bounded map, by Corollary 9.2 if it is also onto, then it has bounded inverse. Then we would have $\|\widehat{f}\|_{c_0(\mathbb{Z})} \ge C\|f\|_{L^1(\mathbb{T})}$ for some fixed C. But then also $\|\widehat{D_n}\|_{c_0(\mathbb{Z})} \ge C\|D_n\|_{L^1(\mathbb{T})}$ which is impossible, since the left is 1 and the right goes to ∞. $\square$

Remark 9.6. Notice that in the above proof we exploited $\|\widehat{D_n}\|_{\ell^\infty(\mathbb{Z})} = 1$ and $\|D_n\|_{L^1(\mathbb{T})} \xrightarrow{n\to+\infty} +\infty$. The two quantities $\|\widehat{D_n}\|_{\ell^{p'}(\mathbb{Z})}$ and $\|D_n\|_{L^p(\mathbb{T})}$ are instead comparable for $1 < p \le 2$ and $p' = \dfrac{p}{p-1}$. It is possible to prove that

$$\text{for } 1 < p < 2 \text{ the map } L^p(\mathbb{T}) \ni f \to \{\widehat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^{p'}(\mathbb{Z}) \tag{9.7}$$

is bounded (it is an immediate consequence of Reistz interpolation theorem and the fact that the map is bounded for $p = 1$ and for $p = 2$). Injectivity is proved in the above theorem. Finally, for the fact that (9.7) is not an isomorphism, see later exercise 19.18.

Example 9.7. It can be shown that the trigonometric series $\sum_{n=2}^{\infty} \frac{\sin(nx)}{\log n}$ is not the Fourier series of an element in $L^1(\mathbb{T})$, while $\sum_{n=2}^{\infty} \frac{\cos(nx)}{\log n}$ is the Fourier series of an element $L^1(\mathbb{T})$.

9.2 The Closed Graph Theorem

Notice that if E and F are normed spaces, then $E \times F$ can be provided with the norm

$$\|(x, y)\| := \|x\|_E + \|y\|_F. \tag{9.8}$$

Exercise 9.8. Show that $E \times F$ is a Banach space if and only if both E and F are Banach spaces.

Theorem 9.9. *Let E and F be Banach spaces and consider a linear $T : E \to F$. If the graph $G(T)$ is closed in $E \times F$ then T is bounded.*

Proof. Being a closed subspace in the Banach space $E \times F$, $G(T)$ is also a Banach space. The projection $G(T) \to E$ is bounded, since $\|x\|_E \le \|x\|_E + \|Tx\|_F$, is one to one and is onto. Hence it is an isomorphism by Corollary 9.2. This means that $E \ni x \to (x, Tx) \in G(T)$ is a bounded map, and hence that there exists $C > 0$ such that $\|x\|_E + \|Tx\|_F \le C\|x\|_E$, and so $\|Tx\|_F \le (C - 1)\|x\|_E$ for any $x \in E$. This implies that T is bounded. $\square$

The following are two examples of linear operators $T : E \to F$ where E is not a Banach space, F is a Banach space, the graph $G(T)$ is closed in $E \times F$ but T is not a bounded operator.

Example 9.10. Let $F = \ell^1(\mathbb{N})$ with its own norm and $E := \{\mathbf{x} = (x_n) \in \ell^1(\mathbb{N}) : \sum_{n=1}^{\infty} n|x_n| < \infty\}$ with the norm of $\ell^1(\mathbb{N})$. Clearly, $E \subsetneq F$ is a dense subspace of F, since $E \supset H$, $H = \{\mathbf{x} = (x_n) \in \ell^1(\mathbb{N}) : x_n = 0$ except for finitely many $n's\}$, which is a dense subspace of $\ell^1(\mathbb{N})$.

Consider now the map $T : E \to F$ defined by $(T\mathbf{x})_n = nx_n$. It is clearly an unbounded map, since otherwise, from

$$T(0, \cdots, 0, \underbrace{1}_{n\text{-th position}}, 0, \cdots) = n(0, \cdots, 0, \underbrace{1}_{n\text{-th position}}, 0, \cdots),$$

we would get the absurd conclusion

$$\begin{aligned}
&\|T(0, \cdots, 0, \underbrace{1}_{n\text{-th position}}, 0, \cdots)\|_{\ell^1(\mathbb{N})} \\
&= n\|(0, \cdots, 0, \underbrace{1}_{n\text{-th position}}, 0, \cdots)\|_{\ell^1(\mathbb{N})} = n \xrightarrow{n \to +\infty} +\infty \le \|T\|_{E \to F} < +\infty.
\end{aligned}$$

Yet the graph $G(T) \subset E \times F$ is closed. Indeed, suppose that $\{(\mathbf{x}_n, T\mathbf{x}_n)\}_{n \in \mathbb{N}}$ has limit $(\mathbf{x}, \mathbf{y})$ in $E \times F$. Then $\mathbf{x}_n \xrightarrow{n \to +\infty} \mathbf{x}$ in $\ell^1(\mathbb{N})$ implies that for any $m \in \mathbb{N}$, we have $\mathbf{x}(m) =$

$\lim\limits_{n\to+\infty} \mathbf{x}_n(m)$. Similarly, $T\mathbf{x}_n \xrightarrow{n\to+\infty} \mathbf{y}$ in $\ell^1(\mathbb{N})$ implies that for any $m \in \mathbb{N}$, we have $\mathbf{y}(m) = \lim\limits_{n\to+\infty} T\mathbf{x}_n(m) = \lim\limits_{n\to+\infty} m\mathbf{x}_n(m) = m\mathbf{x}(m)$. This means that $\mathbf{y} = T\mathbf{x}$, and so that $G(T) \subset E \times F$ is closed.

The map $T : E \to F$ is clearly invertible, with inverse $T^{-1} : F \to E$ defined by $(T^{-1}\mathbf{x})_n = \dfrac{x_n}{n}$. Clearly this map is bounded. And yet T, as we saw above, is not bounded.

A more interesting but similar example, is the following one.

Example 9.11. Let $F = C^0([0,1])$ with the norm $L^\infty([0,1])$ and $E := C^1([0,1])$ with the norm as subspace of F. It has been already proved that $E \subsetneq F$ is a dense subspace of F (Exercise:why?).

Consider now the map $T : E \to F$ defined by $Tf = \dfrac{d}{dt}f$. It is clearly an unbounded map, since $\|Tt^n\|_{L^\infty([0,1])} = n\|t^{n-1}\|_{L^\infty([0,1])} = n$ and so, like above if it was bounded we would have

$$\|Tt^n\|_{L^\infty([0,1])} = n\|t^n\|_{L^\infty([0,1])} = n \xrightarrow{n\to+\infty} +\infty \leq \|T\|_{E\to F} < +\infty.$$

Yet the graph $G(T) \subset E \times F$ is closed. Indeed, if $\left(f_n, \dfrac{d}{dt}f_n\right) \xrightarrow{n\to+\infty} (f, g)$ in $E \times F$, notice that

$$f(t) = \lim_n f_n(t) = \lim f_n(0) + \lim_n \int_0^t Tf_n(s)ds = f(0) + \int_0^t g(s)ds$$

from which we conclude that $f \in C^1([0,1])$ with $\dfrac{d}{dt}f = g$.

Unlike in Example 9.10, here the map $T : E \to F$ is not invertible, since it is not one to one.

10 Projections and complementary subspaces

Definition 10.1. A vector subspace F of a topological vector space E is said *complementary* if it is closed and if there is a closed subspace G of E such that

$$E = F \oplus G, \tag{10.1}$$

that is, $E = F + G$ and $F \cap G = 0$.

Exercise 10.2. Let E be a topological vector space and $E = F \oplus G$ with F and G closed. Show that then E is isomorphic to the product $F \times G$.

In the next two lemmas, we consider to classes of examples.

Lemma 10.3. *Let E be Banach and let F be a subspace of finite dimension. Then F is complementary.*

Proof. Consider a basis f_1, ..., f_n of F and write $x \in F$ as $x = \sum_{j=1}^n x_j f_j$. This defines bounded operators $\phi_j : F \to K$ by $\phi_j x = x_j$ which we extend by Hahn Banach. Set $G := \cap_{j=1}^n \ker \phi_j$. Then $F \cap G = 0$ because if $x = \sum_{j=1}^n x_j f_j$ is such that $\phi_j x = x_j = 0$ then $x = 0$. Furthermore, given $z \in E$ with $\phi_j z = x_j$, set $x = \sum_{j=1}^n x_j f_j$. Then $\phi_j(z - x) = 0$ for all j and so $z = x + (z - x)$ with $x \in F$ and $(z - x) \in G$. □

Lemma 10.4. *Let E be Banach and let F be a closed subspace of finite codimension. Then F is complementary.*

Proof. The space E/F is a finite dimensional vector space and we can consider the projection $\pi : E \to E/F$. Consider elements g_1, ..., $g_n \in E$ which project into a basis of E/F. Then their span G is a closed complement of F. □

Definition 10.5. Given a topological vector space E, an operator $P \in \mathcal{L}(E)$ is a projection if $P^2 = P$.

Exercise 10.6. Show that if P is a projection, also $1 - P$ is a projection.

Exercise 10.7. Given $E = X \oplus Y$ with X and Y closed, show that the maps $P(x+y) := x$ and $Q(x + y) = y$ are projections.

Exercise 10.8. Given a topological vector space E and a closed vector subspace X, then X is complementary if and only if there exists a projection $P \in \mathcal{L}(E)$ such that $PE = X$.

Remark 10.9. It will be obvious, later, that if X is a Hilbert space and Y is a closed subspace of X, Y has a closed complement, thanks to the fact that there exists an orthogonal projection on Y. Remarkably, it can be proved that if X is a Banach space which is not topologically isomorphic to a Hilbert space, there exists in X a closed subspace not complementary. For example, $c_0(\mathbb{N})$ is not complementary in $\ell^\infty(\mathbb{N})$, see [1]. Similarly, $C^0_0(\mathbb{R})$ is not complementary in $L^\infty(\mathbb{R})$.

Lemma 10.10. *Let $T \in \mathcal{L}(E, F)$ be an onto bounded operator between two topological vector spaces. Then the following are equivalent:*

1 T has right inverse (that is, $S \in \mathcal{L}(F, E)$ with $T \circ S = \mathrm{Id}_F$).

2 $\ker T$ is complementary in E.

Proof. If we assume 1, then $S(F)$ is such that $E = \ker T + S(F)$ and $\ker T \cap S(F) = 0$. By $T \in \mathcal{L}(E, F)$ we conclude that $T : S(F) \to F$ is a bounded operator. By hypothesis, $S : F \to S(F) \subseteq E$ is a bounded operator. Since $T \circ S : F \to F$ is the identity, we conclude that $T : S(F) \to F$ is onto. We know from $\ker T \cap S(F) = 0$ that $T : S(F) \to F$ is one to one. So $S : F \to S(F)$ is the inverse of $T : S(F) \to F$. This implies that $T : S(F) \to F$ is an isomorphism between Banach spaces. In particular, $S(F)$ is closed in E, and so is a closed complement of $\ker T$ in E.

If we assume 2, let $E = \ker T \oplus G$. Then $T(E) = T(G) = F$, and $T : G \to F$ is an isomorphism. So there is an inverse. □

Exercise 10.11. Let E be a Banach space which is not topologically isomorphic to a Hilbert space, and let F be a closed subspace of E which is not complementary. Show that the immersion $j : F \hookrightarrow E$ cannot be extended into a bounded operator $E \to E$.

Some of the most important projections come up when dealing with the spectrum.

Exercise 10.12 (Spectral projections)**.** Let X be a Banach space on $\mathbb{C}$, let $A \in \mathcal{L}(X)$, and let γ an counterclockwise oriented closed path which is topologically a circle inside the resolvent set $\rho(A)$. Show that

$$P := -\frac{1}{2\pi \mathrm{i}} \int_\gamma R_A(z) dz, \tag{10.2}$$

is a projection. In particular, show that if $\sigma(A)$ is wholly contained inside the bounded region delimited by γ, then P is the identity operator (this has already been shown in Example 5.40).

Answer. We can represent P also using a different path σ, fully contained in the region enclosed by γ. Then

$$P^2 = \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_\sigma \int_\gamma R_A(z') R_A(z) dz dz'.$$

Now notice that we have the important resolvent identity

$$R_A(z') R_A(z) = (z' - z)^{-1} \left(R_A(z') - R_A(z)\right).$$

So, inserting this in the previous formula, we get

$$\begin{aligned} P^2 &= -\frac{1}{2\pi \mathrm{i}} \int_\sigma dz' R_A(z') \frac{1}{2\pi \mathrm{i}} \int_\gamma (z - z')^{-1} dz + \frac{1}{2\pi \mathrm{i}} \int_\gamma dz R_A(z) \frac{1}{2\pi \mathrm{i}} \int_\sigma (z' - z)^{-1} dz' \\ &= -\frac{1}{2\pi \mathrm{i}} \int_\sigma R_A(z') \mathrm{Ind}(\gamma, z') dz' + \frac{1}{2\pi \mathrm{i}} \int_\gamma R_A(z) \mathrm{Ind}(\sigma, z) dz. \end{aligned}$$

Since each $z \in \gamma$ is in the outer component in the complement of the path σ, we have $\mathrm{Ind}(\sigma, z) \equiv 0$. Since each $z' \in \sigma$ is in the inner component in the complement of the path γ, we have $\mathrm{Ind}(\gamma, z') dz' \equiv 1$. So

$$P^2 = -\frac{1}{2\pi \mathrm{i}} \int_\sigma R_A(z') dz' = P. \tag{10.3}$$

Since the operator in (10.2) is in $\mathcal{L}(X)$, we conclude that (10.2)–(10.3) imply that P is a projection. $\square$

Exercise 10.13. Let X be a Banach space on $\mathbb{C}$, let $A \in \mathcal{L}(X)$ and $z \in \rho(A)$. Show the commutation formula $[A, R_A(z)] = 0$.
Then, for γ a closed path in $\rho(A)$ and for P defined by (10.2) show the commutation formula $[A, P] = 0$.

Answer. We have $(A-z)A = A(A-z)$. So, applying $R_A(z)$ both on the right and on the left, we have

$$R_A(z)(A-z)AR_A(z) = R_A(z)A(A-z)R_A(z) \Longrightarrow AR_A(z) = R_A(z)A$$

that is $[A, R_A(z)] = 0$. Next, $[A, P] = 0$ follows by

$$AP = -\frac{1}{2\pi \mathrm{i}} A \int_\gamma R_A(z)dz = -\frac{1}{2\pi \mathrm{i}} \int_\gamma AR_A(z)dz = -\frac{1}{2\pi \mathrm{i}} \int_\gamma R_A(z)Adz - \frac{1}{2\pi \mathrm{i}} \int_\gamma R_A(z)dzA = PA.$$

$\square$

Example 10.14. Suppose $\sigma(A) = \Sigma_1 \cup \Sigma_2$, where Σ_1 is wholly contained inside the bounded region delimited by γ of Exercise 10.12, while Σ_2 is in the unbounded region Then

$$A = PA + (1-P)A, \tag{10.4}$$

with $\sigma(PA) = \Sigma_1$ and $\sigma((1-P)A) = \Sigma_2$. Finally, the splitting

$$E = \ker P \oplus R(P), \tag{10.5}$$

where $R(P) = PX$, is left invariant by A by Exercise 10.13.

Notice that, by iterating as much as possible (10.4)–(10.5), one gets the *spectral decomposition* of A, which is akin to the decomposition (modulo conjugation) in Jordan blocks of a matrix.

Let us show that restricting A to $F := R(P)$ we have $\Sigma_2 \subseteq \rho(A)$. Recall that, the P in (10.2) reduces to the identity in operator in F. But then, like in the discussion in 5.40 it is possible to show that we can define like in (5.28) the operator

$$T := -\frac{1}{2\pi \mathrm{i}} \int_\gamma (z-\lambda)^{-1} R_A(z)dz \text{ for any } \lambda \in \Sigma_2. \tag{10.6}$$

Then we claim that $T = (A-\lambda)^{-1} \in \mathcal{L}(F)$ for any $\lambda \in \Sigma_2$. Indeed, for σ like above

$$\begin{aligned}
&(A-\lambda)T = \frac{1}{2\pi \mathrm{i}} \int_\sigma (z'-\lambda)R_A(z')dz' \frac{1}{2\pi \mathrm{i}} \int_\gamma (z-\lambda)^{-1} R_A(z)dz \\
&= \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_\sigma \int_\gamma (z'-\lambda)R_A(z')(z-\lambda)^{-1}R_A(z)dzdz' \\
&= -\frac{1}{2\pi \mathrm{i}} \int_\sigma dz'(z'-\lambda)R_A(z') \frac{1}{2\pi \mathrm{i}} \int_\gamma (z-\lambda)^{-1}(z-z')^{-1}dz \\
&+ \frac{1}{2\pi \mathrm{i}} \int_\gamma dz(z-\lambda)^{-1}R_A(z) \frac{1}{2\pi \mathrm{i}} \int_\sigma (z'-\lambda)(z'-z)^{-1}dz' \\
&= -\frac{1}{2\pi \mathrm{i}} \int_\sigma (z'-\lambda)R_A(z')\mathrm{Ind}(\gamma, z')(z'-\lambda)^{-1}dz' + \frac{1}{2\pi \mathrm{i}} \int_\gamma (z-\lambda)^{-1}R_A(z)\mathrm{Ind}(\sigma, z)(z-\lambda)dz.
\end{aligned}$$

Now, like before $\operatorname{Ind}(\sigma, z) \equiv 0$ for $z \in \gamma$ and $\operatorname{Ind}(\gamma, z')dz' \equiv 1$ for $z' \in \sigma$. So, in F

$$(A-\lambda)T = -\frac{1}{2\pi \mathrm{i}} \int_\sigma R_A(z')dz' = 1$$

Since also, by the commutations in Exercise 10.13, we have $T(A-\lambda) = (A-\lambda)T = 1$, our claim is proved.

Example 10.15 (Leray projector). One of the most famous projections in the theory of Partial Differential Equations is the Leray projector. If $L^2(\mathbb{T}^d, \mathbb{R}^d)$ are the L^2 vector fields on $\mathbb{T}^d$, and if $H(\mathbb{T}^d, \mathbb{R}^d)$ are the L^2 vector fields with 0 divergence, that is they satisfy (recall from (7.8)that $\widehat{\nabla \cdot u}(\mathbf{n}) = \mathrm{i}\mathbf{n} \cdot \widehat{u}(\mathbf{n})$)

$$\sum_{j=1}^d n_j \widehat{u}_j(\mathbf{n}) = 0, \text{ where } \mathbf{n} = (n_1, ..., n_d),$$

then $\mathbb{P} : L^2(\mathbb{T}^d, \mathbb{R}^d) \to L^2(\mathbb{T}^d, \mathbb{R}^d)$ is the orthogonal projection on $H(\mathbb{T}^d, \mathbb{R}^d)$ and is defined by

$$\widehat{(\mathbb{P}u)}^j(\mathbf{n}) = \begin{cases} \widehat{u}^j(\mathbf{0}) \text{ if } \mathbf{n} = \mathbf{0} \\ \widehat{u}^j(\mathbf{n}) - \frac{1}{\|\mathbf{n}\|^2_{\mathbb{R}^d}} \sum_{k=1}^d n_j n_k \widehat{u}^k(\mathbf{n}) \text{ if } \mathbf{n} \neq \mathbf{0}. \end{cases} \tag{10.7}$$

There is a version with $\mathbb{T}^d$ replaced by $\mathbb{R}^d$.

Exercise 10.16. a Check that $\mathbb{P}$ is indeed the orthogonal projection of $L^2(\mathbb{T}^d, \mathbb{R}^d)$ on $H(\mathbb{T}^d, \mathbb{R}^d)$.

b Check that $\ker \mathbb{P}$ is formed by the conservative fields in $L^2(\mathbb{T}^d, \mathbb{R}^d)$.

Exercise 10.17. Let X be a topological vector space and $P \in \mathcal{L}(X)$ a projection. Show that $\sigma(P) \subseteq \{0, 1\}$ and that $X = \ker(P) \oplus R(P)$ with $P = 0 \oplus 1$ is its spectral decomposition.

11 Weak $\sigma(E, E')$ topology

Definition 11.1. Given a topological vector space E, we consider the (*weak*) $\sigma(E, E')$ *topology*, that is the topology which has as subbasis of seminorms the family $\{|f|\}_{f \in E'}$.

Exercise 11.2. Show that for any $x_0 \in E$ a basis of neighborhoods of x_0 for the $\sigma(E, E')$ topology is of the form

$$V_{x_0}(f_1, ..., f_n, \epsilon) := \{x : |f_j(x - x_0)| < \epsilon \text{ for } j = 1, ..., n\} \text{ where } \tag{11.1}$$
$$n \in \mathbb{N},\ f_1, ..., f_n \in E' \text{ and } \epsilon > 0.$$

Exercise 11.3. Consider an infinite dimensional normed space E, and suppose that there exists $X \subseteq E'$ countable and dense in E' (i.e. E' is *separable*, c.f. below). Is the topology on E which has as subbasis of seminorms the family $\{|f|\}_{f \in X}$ the same as the $\sigma(E, E')$?

Answer. No, because if yes, then E with the (*weak*) $\sigma(E,E')$ topology would be metrizable by Exercise 4.25. But, by Corollary 11.13 below, this is not true. $\square$

Exercise 11.4. Show that the $\sigma(E,E')$ topology is the weakest topology on E such that all the linear functionals $f \in E'$ are continuous functions.

Exercise 11.5. Show that if E is a topological vector space on $\mathbb{C}$, the two weak $\sigma(E,E')$ topologies, one from linear functionals on $\mathbb{R}$ and the other from linear functionals on $\mathbb{C}$, coincide.

Lemma 11.6. *If E is a locally convex space, then is Hausdorff for the weak $\sigma(E,E')$ topology.*

Proof. We consider first the case $K=\mathbb{R}$. Let $x_0 \neq x_1$ in E. Then we can apply the 2nd geometric form of Hahn–Banach Theorem 6.16 and conclude that there exists $f \in E'$ and $\alpha \in \mathbb{R}$ such that $f(x_0) < \alpha < f(x_1)$. Then $f^{-1}(-\infty,\alpha)$ is an open neighborhood of x_0 and $f^{-1}(\alpha,+\infty)$ is an open neighborhood of x_1 for the weak $\sigma(E,E')$ topology and these two open sets are disjoint. $\square$

Notation 11.7. When a sequence $\{x_n\}$ in X converges to x in a weak topology we will write $x_n \rightharpoonup x$.

Lemma 11.8. *Let E be a topological vector space and let x_n be a sequence in E. Then:*

1 $x_n \rightharpoonup x$ for $\sigma(E,E')$ if and only if $f(x_n) \to f(x)$ for any $f \in E'$.

2 If $x_n \to x$ strongly, then $x_n \rightharpoonup x$ for $\sigma(E,E')$.

Suppose now that E is a normed space.

3 If $x_n \rightharpoonup x$ for $\sigma(E,E')$ then $\{\|x_n\|_E\}$ is bounded and $\|x\|_E \le \liminf \|x_n\|_E$.

4 If $x_n \rightharpoonup x$ for $\sigma(E,E')$ and if $f_n \to f$ in norm in E', then $f_n(x_n) \to f(x)$.

Proof. We prove only 3. For any $f \in E'$ we know that $f(x_n) \to f(x)$ and so that $\{f(x_n)\}$ is bounded. If this holds for any $f \in E'$, this implies by Banach Steinhaus that $\{\|x_n\|_E\}$ is bounded. Next,

$$|f(x)| = \lim_{n\to+\infty} |f(x_n)| = \lim_{k\to+\infty} |f(x_{n_k})|$$

for any subsequence $\{n_k\}$. If we take this subsequence so that $\|x_{n_k}\|_E \xrightarrow{k\to+\infty} \liminf\limits_{n\to+\infty} \|x_n\|_E$, we conclude

$$|f(x)| = \lim_{k\to+\infty} |f(x_{n_k})| \le \|f\|_{E'} \lim_{k\to+\infty} \|x_{n_k}\|_{E'},$$

and so $\|x\|_E \le \liminf\limits_{n\to+\infty} \|x_n\|_E$.

$\square$

Exercise 11.9. Prove that if E is finite dimensional, then the strong topology and the $\sigma(E, E')$ topology coincide.

Theorem 11.10. *Let E be a locally convex topological vector space and consider a convex set $C \subset E$. Then C is closed for the $\sigma(E, E')$ topology if and only if it is closed for the strong topology.*

Proof. Suppose C is strongly closed. Consider $x_0 \in \complement C$. By the 2nd geometric form of Hahn–Banach Theorem 6.16, there is $f \in E'$ and $\alpha \in \mathbb{R}$ with

$$f(x_0) < \alpha < f(x) \text{ for all } x \in C. \tag{11.2}$$

On the other hand

$$V = \{y : f(y) < \alpha\}$$

is an open set for the $\sigma(E, E')$ topology containing x_0, and so in particular it is an open neighborhood of x_0 for the $\sigma(E, E')$ topology. Since by (11.2) we have $V \subseteq \complement C$, we conclude that any point $x_0 \in \complement C$ is an interior point of $\complement C$ for the $\sigma(E, E')$ topology. So $\complement C$ is open for the $\sigma(E, E')$ topology and, hence, C is closed for the $\sigma(E, E')$ topology.

If C is closed for the $\sigma(E, E')$ topology, it is closed also for the, stronger, strong topology. $\square$

Remark 11.11. There is no analogue saying that a convex $C \subset E'$ closed for the strong topology in E' is closed also for the $\sigma(E', E)$ topology introduced in Sect. 12 below. One example is $c_0(\mathbb{N})$, which is a closed vector subspace in $\ell^\infty(\mathbb{N})$ for the strong topology but not for the weak $\sigma(\ell^\infty(\mathbb{N}), \ell^1(\mathbb{N}))$ topology. See Example 12.6 below for the reason.

Lemma 11.12. *Let E be an infinite dimensional Banach space and let U be an open subset for the $\sigma(E, E')$ topology. Then U contains a line.*

Proof. Let $x_0 \in U$. Then U contains a neighborhood of x_0 for the $\sigma(E, E')$ *topology* of the form

$$V := \{x : |f_j(x - x_0)| < \epsilon, \, j = 1, ..., n\} \text{ for some } f_1,...,f_n \in E'. \tag{11.3}$$

Notice that, for $\mathbf{f} : E \to \mathbb{R}^n$ defined by $\mathbf{f}(x) = (f_1(x), ..., f_n(x))$, $\ker \mathbf{f}$ has finite codimension. Since E is infinite dimensional, this means that $\ker \mathbf{f}$ has infinite dimension, and so in particular it contains a line. Finally, $x_0 + \ker \mathbf{f} \subseteq V \subseteq U$. $\square$

Corollary 11.13. *Let E be an infinite dimensional normed space. It is not metrizable for the $\sigma(E, E')$ topology.*

Proof. Suppose by contradiction that there is a metric d and consider the balls $U_n = \{x : d(x, 0) < 1/n\}$. Then, since each U_n is open, it contains a line, and in particular there exists $x_n \in U_n$ with $\|x_n\|_E = n$. Then obviously $x_n \xrightarrow{n \to +\infty} 0$ in (E, d), that is $x_n \rightharpoonup 0$ in the $\sigma(E, E')$ *topology.* But $\|x_n\|_E = n \xrightarrow{n \to +\infty} +\infty$, contradicting Lemma 11.8. $\square$

Lemma 11.14. *If E is an infinite dimension normed space, then the unitary sphere $S = \{x : \|x\|_E = 1\}$ is not closed for the $\sigma(E, E')$ topology, and its closure is $\overline{D_E(0,1)} = \{x : \|x\|_E \le 1\}$.*

Proof. Let $\|x_0\|_E < 1$ and consider a neighborhood V of x_0 of the form (11.3). Let now $y_0 \neq 0$ with $f_j(y_0) = 0$ for all j. Consider $g(t) := \|x_0 + ty_0\|_E$. We have $g(0) < 1$, $\lim_{t\to\infty} g(t) = +\infty$ and so there is $t_0 > 0$ so that $g(t_0) = \|x_0 + t_0 y_0\|_E = 1$. Notice now that $x_0 + ty_0 \in V$ for any t. Hence we see that $S \cap V \neq \emptyset$ for any V's of the form (11.3) Since there is a basis of neighborhoods of x_0 for the $\sigma(E, E')$ topology of the form (11.3), we conclude that any $x_0 \in D_E(0,1)$ is an accumulation point for S for the $\sigma(E, E')$ topology. Then the closure of S for the $\sigma(E, E')$ topology contains $D_E(0,1)$ and the closure $\left.\overline{D_E(0,1)}\right|_{\sigma(E,E')}$ of $D_E(0,1)$ for the $\sigma(E, E')$ topology. The latter is a closed set also for the strong topology of E (all the closed sets for the $\sigma(E, E')$ topology are also closed sets for the strong topology). Then $\left.\overline{D_E(0,1)}\right|_{\sigma(E,E')} \supseteq \overline{D_E(0,1)}$. On the other hand $\overline{D_E(0,1)}$ is closed for the $\sigma(E, E')$ topology, see in Theorem 11.10 above. Hence we have proved that the closure of S for the $\sigma(E, E')$ topology coincides with $\overline{D_E(0,1)}$. □

Remark 11.15. $D_E(0,1)$ has empty interior in the $\sigma(E, E')$ topology, in the infinite dimensional case. Indeed, if $\widetilde{V}$ is an open set for the $\sigma(E, E')$ topology contained in $D_E(0,1)$, it contains an open set V for the $\sigma(E, E')$ topology of the form (11.3) which contains a line. Hence, for no such V we can have $V \subseteq D_E(0,1)$.

Exercise 11.16. Consider a normed space E, and suppose that there exists $X \subseteq E'$ countable and dense in E' and consider the topology τ on E which has as subbasis of seminorms the family $\{|f|\}_{f\in X}$. Show that the topology induced on $D_E(0, R)$ and on $\overline{D_E(0, R)}$ for any $R > 0$ by (E, τ) coincides with the topology induced by the $\sigma(E, E')$ topology. Prove that $D_E(0, R)$ and $\overline{D_E(0, R)}$ with the $\sigma(E, E')$ topology are metrizable.

Example 11.17. While Lemma 11.14 might seem surprising, in fact it is quite natural. To see this consider $f \in L^p(\mathbb{R}^d)$ for $1 < p < \infty$ with $\|f\|_{L^p} = 1$ and let $\{x_n\}$ a sequence in $\mathbb{R}^d$ divergent to infinity. Then obviously $\|f(\cdot - x_n)\|_{L^p} = 1$. We claim that $f(\cdot - x_n) \rightharpoonup 0$ for $\sigma(L^p, (L^p)')$. We will see later that $(L^p(\mathbb{R}^d))' = L^{p'}(\mathbb{R}^d)$. Then our claim is equivalent to the following statement,

$$\langle f(\cdot - x_n), g\rangle \xrightarrow{n\to+\infty} 0 \text{ for all } g \in L^{p'}(\mathbb{R}^d). \tag{11.4}$$

To prove (11.4) suppose that $\Omega_0 := \operatorname{supp} f$ and $\Omega_1 := \operatorname{supp} g$ are both compact. Then $\operatorname{supp} f(\cdot - x_n) = x_n + \Omega_0$ and, since $\{x_n\}$ is divergent to infinity, then there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ we have $(x_n + \Omega_0) \cap \Omega_1 = \emptyset$. But then we conclude that

$$\langle f(\cdot - x_n), g\rangle = 0 \text{ for } n > n_0. \tag{11.5}$$

Now let us assume that f and g are not of compact support. Nonetheless, we will see later that there exist $\widetilde{f}$ and $\widetilde{g}$ in $C_c^\infty(\mathbb{R}^d)$ with

$$\|f - \widetilde{f}\|_{L^p} < \epsilon \text{ and } \|g - \widetilde{g}\|_{L^{p'}} < \epsilon.$$

Then, from

$$\begin{aligned}
&\langle f(\cdot - x_n), g\rangle \\
&= \left\langle \widetilde{f}(\cdot - x_n), \widetilde{g}\right\rangle + \left\langle f(\cdot - x_n) - \widetilde{f}(\cdot - x_n), \widetilde{g} - g\right\rangle + \left\langle f(\cdot - x_n) - \widetilde{f}(\cdot - x_n), g\right\rangle + \langle f(\cdot - x_n), g - \widetilde{g}\rangle
\end{aligned}$$

we obtain

$$\begin{aligned}
&|\langle f(\cdot - x_n), g\rangle| \le |\left\langle \widetilde{f}(\cdot - x_n), \widetilde{g}\right\rangle| + \|f(\cdot - x_n) - \widetilde{f}(\cdot - x_n)\|_{L^p}\|g - \widetilde{g}\|_{L^{p'}} \\
&+ \|f(\cdot - x_n) - \widetilde{f}(\cdot - x_n)\|_{L^p}\|g\|_{L^{p'}} + \|f(\cdot - x_n)\|_{L^p}\|g - \widetilde{g}\|_{L^{p'}} \\
&\le |\left\langle \widetilde{f}(\cdot - x_n), \widetilde{g}\right\rangle| + \epsilon^2 + \epsilon\|g\|_{L^{p'}} + \|f\|_{L^p}\epsilon.
\end{aligned}$$

By the previous argument there exists $n_\epsilon \in \mathbb{N}$ such that for $n > n_\epsilon$ we have $|\left\langle \widetilde{f}(\cdot - x_n), \widetilde{g}\right\rangle| = 0$ we conclude

$$|\langle f(\cdot - x_n), g\rangle| \le \epsilon^2 + \epsilon\|g\|_{L^{p'}} + \|f\|_{L^p}\epsilon \text{ for } n > n_\epsilon.$$

By the arbitrariness of $\epsilon > 0$, this implies (11.4).

Exercise 11.18. What can be said of $\{f(\cdot - x_n)\}_{n\in\mathbb{N}}$ for $x_n \xrightarrow{n\to+\infty} +\infty$ in $L^1(\mathbb{R}^d)$ if $f \not\equiv 0$?

Answer. Then we cannot say that $f(\cdot - x_n) \rightharpoonup 0$. If for example $\int_{\mathbb{R}^d} f(x)dx \neq 0$, then $\langle f(\cdot - x_n), 1\rangle_{L^1\times L^\infty} = \int_{\mathbb{R}^d} f(x)dx$ is incompatible with $f(\cdot - x_n) \rightharpoonup 0$. On the other hand, since for any $\varphi \in C^0_c(\mathbb{R}^d)$ we have $\langle f(\cdot - x_n), \varphi\rangle_{L^1\times L^\infty} \xrightarrow{n\to+\infty} 0$, we cannot have $f(\cdot - x_n) \rightharpoonup g$ for some $g \in L^1(\mathbb{R}^d)$ different from 0.

If $\int_{\mathbb{R}^d} f(x)dx = 0$ it is not restrictive to assume that in a disk D we have $a := \int_D f(x)dx > 0$. Let now

$$X := \bigcup_{n=1}^{\infty} (D + x_n)$$

By taking a subsequence, we can assume that the $\{D + x_n\}$ are disjoint and that

$$\|f(\cdot - (x_n - x_j)\|_{L^1(D)} < 2^{-j-1}a \text{ for all } j \neq n.$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^d} f(x - x_n)\, 1_X(x)dx = \int_D f(x)dx + \sum_{j\ge1, j\neq n} \int_{\mathbb{R}^d} f(x - x_n)\, 1_D(x - x_j)dx \\
&= a - \sum_{j\ge1, j\neq n} \|f(\cdot - (x_n - x_j)\|_{L^1(D)} > a - \sum_{j\ge1, j\neq n} 2^{-j-1}a = 2^{-1}a > 0.
\end{aligned}$$

□

Remark 11.19. Notice that by the Theorem 13.1 and by Theorems 16.5–16.6, for $1 < p < \infty$ the closed ball $\overline{D_{L^p(\mathbb{R}^d)}(0,1)}$, is compact for the $\sigma(L^p(\mathbb{R}^d),(L^p(\mathbb{R}^d))')$ topology which, by Exercise 14.6 and the fact that the $(L^p(\mathbb{R}^d))' = L^{p'}(\mathbb{R}^d)$ are separable, is a compact metric space. We know that given any sequence in a compact metric space, we can extract a convergent subsequence.

The closed ball $\overline{D_{L^1(\mathbb{R}^d)}(0,1)}$, on the other hand, is neither compact, see Theorem 13.1, nor metrizable for the $\sigma(L^1(\mathbb{R}^d),L^\infty(\mathbb{R}^d))$ topology. So, given a sequence in $\overline{D_{L^1(\mathbb{R}^d)}(0,1)}$, we cannot conclude that it has a convergent subsequence for the $\sigma(L^1(\mathbb{R}^d),L^\infty(\mathbb{R}^d)))$ topology. See also Remark 11.29 for a simple bounded sequence in $\ell^1(\mathbb{N})$ which does not have convergent subsequences in the $\sigma(\ell^1(\mathbb{N}),\ell^\infty(\mathbb{N}))$ topology.

Example 11.20. Brezis [3, Exercise 4.38] considers the case of the sequence

$$u_n := n\sum_{j=0}^{n-1}\chi_{\left[\frac{j}{n},\frac{j}{n}+\frac{1}{n^2}\right]}.$$

First of all, it discusses the fact that

$$\lim_{n\to+\infty}\int_0^1 u_n f dx = \int_0^1 f dx \text{ for all } f\in C^0([0,1]). \tag{11.6}$$

This is easy to see, because

$$\int_0^1 u_n f dx = n\sum_{j=0}^{n-1}\int_{\frac{j}{n}}^{\frac{j}{n}+\frac{1}{n^2}} f\left(\frac{j}{n}\right)dx + n\sum_{j=0}^{n-1}\int_{\frac{j}{n}}^{\frac{j}{n}+\frac{1}{n^2}}\left(f(x)-f\left(\frac{j}{n}\right)\right)dx.$$

Now, by the fact that $f\in C^0([0,1])$, it is Riemann–integrable, and so

$$n\sum_{j=0}^{n-1}\int_{\frac{j}{n}}^{\frac{j}{n}+\frac{1}{n^2}} f\left(\frac{j}{n}\right)dx = \sum_{j=0}^{n-1}\frac{1}{n}f\left(\frac{j}{n}\right) \xrightarrow{n\to+\infty} \int_0^1 f dx.$$

On the other hand, f is uniformly continuous. So for any $\epsilon > 0$ there exists $\delta > 0$ such that for any interval $I\subset[0,1]$ with $|I| < \delta$ we have $\operatorname{osc}_I f < \epsilon$. So in particular, if $n^2 > 1/\delta$ we have

$$n\sum_{j=0}^{n-1}\int_{\frac{j}{n}}^{\frac{j}{n}+\frac{1}{n^2}}\left|f(x)-f\left(\frac{j}{n}\right)\right|dx \le \epsilon\sum_{j=0}^{n-1}\frac{1}{n} = \epsilon.$$

This proves (11.6).

If now there exists a subsequence with u_{n_k} weakly convergent to some $u\in L^1(0,1)$, from (11.6) it must be $u=1$. On the other hand $u_{n_k}\not\rightharpoonup 1$. To see this notice that $|\text{supp } u_n| = \frac{1}{n}$. Choosing a further subsequence, we can assume that $\sum_k |\text{supp } u_{n_k}| < 1$. So $\bigcup_k \text{supp } u_{n_k} \subsetneq (0,1)$. So, if say $f(x) = \begin{cases}1 \text{ if } x\notin\bigcup_k \text{supp } u_{n_k}\\ 0 \text{ if } x\in\bigcup_k \text{supp } u_{n_k}\end{cases}$ we have $\langle u_{n_k}, f\rangle = 0$ for all k while $\langle 1, f\rangle > 0$.

Example 11.21. Let now $f_0, f \in D_{L^p(\mathbb{R}^d)}(0,1)$ with compact support and with $\|f_0\|_{L^p}^p + \|f\|_{L^p}^p = 1$. Then, since for $n \gg 1$ the supports of f_0 and $f(\cdot - x_n)$ are disjoint, for $n \gg 1$

$$\|f_0 + f(\cdot - x_n)\|_{L^p} = \left(\int_{\mathbb{R}^d} |f_0(x) + f(x - x_n)|^p dx \right)^{\frac{1}{p}}$$
$$= \left(\int_{\mathbb{R}^d} |f_0(x)|^p dx + \int_{\mathbb{R}^d} |f(x - x_n)|^p dx \right)^{\frac{1}{p}} = \sqrt[p]{\|f_0\|_{L^p}^p + \|f\|_{L^p}^p} = 1$$

while $f_0 + f(\cdot - x_n) \rightharpoonup f_0$ for $\sigma(L^p, (L^p)')$.

Example 11.22. More generally, for $f_0, f \in D_{L^p(\mathbb{R}^d)}(0,1)$ with $\|f_0\|_{L^p}^p + \|f\|_{L^p}^p = 1$ and supp f compact we claim

$$\lim_{n\to+\infty} \|f_0 + f(\cdot - x_n)\|_{L^p} = \sqrt[p]{\|f_0\|_{L^p}^p + \|f\|_{L^p}^p} = 1 \tag{11.7}$$

Indeed, for $\epsilon > 0$ let $g_\epsilon \in C_c^\infty(\mathbb{R}^d)$ with $\|f_0 - g_\epsilon\|_{L^p} < \epsilon$. Then

$$\|g_\epsilon + f(\cdot - x_n)\|_{L^p} - \|f_0 - g_\epsilon\|_{L^p} \le \|f_0 + f(\cdot - x_n)\|_{L^p} \le \|g_\epsilon + f(\cdot - x_n)\|_{L^p} + \|f_0 - g_\epsilon\|_{L^p}$$

yields

$$\sqrt[p]{\|g_\epsilon\|_{L^p}^p + \|f\|_{L^p}^p} - \epsilon \le \liminf_{n\to+\infty} \|f_0 + f(\cdot - x_n)\|_{L^p} \le \limsup_{n\to+\infty} \|f_0 + f(\cdot - x_n)\|_{L^p} \le \|g_\epsilon + f(\cdot - x_n)\|_{L^p} + \epsilon$$

Taking the limit $\epsilon \to 0^+$ we obtain (11.7).

Then

$$\frac{f_0 + f(\cdot - x_n)}{\|f_0 + f(\cdot - x_n)\|_{L^p}} \rightharpoonup f_0$$

So we have proved that for any $f_0 \in D_{L^p(\mathbb{R}^d)}(0,1)$ there is a sequence $\{f_n\}$ with $\|f_n\|_{L^p} = 1$ such that $f_n \rightharpoonup f_0$ for $\sigma(L^p, (L^p)')$.

Example 11.23. In the previous examples we exploited the group action of $\mathbb{R}^d$ on $L^p(\mathbb{R}^d)$, specifically spacial translations. Dilation provides another example of group action. Let for example $f_\lambda(x) := \lambda^{\frac{d}{p}} f(\lambda x)$. Notice that $\|f_\lambda\|_{L^p} = \|f\|_{L^p}$ and let again assume $1 < p < \infty$. We claim that

$$f_{\lambda_n} \rightharpoonup 0 \text{ in } L^p(\mathbb{R}^d) \text{ if } \lambda_n \xrightarrow{n\to+\infty} +\infty. \tag{11.8}$$

Suppose that $f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and take $g \in L^{p'}(\mathbb{R}^d)$. Suppose initially that $g \in C_c^0(\mathbb{R}^d)$. Then by dominated convergence we have

$$\int_{\mathbb{R}^d} \lambda_n^{\frac{d}{p}} f(\lambda_n x) g(x) dx = \lambda_n^{-d\left(1-\frac{1}{p}\right)} \int_{\mathbb{R}^d} f(x) g\left(\frac{x}{\lambda_n}\right) dx \tag{11.9}$$
$$= \int_{\mathbb{R}^d} \lambda_n^{\frac{d}{p}} f(\lambda_n x) g(x) dx = \lambda_n^{-\frac{d}{p'}} \int_{\mathbb{R}^d} f(x) g\left(\frac{x}{\lambda_n}\right) \xrightarrow{n\to+\infty} \int_{\mathbb{R}^d} f(x) dx g(0) \lim_{n\to+\infty} \lambda_n^{-\frac{d}{p'}} = 0.$$

By a density argument it is easy to conclude (11.8).
We now claim that

$$f_{\lambda_n} \rightharpoonup 0 \text{ in } L^p(\mathbb{R}^d) \text{ if } \lambda_n \xrightarrow{n\to+\infty} 0^+. \tag{11.10}$$

By the above computation

$$\langle f_{\lambda_n}, g\rangle = \left\langle f, g_{\frac{1}{\lambda_n}} \right\rangle \xrightarrow{n\to+\infty} 0,$$

where $g_{\frac{1}{\lambda_n}} \rightharpoonup 0$ in $L^{p'}(\mathbb{R}^d)$ by (11.8). This yields (11.10).

Exercise 11.24. What can be said of $\{\lambda_n^d f(\lambda_n x)\}_{n\in\mathbb{N}}$ for $\lambda_n \xrightarrow{n\to+\infty} +\infty$ in $L^1(\mathbb{R}^d)$ if $f \not\equiv 0$?

Answer. For $g \in BC^0(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \lambda_n^d f(\lambda_n x) g(x) dx = \int_{\mathbb{R}^d} f(x) g\left(\frac{x}{\lambda_n}\right) dx \xrightarrow{n\to+\infty} \int_{\mathbb{R}^d} f(x) dx g(0). \tag{11.11}$$

This shows that for $\int f \neq 0$, then if $\lambda_n \xrightarrow{n\to+\infty} +\infty$ it is not true that $\lambda_n^d f(\lambda_n \cdot) \rightharpoonup 0$. Rather, as measures, $\lambda_n^d f(\lambda_n x)dx$ converge to $\left(\int f dx\right) \delta(x) dx$, with $\delta(x)$ the Dirac delta centered in 0 (see next semester). Notice that for any $g \in BC^0(\mathbb{R}^d)$ with $g(0) = 0$ the limit in (11.11) is 0, and so there cannot be any $0 \neq u \in L^1(\mathbb{R}^d)$ with $\lambda_n^d f(\lambda_n \cdot) \rightharpoonup u$.

Let now $\int f = 0$. It is not restrictive to assume that on a closed disk $D \subset \mathbb{R}^d$ not containing 0, we have $a := \int_D f(x)dx > 0$. Let now

$$X := \bigcup_{n=1}^{\infty} \lambda_n^{-1} D$$

By taking a subsequence, we can assume that the $\{\lambda_n^{-1} D\}$ are disjoint and that

$$\|f\|_{L^1(\lambda_n \lambda_j^{-1} D)} < 2^{-j-1} a \text{ for all } j \neq n.$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^d} \lambda_n^d f(\lambda_n x) 1_X(x) dx = \int_D f(x) dx + \sum_{j\geq 1, j\neq n} \int_{\mathbb{R}^d} \lambda_n^d f(\lambda_n x) 1_D(\lambda_j x) dx \\
&= a + \sum_{j\geq 1, j\neq n} \int_{\mathbb{R}^d} f(x) 1_D(\lambda_n^{-1} \lambda_j x) dx = a + \sum_{j\geq 1, j\neq n} \int_{\lambda_n \lambda_j^{-1} D} f(x) dx \\
&= a - \sum_{j\geq 1, j\neq n} \|f\|_{L^1(\lambda_n \lambda_j^{-1} D)} > a - \sum_{j\geq 1, j\neq n} 2^{-j-1} a = 2^{-1} a > 0.
\end{aligned}$$

Another construction is the following, if we have a nonzero $f \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} f = 0$ and such that there exists an infinite cone C in $\mathbb{R}^d$ with tip $0 \in \mathbb{R}^d$ such that $\int_C f > 0$.

Then, for any sequence $\lambda_n \xrightarrow{n\to+\infty} +\infty$, we do not have $\lambda_n^d f(\lambda_n\cdot) \rightharpoonup 0$ for the $\sigma(L^1, L^\infty)$ topology. Indeed,

$$\int_{\mathbb{R}^d} \lambda_n^d f(\lambda_n x) 1_C(x) dx = \int_C \lambda_n^d f(\lambda_n x) dx = \int_C f(x) dx > 0.$$

$\square$

Remark 11.25. Consider the sequence $\{\lambda_n^d f(\lambda_n x)\}_{n\in\mathbb{N}}$ for $\lambda_n \xrightarrow{n\to+\infty} +\infty$ and $f \in L^1(\mathbb{R}^d)$ with $\|f\|_{L^1(\mathbb{R}^d)} = 1$. Notice the crucial difference between thinking $\overline{D_{L^1(\mathbb{R}^d)}(0,1)}$ in $(C_0^0(\mathbb{R}^d))'$, where it is relatively compact and metrizable for the $\sigma\left((C_0^0(\mathbb{R}^d))', C_0^0(\mathbb{R}^d)\right)$ topology, and in $(L^\infty(\mathbb{R}^d))'$, where it is relatively compact but not metrizable for the $\sigma\left((L^\infty(\mathbb{R}^d))', L^\infty(\mathbb{R}^d)\right)$ topology.

Corollary 11.26. *Let E be a locally convex space. If $\phi : E \to (-\infty, +\infty]$ is convex, then it is lower semi continuous for the $\sigma(E, E')$ topology if and only if it is lower semi continuous for the strong topology.*

Proof. Indeed for any a, $C = \{x : \phi(x) \leq a\}$ is convex since ϕ is convex, and is closed in one topology if and only if it is closed for the other.

$\square$

Notice that in a normed space E, the fact that $x_n \rightharpoonup x$ implies $\|x\|_E \leq \liminf \|x_n\|_E$ follows from the fact that $\phi(x) = \|x\|_E$ is convex and continuous (and therefore also lower semi continuous) in the strong topology.

Corollary 11.27. *Let E and F be two Banach space. Then, a linear map $T : E \to F$ is continuous in the strong topologies if and only if it is continuous from the $\sigma(E, E')$ to the $\sigma(F, F')$ topologies.*

Proof. Suppose T is continuous for the strong topologies. Then, for any $f \in F'$, the map $x \to f(Tx)$ is continuous in E. Hence $f \circ T \in E'$. Notice that if τ is the weakest topology in E for which $f \circ T \in E'$ for any $f \in F'$, this is exactly the weakest topology τ' in E which makes $T : E \to (F, \sigma(F, F'))$ continuous. Indeed, the open sets for τ' are of the form $T^{-1}A$, with A open set in $(F, \sigma(F, F'))$, and the open sets of the latter are generated by $f^{-1}(I)$, with $f \in F'$ and I open in $\mathbb{R}$. So, the open sets for τ' in E are generated by $T^{-1}f^{-1}(I) = (f \circ T)^{-1}(I)$, and hence they coincide with the open sets of τ. So $\tau = \tau'$. So $(E, \tau) \xrightarrow{T} (F, \sigma(F, F'))$ is continuous. On the other hand, the $\sigma(E, E')$ topology is obviously stronger than the τ topology, so we conclude that $(E, \sigma(E, E')) \xrightarrow{T} (F, \sigma(F, F'))$ is continuous. Notice that for this part of the proof, we did not use the Banach structure of E and F.

For the opposite direction, $G(T)$ is a vector subspace and so a convex subspace of $E \times F$. Furthermore, the continuity of $(E, \sigma(E, E')) \xrightarrow{T} (F, \sigma(F, F'))$ implies that $G(T)$ is closed in $E \times F$ for the $\sigma(E, E') \times \sigma(F, F')$ topology (for any continuous map $f : X \to Y$ between two topological spaces, the graph of f is closed in $X \times Y$). Furthermore, the $\sigma(E, E') \times \sigma(F, F')$ topology coincides with the $\sigma(E \times F, (E \times F)')$ topology. Then, by Theorem 11.10, $G(T)$

is closed in $E \times F$ for the strong topology. Then T is continuous for the strong topologies by the closed graph theorem 9.9.

$\square$

Exercise 11.28. a For $1 < p < \infty$ find that there are sequences in $\ell^p(\mathbb{N})$ converging $\sigma(\ell^p(\mathbb{N}), \ell^{p'}(\mathbb{N}))$ weakly to 0 but not strongly.

b Show that a sequence in $\ell^1(\mathbb{N})$ converging $\sigma(\ell^1(\mathbb{N}), \ell^\infty(\mathbb{N}))$ weakly to 0, it does so also strongly.

Answer. For **a** is enough to consider sequences of the form $\{f(\cdot - n)\}_{n\in\mathbb{N}}$.

Let us turn to **b**. Suppose the statement is false. Then it is easy to see that there is a sequence $\{f_n(\cdot)\}_{n\in\mathbb{N}}$ in $\ell^1(\mathbb{N})$ such that $f_n \rightharpoonup 0$ but $\|f_n\|_{\ell^1(\mathbb{N})} \geq 1$ for all $n \in \mathbb{N}$. It is easy to see that $f_n(m) \xrightarrow{n\to+\infty} 0$ for any $m \in \mathbb{N}$. Then it is possible to define a sequence of disjoint intervals $\{[N_k, M_k]\}_{k\in\mathbb{N}}$ such that $M_k < N_{k+1}$ such that there is a subsequence $\{f_{n_k}(\cdot)\}_{k\in\mathbb{N}}$ and such that

$$\sum_{j=N_k}^{M_k} |f_{n_k}(j)| > \frac{1}{2} \text{ and } \sum_{j\notin[N_k,M_k]} |f_{n_k}(j)| < \frac{1}{4}$$

and define $g \in \ell^\infty(\mathbb{N})$ by

$$g(j) = \begin{cases} \mathrm{signf_{n_k}(j) \text{ for } j \in [N_k, M_k]} \\ 0 \text{ for } \text{ for } j \notin \bigcup_{k=1}^{\infty}[N_k, M_k]. \end{cases}$$

Then $\|g\|_{\ell^\infty(\mathbb{N})} = 1$ and

$$\langle f_{n_k}, g\rangle = \sum_{j=N_k}^{M_k} f_{n_k}(j)g(j) + \sum_{j\notin[N_k,M_k]} f_{n_k}(j)g(j) \geq \sum_{j=N_k}^{M_k} |f_{n_k}(j)| - \sum_{j\notin[N_k,M_k]} |f_{n_k}(j)| > \frac{1}{4}.$$

And so it is not true that $\langle f_{n_k}, g\rangle \xrightarrow{k\to+\infty} 0$ and that $f_{n_k} \rightharpoonup 0$.

Remark 11.29. Notice that the sequence $\{e_n\}_{n\in\mathbb{N}}$ in $\ell^1(\mathbb{N})$ is obviously not convergent strongly and so, by item **b** in Exercise 11.28, neither weakly. Notice the connection with Remark 11.19.

12 Weak $\sigma(E', E)$ topology

We will consider a Banach space E. Then we know that E' has a structure of Banach space. On the other hand E' has also the $\sigma(E', E'')$ topology. We will consider on E' also the weak $\sigma(E', E)$ topology.

Definition 12.1. Given E', the weak $\sigma(E', E)$ topology, has a subbasis of seminorms the family $\left\{\left|\langle x, \cdot\rangle_{E\times E'}\right|\right\}_{x\in E}$.

Exercise 12.2. Suppose that E is infinite dimensional and there exists a subset $X \subseteq E$ countable and dense in E (i.e. E is *separable*, c.f. below). Is the topology on E' which has as sub–basis of seminorms the family $\left\{ |\langle x, \cdot \rangle_{E \times E'}| \right\}_{x \in X}$ the same as the $\sigma(E', E)$?

Answer. No, because if yes, then E' with the (*weak*) $\sigma(E', E)$ topology would be metrizable by Exercise**??**. But, by Exercise 12.7 below, this is not true. □

Lemma 12.3. *E' is Hausdorff for the weak $\sigma(E', E)$ topology.*

Proof. Given $f_0 \neq f_1$ in E', there exists $x \in E$ such that $f_0(x) \neq f_1(x)$. It is not restrictive to assume that $f_0(x) < \alpha < f_1(x)$ for some $\alpha \in \mathbb{R}$. But then

$$\{f \in E' : f(x) < \alpha\} \text{ resp. } \{f \in E' : f(x) > \alpha\}$$

are disjoint open neighborhoods of f_0 resp. f_1. □

Exercise 12.4. Consider E' with the weak $\sigma(E', E)$ topology. Then show for any $f_0 \in E$ a basis of neighborhoods of f_0 is of the form

$$V_{f_0}(x_1, ..., x_n, \epsilon) := \{f : |f(x_j) - f_0(x_j)| < \epsilon \text{ for } j = 1, ..., n\} \text{ where} \\ n \in \mathbb{N},\ x_1, ..., x_n \in E \text{ and } \epsilon > 0. \tag{12.1}$$

Lemma 12.5. *Let f_n be a sequence in E'. Then:*

1 $f_n \rightharpoonup f$ for $\sigma(E', E)$ if and only if $f_n(x) \to f(x)$ for any $x \in E$.

2 If $f_n \to f$ strongly, then $f_n \rightharpoonup f$ for $\sigma(E', E)$).
If $f_n \rightharpoonup f$ for $\sigma(E', E''))$, then $f_n \rightharpoonup f$ for $\sigma(E', E)$).

3 If $f_n \rightharpoonup f$ for $\sigma(E', E)$ then $\{\|f_n\|_{E'}\}$ is bounded and $\|f\|_{E'} \leq \liminf \|f_n\|_{E'}$.

4 If $f_n \rightharpoonup f$ for $\sigma(E', E)$ and if $x_n \to x$ strongly, then $f_n(x_n) \to f(x)$.

Proof. We prove only 3. For any $x \in E$ we know that $f_n(x) \to f(x)$ and so that $\{f_n(x)\}$ is bounded. If this holds for any $x \in E$, this implies by Banach Steinhaus that $\{\|f_n\|_{E'}\}$ is bounded. Next,

$$|f(x)| = \lim_{n \to +\infty} |f_n(x)| = \lim_{k \to +\infty} |f_{n_k}(x)|$$

for any subsequence $\{n_k\}$. If we take this subsequence so that $\|f_{n_k}\|_{E'} \xrightarrow{k \to +\infty} \liminf_{n \to +\infty} \|f_n\|_{E'}$, we conclude

$$|f(x)| = \lim_{k \to +\infty} |f_{n_k}(x)| \leq \|x\|_E \lim_{k \to +\infty} \|f_{n_k}\|_{E'} = \|x\|_E \|f\|_{E'},$$

and so $\|f\|_{E'} \leq \liminf_{n \to +\infty} \|f_n\|_{E'}$.

□

Example 12.6. Let $c_0(\mathbb{N}) \ni \mathbf{x}_n := (\underbrace{1, ..., 1}_{n \text{ times}}, 0, ...\}$. Then, for any $\xi \in (c_0(\mathbb{N}))' = \ell^1(\mathbb{N})$, we have $\langle \mathbf{x}_n, \xi \rangle_{c_0(\mathbb{N}) \times \ell^1(\mathbb{N})} \xrightarrow{n \to +\infty} \sum_{j=1}^{\infty} \xi(j)$. This implies that $\mathbf{x}_n \rightharpoonup \mathbf{x}_\infty = (1, 1, 1, 1, ...)$ in $\sigma(\ell^\infty(\mathbb{N}), \ell^1(\mathbb{N}))$. Obviously $\{\mathbf{x}_n\}_{n \in \mathbb{N}}$ is not a Cauchy sequence in $c_0(\mathbb{N})$. Notice that $c_0(\mathbb{N})$ is closed for the strong topology in $\ell^\infty(\mathbb{N})$, but not for the $\sigma(\ell^\infty(\mathbb{N}), \ell^1(\mathbb{N}))$ topology.

Exercise 12.7. Let E be an infinite dimensional normed space. Is E' metrizable for the $\sigma(E', E)$ *topology*?

Answer. Suppose by contradiction that there is a metric d and consider the balls $U_n = \{f : d(f, 0) < 1/n\}$. Each U_n is open. Then, each U_n must contain a line. Indeed, each U_n contains a set of the form (12.1), which in turn, is an open set also for the $\sigma(E', E'')$ and hence, by Lemma 11.12, contains a line. Then there exists $f_n \in U_n$ with $\|f_n\|_{E'} = n$.Then obviously $f_n \xrightarrow{n \to +\infty} 0$ in (E', d), that is $f_n \rightharpoonup 0$ in the $\sigma(E', E)$ *topology.* But $\|f_n\|_{E'} = n \xrightarrow{n \to +\infty} +\infty$, contradicting Lemma 12.5. □

Example 12.8. Let $f \in L^\infty(\mathbb{R}^d)$ with supp f compact. Then if $\lambda_n \xrightarrow{n \to +\infty} +\infty$ we have $f(\lambda_n \cdot) \rightharpoonup 0$ in the $\sigma(L^\infty, L^1)$ topology. Indeed, for any $g \in C_c^0(\mathbb{R}^d)$

$$\left| \int_{\mathbb{R}^d} f(\lambda_n x) g(x) dx \right| = \lambda_n^{-d} \left| \int_{\mathbb{R}^d} f(x) g\left(\frac{x}{\lambda_n}\right) dx \right| \leq \lambda_n^{-d} \|f\|_\infty \|g\|_\infty \xrightarrow{n \to +\infty} 0.$$

By the density of $C_c^0(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d)$ this yields the limit $f(\lambda_n \cdot) \rightharpoonup 0$ in the $\sigma(L^\infty, L^1)$ topology.

Proposition 12.9. *Given $\phi : E' \to \mathbb{R}$ linear and continuous for the $\sigma(E', E)$ topology, then there is $x \in E$ such that $\phi(f) = f(x)$ for any $f \in E'$.*

This uses the following lemma.

Lemma 12.10. *Let $f_1, \dots f_n, f$ linear forms on a vector space X such that $f_j(x) = 0$ for all x implies $f(x) = 0$. Then f is a linear combination of the f_j's.*

Proof of the Lemma. Consider the map

$$F : X \to \mathbb{R}^{n+1}, \, F(x) := (f(x), f_1(x), \dots, f_n(x)).$$

Then $a = (1, 0, \dots, 0)$ does not belong to $F(X)$, which is a vector space. So there exists a linear map $\mathbb{R}^{n+1} \to \mathbb{R}$

$$(x_0, x_1, ..., x_n) \to \lambda x_0 + \lambda_1 x_1 + ... + \lambda_n x_n$$

which separates a and $F(X)$. In particular, it is not restrictive to assume for all$x \in X$

$$\lambda < \alpha < \lambda f(x) + \sum \lambda_j f_j(x).$$

Then, $\lambda f(x) + \sum \lambda_j f_j(x) = 0$ for all$x \in X$ and $\alpha < 0$ and so $\lambda < 0$. □

Proof of Proposition 12.9. Let $|\phi(f)| < 1$ for $|f(x_j)| < \epsilon$, for $j = 1, ..., n$. In particular, if $f \in E'$ is such that $f(x_j) = 0$ for $j = 1, ..., n$, then we have $|\phi(tf)| = t|\phi(f)| < 1$ for all $t > 0$, and this can only happen if $\phi(f) = 0$. Then use the lemma for $X = E'$ and conclude that

$$\phi(f) = \sum_{j=1}^{n} \lambda_j f(x_j) = f(x) \text{ for } x = \sum_{j=1}^{n} x_j.$$

□

Corollary 12.11. *If H is an hyperplane in E' closed for $\sigma(E', E)$, then it is of the form $\{f \in E' : f(x) = a\}$ for some $x \in E$ and $a \in \mathbb{R}$.*

Proof. By definition, see Def. 2.18, H is the set of solutions of $\phi(f) = a$ for a linear map ϕ and a fixed a. By Exercise 2.19 and by the fact that $(E', \sigma(E', E))$ is locally convex, ϕ is continuous for the $\sigma(E', E)$ topology. Then, by Proposition 12.9, there exists $x \in E$ such that $\phi(f) = f(x)$ for any $f \in E'$. □

Theorem 12.12 (Banach Alaoglu)**.** *$\{f : \|f\|_{E'} \leq 1\}$ is compact for $\sigma(E', E)$ topology.*

Proof. We consider the map $\Phi : E' \to \mathbb{R}^E$ defined by $f \to f(x)$ for $x \in E$. We claim that this map establishes a homeomorphism $E' \to \Phi(E')$, where $\mathbb{R}^E$ has product topology and $\Phi(E')$ is a subspace of $\mathbb{R}^E$. First of all Φ is continuous, because so is $f \to f(x)$ for $x \in E$. Next, it is easy to see that Φ is injective. Obviously, Φ is onto on the image. Now we need to show that $\Phi^{-1} : \Phi(E') \to (E', \sigma(E', E))$ is continuous. To prove this we need to show that $\Phi^{-1}(\omega)(x) =: \omega_x$ is continuous for any x. But this is so, because $\omega \to \omega_x$ is a continuous projection (the restriction on $\Phi(E')$ of the continuous projection $\mathbb{R}^E \to \mathbb{R}$ which associates to each element its x–th coordinate). Hence we have proved that $\Phi : E' \to \Phi(E')$ is a homeomorphism.
Now we consider $\Phi(\{f : \|f\|_{E'} \leq 1\})$ and claim it is $K_1 \cap K_2$ with

$$K_1 = \{\omega : |\omega_x| \leq \|x\|_E \ \forall \ x\}, K_2 = \{\omega : \omega_{x+y} = \omega_x + \omega_y, \omega_{\lambda x} = \lambda\omega_x \forall x, y \in E, \lambda \in \mathbb{R}\}.$$

This is obvious. First of all $\mathbb{R}^E$ can be identified with the set of the functions $E \to \mathbb{R}$. K_2 can be identified with the functions which are linear. Finally, $K_1 \cap K_2$ can be identified with the linear operators $E \to \mathbb{R}$ with norm ≤ 1. Now we show that $K_1 \cap K_2$ is compact in $\mathbb{R}^E$ by showing that K_2 is closed and K_1 is compact. K_2 is closed because $K_2 = \cap A_{x,y} \cap B_{\lambda,x}$ with $A_{x,y}$ defined by the scalar equation $\omega_{x+y} - \omega_x - \omega_y = 0$ and $B_{\lambda,x}$ defined by the scalar equation $\omega_{\lambda x} - \lambda\omega_x = 0$, which are closed sets because involve continuous functions for the product topology. On the other hand, K_1 is the product of the compact sets $[-\|x\|_E, \|x\|_E]$ ($\{0\}$ when $x = 0$) and, by Tychonoff's theorem, K_1 is compact.

□

Example 12.13. Consider $E = \ell^\infty(\mathbb{N})$ and the sequence $\{e_n\}$ in E', where we notice that $e_n \in \ell^1(\mathbb{N}) \subset E'$. Then $\|e_n\|_{E'} = \|e_n\|_{\ell^1(\mathbb{N})} = 1$. The sequence $\{e_n\}$ does not have subsequences convergent weakly for the $\sigma(E', E)$ topology. In fact, for any given subsequence $\{e_{n_k}\}$, let $\xi \in \ell^\infty(\mathbb{N})$ be defined by

$$\xi(m) = \begin{cases} 0 \text{ if } m \neq n_k \text{ for all } k \\ (-1)^k \text{ if } m = n_k. \end{cases}$$

Then $\|\xi\|_E = 1$ and, clearly, $\langle e_{n_k}, \xi \rangle_{E' \times E} = \langle e_{n_k}, \xi \rangle_{\ell^1(\mathbb{N}) \times \ell^\infty(\mathbb{N})} = (-1)^k$ is not convergent. Hence, there is no subsequence $\{e_{n_k}\}$ convergent weakly for the $\sigma(E', E)$ topology. This is related to the fact that E' with the $\sigma(E', E)$ topology is not metrizable.

Exercise 12.14. Consider a normed space E, and suppose that there exists $X \subseteq E$ countable and dense in E and consider the topology τ on E' which has as subbasis of seminorms the family $\{| \langle \cdot, x \rangle_{E' \times E} |\}_{x \in X}$. Show that the topology induced on $D_{E'}(0, R)$ and on $\overline{D_{E'}(0, R)}$ for any $R > 0$ by $(E,' \tau)$ coincides with the topology induced by the $\sigma(E', E)$ topology. Prove that $D_{E'}(0, R)$ and $\overline{D_{E'}(0, R)}$ with the $\sigma(E', E)$ topology are metrizable.

Example 12.15. For $n \in \mathbb{N}$ let $\phi_n \in (\ell^\infty(\mathbb{N}))'$ defined by

$$\langle \phi_n, f \rangle_{(\ell^\infty(\mathbb{N}))' \times \ell^\infty(\mathbb{N})} = \frac{f(1) + ... + f(n)}{n}.$$

There are no subsequences weakly convergent for the $\sigma\left((\ell^\infty(\mathbb{N}))', \ell^\infty(\mathbb{N})\right)$ weak topology. Suppose, by contradiction, that ϕ_{n_k} is such a subsequence. Then, by taking a further subsequence, we can assume $\frac{n_k}{n_{k-1}} \xrightarrow{k \to +\infty} +\infty$. Setting $n_0 = 0$, we define

$$\xi(m) = (-1)^k \text{ if } m \in [n_{k-1} + 1, n_k].$$

Notice that $\|\xi\|_{\ell^\infty(\mathbb{N})} = 1$ and that $\langle \phi_{n_k}, \xi \rangle \in [-1, 1]$ for all k. Now we have

$$\begin{aligned} \langle \phi_{n_k}, \xi \rangle &= \frac{\sum_{j=1}^{n_{k-1}}}{n_k} + (-1)^k \frac{n_k - n_{k-1} - 1}{n_k} = \left\langle \phi_{n_{k-1}}, \xi \right\rangle \frac{n_{k-1}}{n_k} + (-1)^k - (-1)^k \frac{n_{k-1}}{n_k} - (-1)^k \frac{1}{n_k} \\ &= (-1)^k + o(1) \text{ where } o(1) \xrightarrow{k \to +\infty} 0. \end{aligned}$$

This shows that ϕ_{n_k} is not weakly convergent for the $\sigma\left((\ell^\infty(\mathbb{N}))', \ell^\infty(\mathbb{N})\right)$ weak topology.

Remark 12.16. See also Lemma 14.8 for a result of existence of weakly convergent subsequences in the context of reflexive Banach spaces.

13 Reflexive Spaces

Let E be a Banach space. Let $J : E \to E''$ be the natural immersion. It is a continuous injection for the strong topology. We say that E is *reflexive* if J is an isomorphism.

Theorem 13.1 (Kakutani)**.** *E is reflexive if and only if $\overline{D_E(0, 1)}$ is compact for the $\sigma(E, E')$ topology.*

Proof. If E is reflexive and so, by definition, $J : E \to E''$ is an isomorphism (for the strong topologies), then $J(\overline{D_E(0,1)}) = \overline{D_{E''}(0,1)}$. We know that $\overline{D_{E''}(0,1)}$ is compact for $\sigma(E'', E')$ by the Banach Alaoglu Theorem. So we need to show that $J^{-1} : (E'', \sigma(E'', E')) \to (E, \sigma(E, E'))$ is continuous. It is enough to show that $E'' \ni \xi \to \langle f, J^{-1}\xi\rangle_{E'\times E}$ is $\sigma(E'', E')$ continuous for any fixed $f \in E'$. We have $E'' \ni \xi \to \langle f, J^{-1}\xi\rangle_{E'\times E} = \langle \xi, f\rangle_{E''\times E'}$ and the latter is continuous in ξ for $\sigma(E'', E')$, by definition. This completes the proof that E reflexive implies $\overline{D_E(0,1)}$ compact for the $\sigma(E, E')$ topology.
Now we need to show that $\overline{D_E(0,1)}$ compact for the $\sigma(E, E')$ topology implies E reflexive.

Lemma 13.2 (Helly)**.** *Let E be Banach. Fix f_1,...,f_n in E' and $a_j \in \mathbb{R}$, $j = 1, ..., n$. The following statements are equivalent:*

1 For any $\epsilon > 0$ there is $x_\epsilon \in E$ such that $\|x_\epsilon\|_E \le 1$, $|f_j(x_\epsilon) - a_j| < \epsilon$ for all $j = 1, ..., n$.

2 $\left|\sum_{j=1}^n b_j a_j\right| \le \left\|\sum_{j=1}^n b_j f_j\right\|_{E'}$ *for all $b_j \in \mathbb{R}$, $j = 1, ..., n$.*

Proof. We first consider $1 \Longrightarrow 2$. Indeed, it is clear that

$$\left|\sum_{j=1}^n b_j a_j\right| = \lim_{\epsilon\to 0^+} \left|\sum_{j=1}^n b_j f_j(x_\epsilon)\right| \le \left\|\sum_{j=1}^n b_j f_j\right\|_{E'} .$$

Next us consider $2 \Longrightarrow 1$. If we set $F = (f_1, ..., f_n) : E \to \mathbb{R}^n$, Claim 1 means that $a := (a_1, ..., a_n) \in \mathbb{R}^n$ is $a \in \overline{F(D_E(0,1))}$. Proceeding by contradiction, we assume that $a \notin \overline{F(D_E(0,1))}$. Then, there exists a vector $b := (b_1, ..., b_n) \in \mathbb{R}^n$ and an $\alpha \in \mathbb{R}$ with

$$\sum_{j=1}^n b_j f_j(x) < \alpha < \sum_{j=1}^n b_j a_j \text{ for all } x \in D_E(0,1).$$

Clearly, since the left hand side is 0 at $x = 0$, we have $\alpha > 0$. Furthermore, by linearity, we get

$$\left|\sum_{j=1}^n b_j f_j(x)\right| < \alpha < \sum_{j=1}^n b_j a_j = \left|\sum_{j=1}^n b_j a_j\right| \text{ for all } x \in D_E(0,1).$$

This implies the following, which contradicts Claim 2, and so it is absurd,

$$\left\|\sum_{j=1}^n b_j f_j\right\|_{E'} \le \alpha < \left|\sum_{j=1}^n b_j a_j\right| .$$

□

Lemma 13.3 (Goldstine)**.** *Let E be a Banach space. Then $JD_E(0,1)$ is dense in $D_{E''}(0,1)$ for $\sigma(E'',E')$.*

Proof. Let $\xi \in D_{E''}(0,1)$ and consider V a neighborhood of ξ for $\sigma(E'',E')$, given by

$$V = \{\eta \in E'' : |\langle \eta - \xi, f_j \rangle_{E'' \times E'}| < \epsilon, j = 1, ..., n\}.$$

We need to find $x \in D_E(0,1)$ with $Jx \in V$, that is such that

$$|\langle Jx - \xi, f_j \rangle_{E'' \times E'} = |\langle x, f_j \rangle_{E \times E'} - \langle \xi, f_j \rangle_{E'' \times E'}| < \epsilon \text{ for all } j = 1, ..., n.$$

Set $a_j = \langle \xi, f_j \rangle$. Now, for any $b_j, j = 1, ..., n$ we have

$$\left| \sum_{j=1}^{n} b_j a_j \right| = \left| \left\langle \xi, \sum_{j=1}^{n} b_j f_j \right\rangle_{E'' \times E'} \right| \leq \left\| \sum_{j=1}^{n} b_j f_j \right\|_{E'} .$$

Then by previous lemma there is $x_\epsilon \in D_E(0,1)$ such that $|\langle x_\epsilon, f_j \rangle_{E \times E'} - a_j| < \epsilon, j = 1, ..., n$. $\square$

Remark 13.4. Notice that, if E is a not reflexive Banach space, by Lemma 13.3 we have JE dense in E'' for the $\sigma(E'',E')$ topology, with JE a closed space of E'' for the strong topology (because $J : E \to E''$ is an isometry and E is complete). So, like in Example 12.6, we have another example of strongly closed convex set (here in E'') which is not closed for the $*$ topology (here the $\sigma(E'',E')$ topology), in contrast to what happens in E for the $\sigma(E,E')$ topology, c.f. Theorem 11.10.

End of proof of the theorem. We are assuming that $\overline{D_E(0,1)}$ compact for the $\sigma(E,E')$ topology . $J : E \to E''$ is continuous for the strong topologies and so, by Corollary 11.27, for $\sigma(E,E') \to \sigma(E'',E''')$. This implies that $J : (E, \sigma(E,E')) \to (E'', \sigma(E'',E'))$ is continuous, because the $\sigma(E'',E')$ topology is weaker than the $\sigma(E'',E''')$ topology.
Since the image of a compact set for a continuous function is compact, we conclude that $J\overline{D_E(0,1)}$ is compact for the $\sigma(E'',E')$ topology. Since $JD_E(0,1)$ (and by consequence also $J\overline{D_E(0,1)}$) is, by the previous lemma, dense in $\overline{D_{E''}(0,1)}$, then $J\overline{D_E(0,1)} = \overline{D_{E''}(0,1)}$. But this implies that $JE = E''$ and so, that E is reflexive. $\square$

Lemma 13.5.

1 E Banach and M closed subspace of E. Then if E is reflexive, also M is reflexive.

2 E is reflexive if and only if E' is reflexive.

Proof. The topologies $\sigma(E,E')$ and $\sigma(M,M')$ coincide on M (indeed at first sight $\sigma(M,M')$ is stronger than $\sigma(E,E')$ because any element in E' leads to an element in M'. By Hahn Banach the two topologies coincide). $\overline{D_E(0,1)}$ is compact for $\sigma(E,E')$ implies that $\overline{D_M(0,1)}$, which is a closed subset of $\overline{D_E(0,1)}$, is compact for the $\sigma(E,E')$ topology, and so also for the $\sigma(M,M')$ topology on M. This sets the 1st claim.

We consider now the 2nd claim.
Assume E is reflexive. $\overline{D_{E'}(0,1)}$ is compact for $\sigma(E',E) = \sigma(E',E'')$ (by Banach Alaoglu and by reflexivity). Hence E' is reflexive.

Assume E' is reflexive. Then, by the previous argument, E'' is reflexive. JE is closed in E'' in the strong topology since J is an isometry in the strong topology. Then, by the 1st claim of this lemma, JE is reflexive, and so is E.

Lemma 13.6. *Let E be Banach reflexive and let $K \subset E$ be a bounded, closed convex set. Then K is compact for $\sigma(E,E')$.*

Proof. Since K is bounded, there is a constant $m > 0$ such that $K \subset m\overline{D_E}$ and since the latter is compact for $\sigma(E,E')$ and K, by Theorem 11.10, is closed for $\sigma(E,E')$, K is also compact. $\square$

Example 13.7. Notice that for $f \in L^1(\mathbb{R}^d)$ and for $\lambda_n \xrightarrow{n\to+\infty} +\infty$ the sequence ${\lambda_n}^d f(\lambda_n \cdot)dx$ converges as a measure to $\int_{\mathbb{R}^d} f dx \delta(x) dx$. Notice that $L^1(\mathbb{R}^d)$ is not reflexive and so Lemma 13.6 does not apply.

Corollary 13.8. *Let E be a reflexive Banach space and let $A \subset E$ be a closed, convex non empty set. Let $\phi : A \to (-\infty, +\infty]$ be convex lower semi continuous with*

$$\lim_{\|x\|_E \to \infty, x \in A} \phi(x) = +\infty.$$

Then there is a point of minimum $x_0 \in A$.

Proof. If we consider any $x_0 \in A$ and we set $\lambda_0 = \phi(x_0)$, then $K_0 = A \cap \phi^{-1}((-\infty, \lambda_0])$ is compact for the $\sigma(E,E')$ topology. Indeed, the fact that ϕ is lower semicontinuous implies that K_0 is closed in A, and also in E. The behaviour at infinity of ϕ implies that K_0 is bounded. Finally, the convexity of ϕ and of A, imply that K_0 is convex. Then, by the previous lemma, it follows that K_0 is compact. Let us take now a sequence $\lambda_n := \phi(x_n) \xrightarrow{n\to+\infty} \inf \phi(K)$ with $\{x_n\}$ a sequence in K. We can always assume that it is strictly decreasing, since otherwise the existence of a minimum point is obvious. Then $\{K_n\}$ is a strictly decreasing sequence of compact subsets of K_0. The intersection $K := \bigcap_{n=0}^{\infty} K_n$ cannot be empty, by the finite collection property, see in Exercise 1.10. So the points in K are absolute minimums.

$\square$

14 Separable spaces

A topological space is separable if it contains a countable dense set. For example, $C^0([0,1])$ is separable because $\mathbb{R}[x]$ is dense and has a countable dense subset.

Lemma 14.1. *For E a Banach space, if E' is separable, then E is separable.*

Proof. Let $\{f_n\}$ be dense in E'. We can consider a sequence $x_n \in E$ with $\|x_n\|_E = 1$ with $f_n(x_n) \geq \|f_n\|_{E'}/2$. Then the closure L of the $\mathrm{Span}\{x_n : n \in \mathbb{N}\}$ is separable. If $L \subsetneqq E$ there exists $f \in E'\backslash 0$ such that $f(x_n) = 0$ for all n. Since there is a subsequence $f_{n_k} \xrightarrow{k \to +\infty} f$ in E' we have

$$\|f_{n_k} - f\|_{E'} \geq f_{n_k}(x_{n_k}) - f(x_{n_k}) = f_{n_k}(x_{n_k}) \geq \|f_{n_k}\|_{E'}/2$$

in the limit we get $0 \geq \|f\|_{E'}$, which is a contradiction.

□

Exercise 14.2. Show that E is a reflexive and separable Banach space if and only if E' is a reflexive and separable Banach space.

Lemma 14.3. *$L^\infty(-1,1)$ is not separable.*

Proof. For any $a \in (-1,1)$ consider $I_a = (-|a|, |a|)$ and consider $D_{L^\infty(-1,1)}(\chi_{I_a}, 1/2)$.
We claim that

$$D_{L^\infty(-1,1)}(\chi_{I_a}, 1/2) \cap D_{L^\infty(-1,1)}(\chi_{I_b}, 1/2) = \emptyset \text{ for any } a \neq b \ . \tag{14.1}$$

Indeed, if there was an f such that $\|f - \chi_{I_a}\|_\infty < 1/2$ and $\|f - \chi_{I_b}\|_\infty < 1/2$ then by the triangular inequality would imply $\|\chi_{I_b} - \chi_{I_a}\|_\infty < 1$. However, we know we have $\|\chi_{I_b} - \chi_{I_a}\|_\infty = 1$, so (14.1) is true.
So $\{D_{L^\infty(-1,1)}(\chi_{I_a}, 1/2)\}_{a \in (-1,1)}$ is an uncountable family of open sets pairwise disjoint. If there existed a dense countable set $f_n \in L^\infty(-1,1)$ we would have an injection $I \to \mathbb{N}$ which of course is impossible. □

Example 14.4. Notice that $E := L^1(-1,1)$ is separable while $E' = L^\infty(-1,1)$ is not separable, so the implication E' separable $\Rightarrow E$ separable cannot be reversed.

Example 14.5. Consider a space $L^\infty(X, \mathbb{C})$. Then the subspace of $L^\infty(X)$ generated by the χ_E, for all measurable E, is dense in L^∞.
Indeed let $g \in L^\infty(X, \mathbb{C})$ decompose the ball $\|z\|_{\mathbb{C}} \leq \|g\|_\infty$ into a finite partition $A_1 \cup ... \cup A_n$ of disjoint measurable sets of diameter $< \epsilon$. Then set $E_j = g^{-1}(A_j)$ and fix $a_j \in A_j$. Then $\|g - \sum_{j=1}^n a_j \chi_{E_j}\|_\infty < \epsilon$.

Exercise 14.6. Consider a normed space E, and suppose that there exists $X \subseteq E'$ countable and dense in E' and consider the topology τ on E which has as subbasis of seminorms the family $\{|f|\}_{f \in X}$. Show that the topology induced on $D_E(0,R)$ and on $\overline{D_E(0,R)}$ for any $R > 0$ by (E, τ) coincides with the topology induced by the $\sigma(E, E')$ topology. Prove that $D_E(0,R)$ and $\overline{D_E(0,R)}$ with the $\sigma(E,E')$ topology are metrizable.

Remark 14.7. We have discussed in Exercise 14.6 that if E' is separable then $D_E(0,1)$ with the $\sigma(E,E')$ topology is metrizable. In fact, the viceversa is also true, so that $D_E(0,1)$ with the $\sigma(E,E')$ topology is metrizable if and only if E' is separable, see Brezis [3, Theorem 3.29].
Similarly, we have discussed in Exercise 12.14 that if E' is separable then $D_{E'}(0,1)$ with the $\sigma(E',E)$ topology is metrizable. In fact, the viceversa is also true, so that $D_{E'}(0,1)$ with

the $\sigma(E', E)$ topology is metrizable if and only if E is separable, see Brezis [3, Theorem 3.28].

Lemma 14.8. *Let $\{x_n\}$ be a bounded sequence in a reflexive Banach space E. Then there exists a subsequence $\{x_{n_k}\}$ weakly convergent in the $\sigma(E, E')$ topology.*

Proof. Consider the closure F in E of the space $\mathrm{Sp}\{x_n : n \in N\}$ generated by the elements of the sequence. Then, by Lemma 13.5 the space F is reflexive. It is obviously separable. Hence by Exercise 14.2, F' is reflexive and separable. Then there exists a subsequence $\{x_{n_k}\}$ weakly convergent in the $\sigma(F, F')$ topology. But, as we remarked in the proof of Lemma 13.5, this is the same as the convergence in the $\sigma(E, E')$ topology. $\square$

Exercise 14.9. Let X be a Banach space, X' its dual space, $\langle \cdot, \cdot \rangle_{X' \times X}$ the duality product, and $D_{X'}(0, 1)$ the unit ball in X'. Consider a bounded sequence $\{x_n, n \in \mathbb{N}\} \subset X$ such that

$$\forall x' \in \partial D_{X'}(0, 1) \text{ the sequence } \langle x', x_n \rangle_{X' \times X} \quad \text{converges.}$$

a Show that if X is reflexive, then x_n is weakly convergent in X.

b Is the above conclusion necessarily true if X is not reflexive? Prove it if it is true, or find a counterexample if it is false.

Answer. For definiteness, let X be a Banach space. A function $\varphi : X' \to \mathbb{R}$ remains defined. It is elementary that φ is a linear map. Since $\{x_n, n \in \mathbb{N}\} \subset X$ is bounded, then the associated sequence $\{Jx_n, n \in \mathbb{N}\} \subset X''$ is bounded. It is elementary to conclude that $\varphi \in X''$ and that $Jx_n \rightharpoonup \varphi$ for the $\sigma(X'', X')$ topology. If X is reflexive, then $J : (X, \sigma(X, X')) \to (X'', \sigma(X'', X'))$ is an isomorphism, and thus $x_n \rightharpoonup x$ in X for the $\sigma(X'', X')$ topology and for the $x \in X$ s.t. $Jx = \varphi$.

Let us now give a counterexample for a X not reflexive. Referring to Example 12.6 let $X = c_0(\mathbb{N})$, $X' = \ell^1(\mathbb{N})$ and $X'' = \ell^\infty(\mathbb{N})$, and recall the sequence $c_0(\mathbb{N}) \ni \mathbf{x}_n := (\underbrace{1, ..., 1}_{n \text{ times}}, 0, ...\}$ for which $\mathbf{x}_n \rightharpoonup \mathbf{x}_\infty = (1, 1, 1, 1, ...)$ in $\sigma(\ell^\infty(\mathbb{N}), \ell^1(\mathbb{N}))$. Notice that $\{\mathbf{x}_n\}$ is bounded in X and $\langle \mathbf{x}', \mathbf{x}_n \rangle_{X' \times X} \xrightarrow{n \to +\infty} \sum_{j=1}^\infty x'(j) = \langle \mathbf{x}', \mathbf{x}_\infty \rangle_{X' \times X''}$ for all $x' \in X' = \ell^1(\mathbb{N})$. So this gives a counterexample.

More generally, if $Jx_n \rightharpoonup x''$ for the $\sigma(X'', X')$ topology for a $x'' \notin R(J)$, then we get a counterexample to the claim. Then one can ask if all the not reflexive X yield a counterexample. Notice that by Lemma 13.3 we have that $JD_X(0, 1)$ is dense in $D_{X''}(0, 1)$ for the $\sigma(X'', X')$ topology. So, for $x'' \in D_{X''}(0, 1) \backslash JD_X(0, 1)$ we can ask if there is a sequence $\{x_n\}$ in $D_X(0, 1)$ such that $Jx_n \rightharpoonup x''$ for the $\sigma(X'', X')$ topology. If $D_{X''}(0, 1)$ is metrizable for the topology induced by the $\sigma(X'', X')$ topology, this is the case. Notice that in the counterexample given above, $X' = \ell^1(\mathbb{N})$ is separable, and so $D_{X''}(0, 1)$ is metrizable for the $\sigma(X'', X')$ topology. $\square$

15 Uniformly convex spaces

E Banach is said uniformly convex if for any $\epsilon > 0$ there is $\delta > 0$ such that for any $\|x\|_E \leq 1$, $\|y\|_E \leq 1$ and $\|x-y\|_E > \epsilon$ we have $\|\frac{x+y}{2}\|_E < 1-\delta$.

So for instance $\mathbb{R}^2$ with $|x| = (x_1^2 + x_2^2)^{\frac{1}{2}}$ is uniformly convex while $|x| = |x_1| + |x_2|$ is not uniformly convex.

Theorem 15.1 (Milman–Pettis)**.** *A uniformly convex Banach space E is reflexive.*

Proof. Let $\xi \in E''$ with $\|\xi\|_{E''} = 1$. We want to show $\xi \in J\overline{D_E(0,1)}$. Since $J\overline{D_E(0,1)}$ is closed for the strong topology in E'', it is enough to show that for any $\epsilon > 0$ there is $x \in \overline{D_E(0,1)}$ such that $\|\xi - Jx\|_{E''} \leq \epsilon$. Consider the $\delta > 0$ associated to $\epsilon > 0$ from the definition of uniform convexity, and let $f \in E'$ such that

$$\langle \xi, f\rangle_{E''\times E'} > 1 - \frac{\delta}{2} \text{ and } \|f\|_{E'} = 1, \tag{15.1}$$

which exists by $\|\xi\|_{E''} = 1$. Set

$$V = \{\eta \in E'' : |\langle \eta - \xi, f\rangle| < \frac{\delta}{2}\}.$$

V is a neighborhood of ξ for the $\sigma(E'', E')$ topology and $V \cap D_{E''}(0,1)$ is a non–empty open set for the $\sigma(E'', E')$ topology in $D_{E''}(0,1)$. Since $JD_E(0,1)$ is dense (by Goldstine) in $\overline{D_{E''}(0,1)}$ for $\sigma(E'', E')$, there is a $x \in D_E(0,1)$ with $Jx \in V$. We will show $\xi \in Jx + \epsilon\overline{D_{E''}(0,1)}$. Suppose that this is not the case and let W be the complement of $Jx+\epsilon\overline{D_{E''}(0,1)}$ in E''. W is open for $\sigma(E'', E')$, because $Jx + \epsilon\overline{D_{E''}(0,1)}$ is compact, and so closed, by Banach Alaoglu. Then $\xi \in W \cap V$ and so $W \cap V$ is nonempty and $\sigma(E'', E')$ open. It is also strongly open and, since ξ is an accumulation point for $D_{E''}(0,1)$ in the strong topology E'', it follows that $W \cap V \cap D_{E''}(0,1) \neq \emptyset$. Once again, since by Goldstine $JD_E(0,1)$ is dense in $D_{E''}(0,1)$ for $\sigma(E'', E')$, we have $W \cap V \cap JD_E(0,1) \neq \emptyset$. So let $\hat{x} \in D_E(0,1)$ so that $J\hat{x} \in W \cap V$. We have

$$\begin{aligned} &|\langle x, f\rangle_{E\times E'} - \langle \xi, f\rangle_{E''\times E'}| < \frac{\delta}{2} \text{ by } Jx \in V \\ &|\langle \hat{x}, f\rangle_{E\times E'} - \langle \xi, f\rangle_{E''\times E'}| < \frac{\delta}{2} \text{ by } J\hat{x} \in V. \end{aligned} \tag{15.2}$$

Then

$$2 - \delta < 2\langle \xi, f\rangle_{E\times E'} \leq \langle x + \hat{x}, f\rangle_{E\times E'} + \delta \leq \|x + \hat{x}\|_E + \delta, \tag{15.3}$$

where the upper bound is obtained summing in (15.2), while the lower bound uses (15.1). Since $J\hat{x} \in W$ we have $J\hat{x} \notin Jx + \epsilon\overline{D_{E''}(0,1)}$. Since $J : E \to E''$ is an isometry, this implies $\|x - \hat{x}\|_E > \epsilon$. But this implies $\|\frac{x+\hat{x}}{2}\|_E < 1-\delta$ which, inserted in (15.3), yields $2-\delta < 2-\delta$, which is absurd.

Hence we have proved that any in $\xi \in E''$ with $\|\xi\|_{E''} = 1$ is in $J\overline{D_E(0,1)}$. Notice that this implies that $E'' = R(J)$. So, J is an isomorphism from E to E''. $J\overline{D_E(0,1)}$

□

16 L^p spaces

Let us consider a measure space (X, μ) with a positive measure μ and let let for $1 \le p < \infty$

$$L^p(X, d\mu) = \{f \text{ measurable s.t.} |f|^p \in L^1(X, d\mu)\}$$
$$L^\infty(X, d\mu) = \{f \text{ measurable s.t. a.e. } |f(x)| \le C \text{ for some } C < \infty\}.$$

Recall that

$$\|f\|_{L^p(X,d\mu)} := \left(\int_X |f(x)|^p d\mu \right)^{\frac{1}{p}} \text{ for } p < \infty \text{ and}$$
$$\|f\|_{L^\infty(X,d\mu)} := \sup\{c \ge 0 : \mu(\{x : |f(x)| \ge c\}) > 0\}.$$

Theorem 16.1 (Hölder inequality)**.** *Let $f \in L^p(X, d\mu)$ and $g \in L^{p'}(X, d\mu)$ with $1 = \frac{1}{p} + \frac{1}{p'}$. Then $fg \in L^1(X, d\mu)$ and*

$$|fg|_{L^1(X,d\mu)} \le |f|_{L^p(X,d\mu)} |g|_{L^{p'}(X,d\mu)} \text{ (Hölder Inequality)} \tag{16.1}$$

Proof. Cases $p = 1, \infty$ are easy. Let $1 < p < \infty$. We have

$$|ab| \le \frac{|a|^p}{p} + \frac{|b|^{p'}}{p'} \text{ (Young's Inequality)} \tag{16.2}$$

which follows from the concavity of $\log : \mathbb{R}_+ \to \mathbb{R}$ and

$$\log \left(\frac{|a|^p}{p} + \frac{|b|^{p'}}{p'} \right) \ge \frac{1}{p} \log |a|^p + \frac{1}{p'} \log |a|^{p'} = \log |ab|.$$

So point–wise we have

$$|f(x)g(x)| \le \frac{|f(x)|^p}{p} + \frac{|g(x)|^{p'}}{p'}$$

which shows that $|fg| \in L^1(X, d\mu)$. Then

$$|fg|_{L^1(X,d\mu)} \le \frac{|f|^p_{L^p(X,d\mu)}}{p} + \frac{|g|^{p'}_{L^{p'}(X,d\mu)}}{p'}.$$

Also, for any $\lambda > 0$, we have

$$|fg|_{L^1(X,d\mu)} \le \frac{\lambda^p |f|^p_{L^p(X,d\mu)}}{p} + \frac{|g|^{p'}_{L^{p'}(X,d\mu)}}{p' \lambda^{p'}} \le \lambda^p |f|^p_{L^p(X,d\mu)} + \frac{|g|^{p'}_{L^{p'}(X,d\mu)}}{\lambda^{p'}} . \tag{16.3}$$

Choose λ so that the two terms in the r.h.s. are equal. Then

$$\lambda^p = \frac{|g|_{L^{p'}(X,d\mu)}}{|f|^{p-1}_{L^p(X,d\mu)}} = \frac{|g|_{L^{p'}(X,d\mu)}}{|f|^{\frac{p}{p'}}_{L^p(X,d\mu)}}$$

will do, since $\lambda^p |f|^p_{L^p(X,d\mu)} = |f|_{L^p(X,d\mu)} |g|_{L^{p'}(X,d\mu)}$ and

$$\frac{|g|^{p'}_{L^{p'}(X,d\mu)}}{\lambda^{p'}} = \frac{|g|^{p'}_{L^{p'}(X,d\mu)}}{\frac{|g|^{\frac{p'}{p}}_{L^{p'}(X,d\mu)}}{|f|_{L^p(X,d\mu)}}} = |f|_{L^p(X,d\mu)} |g|_{L^{p'}(X,d\mu)}.$$

Inserting in the 1st inequality in (16.3), we obtain (16.1).

□

Theorem 16.2 (Minkowsky inequality)**.** *Let $f, g \in L^p(X, d\mu)$. Then $f + g \in L^p(X, d\mu)$ with*

$$|f + g|_{L^p(X,d\mu)} \leq |f|_{L^p(X,d\mu)} + |g|_{L^p(X,d\mu)} \text{ (Minkowsky Inequality).} \tag{16.4}$$

Proof. Case $p = 1, \infty$ easy.
Let $1 < p < \infty$. By triangular inequality,

$$\begin{aligned} &|f(x) + g(x)|^p \leq (|f(x)| + |g(x)|)^p \leq (2 \max\{|f(x)|, |g(x)|\})^p \\ &\leq 2^p (\max\{|f(x)|, |g(x)|\})^p \leq 2^p (|f(x)|^p + |g(x)|^p). \end{aligned} \tag{16.5}$$

Then $f + g \in L^p(X, d\mu)$. Now

$$\begin{aligned} &\int |f(x) + g(x)|^p d\mu = \int |f(x) + g(x)|^{p-1} |f(x) + g(x)| d\mu \\ &\leq \int |f(x) + g(x)|^{p-1} (|f(x)| + |g(x)|) d\mu = \int |f(x) + g(x)|^{p-1} |f(x)| d\mu \\ &+ \int |f(x) + g(x)|^{p-1} |g(x)| d\mu. \end{aligned}$$

By Hölder

$$\begin{aligned} &\|f + g\|^p_{L^p(X,d\mu)} = \int |f(x) + g(x)|^p d\mu \\ &\leq \||f + g|^{p-1}\|_{L^{p'}(X,d\mu)} \|f\|_{L^p(X,d\mu)} + \||f + g|^{p-1}\|_{L^{p'}(X,d\mu)} \|g\|_{L^p(X,d\mu)} \\ &= \|f + g\|^{p-1}_{L^{p'(p-1)}(X,d\mu)} \left(\|f\|_{L^p(X,d\mu)} + \|g\|_{L^p(X,d\mu)}\right) \\ &= \|f + g\|^{p-1}_{L^p(X,d\mu)} \left(\|f\|_{L^p(X,d\mu)} + \|g\|_{L^p(X,d\mu)}\right). \end{aligned}$$

So, after simplification,

$$\|f + g\|_{L^p(X,d\mu)} \leq \|f\|_{L^p(X,d\mu)} + \|g\|_{L^p(X,d\mu)}.$$

□

Example 16.3. Notice that there are sequences $f_n \xrightarrow{n\to+\infty} f$ in $L^p(0,1)$ with $1 \le p < \infty$, with $f_n(x) \not\to f(x)$ for all $x \in [0,1]$. Indeed, consider a sequence $\{I_n\}$ formed by the intervals $\left[\frac{j-1}{n}, \frac{j}{n}\right]$ for $j = 1, ..., n$ and for $n \in \mathbb{N}$. Then $1_{I_n} \xrightarrow{n\to+\infty} 0$ in $L^p(0,1)$, but for any $x \in [0,1]$ the sequence $\{1_{I_n}(x)\}$ is not convergent.

Theorem 16.4. *$L^p(X, d\mu)$ for $1 \le p \le \infty$ is a Banach space.*

Proof. Consider first L^∞. We consider a Cauchy sequence f_n. Then for any $k \in \mathbb{N}$ there is N_k such that for $n, m \ge N_k$ we have $\|f_n - f_m\|_\infty < 1/k$. Hence for a 0 measure set E_k, for all $x \in X - E_k$ we have $|f_n(x) - f_m(x)| < 1/k$ for $n, m \ge N_k$. For any $x \in X - \cup E_k$ there is a limit $f(x)$ such that $|f_n(x) - f(x)| \le 1/k$ for $n \ge N_k$. So $\|f_n - f\|_\infty \le 1/k$ for $n \ge N_k$ and hence $\|f_n - f\|_\infty \xrightarrow{n\to+\infty} 0$.

Consider L^p with $p < \infty$ and a Cauchy sequence f_n. Taking a subsequence, we can suppose we have a sequence with $\|f_n - f_m\|_p < 2^{-n}$ for $m \ge n$. Consider the *telescopic* series

$$f_1 + \sum_{n=1}^{\infty} (f_{n+1} - f_n). \tag{16.6}$$

Then the partial sums $g_n(x) := |f_1(x)| + \sum_{j=1}^n |f_{j+1}(x) - f_j(x)|$ are such that $\|g_n\|_p < C$ for a fixed C. By the monotone convergence theorem, then they converge a.e. and

$$\lim_{n\to+\infty} \int_X |g_n(x)|^p d\mu = \int_X |g(x)|^p d\mu.$$

This implies the pointwise convergence a.e. of the telescopic series (16.6) to f. For $m \ge n$

$$|f_n(x) - f_m(x)| \le \sum_{j=n}^{m-1} |f_{j+1}(x) - f_j(x)| \le g(x) - g_{n-1}(x) \le g(x)$$

and so for $m \to \infty$, $|f_n(x) - f(x)| \le g(x)$ a.e. Then $f \in L^p$ and by dominated convergence $f_n \to f$ in L^p. ☐

Theorem 16.5. *L^p for $2 \le p < \infty$ is reflexive.*

Proof. We have Clarkson inequality (see proof below)

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \le \frac{1}{2} \left(\|f\|_p^p + \|g\|_p^p \right) \text{ for } 2 \le p < \infty.$$

Assuming Clarkson inequality we prove that for $2 \le p < \infty$ then L^p is uniformly convex. Indeed, for $\|f\|_p \le 1$, $\|g\|_p \le 1$, and $\|f - g\|_p > \epsilon$, then

$$\left\| \frac{f+g}{2} \right\|_p^p \le 1 - \frac{\epsilon^p}{2^p} \Rightarrow \left\| \frac{f+g}{2} \right\|_p \le 1 - \left(1 - \left(1 - \frac{\epsilon^p}{2^p} \right)^{\frac{1}{p}} \right)$$

so we conclude that L^p is uniformly convex, and hence reflexive by Milman–Pettis.

We turn to the proof of the Clarkson inequality, which is a consequence of

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \le \frac{1}{2}\left(|a|^p + |b|^p\right) \text{ for } 2 \le p < \infty.$$

We have

$$\alpha^p + \beta^p \le (\alpha^2 + \beta^2)^{\frac{p}{2}} \text{ for } 2 \le p < \infty$$

which in turn is a consequence for $q = p/2$ and for $a = \alpha^2$ and $b = \beta^2$, of

$$a^q + b^q \le (a+b)^q \text{ for } 1 \le q < \infty, \text{ which is equivalent to}$$

$$\left(\frac{a}{a+b}\right)^q + \left(\frac{b}{a+b}\right)^q \le \frac{a}{a+b} + \frac{b}{a+b} = 1.$$

For $\alpha = \left|\frac{a+b}{2}\right|$ and $\beta = \left|\frac{a-b}{2}\right|$

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \le \left(\left|\frac{a+b}{2}\right|^2 + \left|\frac{a-b}{2}\right|^2\right)^{\frac{p}{2}} = \left(\frac{a^2}{2} + \frac{b^2}{2}\right)^{\frac{p}{2}} \le \frac{a^p}{2} + \frac{b^p}{2},$$

where the last inequality follows by the convexity of $t \to t^{\frac{p}{2}}$. □

Theorem 16.6. *L^p for $1 < p < 2$ is reflexive.*

Proof. For any $1 < p < +\infty$, for any $f \in L^p$ there is an element Tf in $(L^{p'})'$ defined by $\langle Tf, g\rangle_{(L^{p'})'\times L^{p'}} = \int fg$. Then by Hölder $\|Tf\|_{(L^{p'})'} \le \|f\|_p$ and, setting $g(x) = |f(x)|^{p-2}f(x) \in L^{p'}$, we see $\langle Tf, g\rangle_{(L^{p'})'\times L^{p'}} = \|f\|_p^p$ and $\|g\|_{p'} = \|f\|_p^{p-1}$ so $\|Tf\|_{(L^{p'})'} \ge \|f\|_p$ and hence $\|Tf\|_{(L^{p'})'} = \|f\|_p$ for all $1 < p < +\infty$ and

$$T : L^p \to (L^{p'})' \text{ is an isometry for all } 1 < p < +\infty. \tag{16.7}$$

So $R(T)$ is a closed subspace of $(L^{p'})'$. Now let $1 < p < 2$. By Theorem 16.5, $L^{p'}$ is reflexive. By Lemma 13.5 this is equivalent to the fact that $(L^{p'})'$ is reflexive. Furthermore, Lemma 13.5 guarantees that the closed subspaces of $(L^{p'})'$, and so also $R(T)$, are reflexive. In turn, since $T : L^p \to R(T)$ is an isomorphism, this implies that L^p is reflexive for $1 < p < 2$. □

Theorem 16.7 (Riesz representation theorem)**.** *Let $1 < p < \infty$ and let $\phi \in (L^p(X))'$. Then there is $u \in L^{p'}(X)$ such that*

$$\phi(f) = \int_X uf \quad \forall f \in L^p.$$

Proof. By (16.7), $TL^{p'}$ is a closed subspace of $(L^p)'$. If $TL^{p'} \subsetneq (L^p)'$, then there is a nontrivial $h \in L^p \simeq (L^p)''$ with $\langle Tu, Jh\rangle_{(L^p)'\times(L^p)''} = 0$ for all $u \in L^{p'}$. But

$$\langle Tu, Jh\rangle_{(L^p)'\times(L^p)''} = \langle h, Tu\rangle_{L^p\times(L^p)'} = \int uh = 0 \text{ for all } u \in L^{p'}.$$

If we choose $u(x) = |h(x)|^{p-2}h(x) \in L^{p'}$, then we obtain

$$0 = \int uh = \int |h|^{p-2}h\ h = \int |h|^p \Longrightarrow h = 0 \text{ in } L^p.$$

Thus we get a contradiction and we conclude $TL^{p'} = (L^p)'$. So T is an isometric isomorphism.

$\square$

Theorem 16.8 (Riesz representation theorem)**.** *Let $\phi \in (L^1(X))'$ where X is σ–finite. Then there is $u \in L^\infty(X)$ such that*

$$\phi(f) = \int uf \quad \forall f \in L^1(X).$$

Proof. Here σ–finite means that $X = \cup_{1\le n<N} X_n$, with $N \in \mathbb{N} \cup \{\infty\}$ and with each X_n of finite measure. We can assume that the sequence X_n is increasing with n. Then it is possible to define a $w \in L^2(X)$ such that for any n there exists $C_n > 0$ such that $w(x) > C_n > 0$ for all $x \in X_n$. Indeed, we can choose $c_n > 0$ with $\sum c_n^2 < \infty$ and then define $w(x) = c_1$ in X_1 and $w(x) = c_n$ in $X_n \backslash X_{n-1}$.
Next, the map $f \in L^2(X) \to \langle \phi, fw\rangle_{(L^1(X))'\times L^1(X)}$ is bounded. So there exists $g \in L^2(X)$ such that

$$\langle \phi, fw\rangle_{(L^1(X))'\times L^1(X)} = \int fg \quad \forall f \in L^2(X).$$

Set now $u = \frac{g}{w}$, which is measurable. Then

$$|\int fg| = |\int fwu| = |\langle \phi, fw\rangle_{(L^1(X))'\times L^1(X)}| \le |\phi|_{(L^1(X))'} |fw|_{L^1}.$$

Notice that $u\chi_{X_n} \in L^2(X)$ for any $n < N$.

We claim that $\|u\|_\infty \le \|\phi\|_{(L^1)'}$. Indeed let $C > |\phi|_{(L^1)'}$ and let

$$A_\pm = \{x : \pm u(x) > C\}.$$

We will show that $|A_+| = 0$ with the argument for $|A_-| = 0$ similar. If $|A_+| > 0$, then there exists n with $|A_+ \cap X_n| > 0$ and

$$\begin{aligned} C\int_{A_+\cap X_n} w < \int_{A_+\cap X_n} wu = \int_{A_+\cap X_n} g = \int \chi_{A_+\cap X_n} g = \langle \phi, \chi_{A_+\cap X_n} w\rangle_{(L^1(X))'\times L^1(X)} \\ \le |\phi|_{(L^1)'} \|\chi_{A_+\cap X_n} w\|_{L^1}, \end{aligned}$$

which yields $C \le |\phi|_{(L^1)'}$ and a contradiction. So now we have $\|u\|_\infty \le \|\phi\|_{(L^1)'}$.

Next, we claim that

$$\langle \phi, f\rangle_{(L^1(X))'\times L^1(X)} = \int fu \text{ for any } f \in L^1(X). \tag{16.8}$$

We have

$$\langle \phi, \chi_{X_n} f\rangle_{(L^1(X))'\times L^1(X)} = \langle \phi, \chi_{X_n} \frac{f}{w} w\rangle_{(L^1(X))'\times L^1(X)} = \int \chi_{X_n} \frac{f}{w} g = \int \chi_{X_n} f u. \tag{16.9}$$

We have $\chi_{X_n} f \xrightarrow{n\to+\infty} f$ in $L^1(X)$, so $\langle \phi, \chi_{X_n} f\rangle_{(L^1(X))'\times L^1(X)} \xrightarrow{n\to+\infty} \langle \phi, f\rangle_{(L^1(X))'\times L^1(X)}$. On the other hand, we already know $u \in L^\infty(X)$, so $fu \in L^1(X)$ and $\chi_{X_n} fu \xrightarrow{n\to+\infty} fu$ in $L^1(X)$ and hence we conclude that $\int \chi_{X_n} fu \xrightarrow{n\to+\infty} \int fu$. So we canclude that taking the limit $n \to \infty$ in (16.9) we obtain (16.8).
Finally, by Hölder

$$|\langle \phi, h\rangle_{(L^1(X))'\times L^1(X)}| = |\int hu| \le \|h\|_1 \|u\|_\infty$$

and we conclude $\|u\|_\infty \ge \|\phi\|_{(L^1)'}$, and so $\|u\|_\infty = \|\phi\|_{(L^1)'}$.

□

Exercise 16.9. Given an open set $\Omega \subseteq \mathbb{R}^d$, show that if $u \in L^p(\Omega)$, then

$$\int_\Omega u f dx = 0 \text{ for all } f \in C_c^0(\Omega) \Longrightarrow u = 0.$$

Answer. If $u \neq 0$, it is not restrictive to assume that there exists a compact set K inside Ω with measure $|K| > 0$ where $u \ge 1$. For any open A with $K \subset A \subset \Omega$ there exists $f_A \in C_c^0(\Omega, [0,1])$ with $f_A = 1$ in K and supp $f_A \subset A$. We can generate a sequence of decreasing A_n with $|A_n \backslash K| \searrow 0$ and $u f_{A_n} \xrightarrow{n\to+\infty} 1_K u$ by dominated convergence. Then we get a contradiction by

$$0 = \int_\Omega u f_{A_n} dx \xrightarrow{n\to+\infty} \int_K u dx \ge |K| > 0.$$

Corollary 16.10. *Given an open set $\Omega \subseteq \mathbb{R}^d$, $C_c^0(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$.*

Proof. Suppose this is not the case and consider the closure $Y := \overline{C_c^0(\Omega)}$. Then there is $0 \neq u \in L^{p'}(\Omega)$ such that $\int_\Omega u f dx = 0$ for all $u \in C_c^0(\Omega)$. By exercise 16.9 we get a contradiction.

□

Exercise 16.11. Show that for $1 \le p < q \le \infty$, then $L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ is a Banach space with norm $\|\cdot\|_{L^p} + \|\cdot\|_{L^q}$ and for any $r \in [p,q]$ we have the bounded immersion $L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$.

Exercise 16.12. Show that for $1 \le p < q \le \infty$, then $L^p(\mathbb{R}^d) + L^q(\mathbb{R}^d)$ is a Banach space with norm

$$\|f\| = \inf\{\|g\|_p + \|h\|_p : f = g + h\}$$

and for any $r \in [p,q]$ we have the bounded immersion $L^r(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) + L^q(\mathbb{R}^d)$.

Exercise 16.13. Show that for $f \in L^p(X)$ for $1 \le p < \infty$ for X of infinite measure but σ–finite, for any $\epsilon > 0$ there exists $A \subset X$ of bounded measure such that

$$\int_{\complement A} |f(x)|^p < \epsilon.$$

Answer. Recall that $X = \cup_n X_n$ (numberable growing union) with all X_n of bounded measure. Then $\chi_{X_n} f \xrightarrow{n\to+\infty} f$ in $L^p(X)$ by dominated convergence. Hence $\chi_{\complement X_n} f \xrightarrow{n\to+\infty} 0$ in $L^p(X)$, and so just take $A = X_n$ for n large enough.

Exercise 16.14. Let X be σ–finite,

$1 \le p < \infty$ and suppose $\sup_{n\in\mathbb{N}} \|f_n\|_p < \infty$ and $f_n \xrightarrow{n\to+\infty} f$ a.e. Show the following.

a We have $f_n \rightharpoonup f$ in $L^p(X)$ for $1 < p < \infty$.

b Statement **a** is not true in $L^1(X)$ (if X is an infinite set).

c Statement **a** is true in $L^\infty(X)$ for the $\sigma(L^\infty, L^1)$ topology

Answer. **a** First of all $f \in L^p(X)$ with $\|f\|_p \le \liminf \|f_n\|_p$ by the Fathou Lemma. Let $g \in L^{p'}(X)$. Then by Exercise 16.13, there exists A with $|A| < \infty$, such that $\int_{\complement A} |g|^{p'} < \epsilon$. Furthermore, for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $B \subseteq A$ with $|B| < \delta$ we have $\int_B |g|^{p'} < \epsilon$. Finally, By Egorov Theorem, there exists $\widetilde{B} \subseteq A$ with $|\widetilde{B}| < \delta$ such that $f_n \xrightarrow{n\to+\infty} f$ uniformly in $A \backslash \widetilde{B}$. Then

$$\langle f_n - f, g\rangle = \left\langle f_n - f, \chi_{A\backslash \widetilde{B}} g\right\rangle + \left\langle f_n - f, \chi_{\widetilde{B}} g\right\rangle + \langle f_n - f, \chi_{\complement A} g\rangle$$

Since

$$|\left\langle f_n - f, \chi_{\widetilde{B}} g\right\rangle| \le 2 \sup \|f_n\|_{L^p} \epsilon \text{ and } |\langle f_n - f, \chi_{\complement A} g\rangle| \le 2 \sup \|f_n\|_{L^p}$$

and

$$|\left\langle f_n - f, \chi_{A\backslash \widetilde{B}} g\right\rangle| \le \|f_n - f\|_{L^\infty(A\backslash \widetilde{B})} \|g\|_{p'} \xrightarrow{n\to+\infty} 0,$$

where ϵ is arbitrary, it follows $\langle f_n - f, g\rangle \xrightarrow{n\to+\infty} 0$.

b As an example, we know from Exercise 11.24 that for $L^1(\mathbb{R}^d) \ni f \neq 0$, then if $\lambda_n \xrightarrow{n\to+\infty} +\infty$ it is not true that ${\lambda_n}^d f(\lambda_n \cdot) \rightharpoonup 0$. Now take for example any f with supp f compact, and then ${\lambda_n}^d f(\lambda_n x) \xrightarrow{n\to+\infty} 0$ for any $x \neq 0$.

Exercise 16.15. Consider $L^2([-\pi, \pi])$.

a Then $\cos(n\cdot) \rightharpoonup 0$ in $L^2([-\pi, \pi])$.

b It is not true that $\cos(nx) \xrightarrow{n\to+\infty} 0$ a.e.

Answer. We have $\int_{-\pi}^{\pi}\cos(nx)f(x)dx = \pi a_n \xrightarrow{n\to+\infty} 0$ by the Riemann–Lebesgue Lemma, see (7.11). If we had $\cos(nx) \xrightarrow{n\to+\infty} 0$ a.e., then also $\sin(nx) \xrightarrow{n\to+\infty} 0$ a.e., but $\sin^2(nx)+\cos^2(nx)\equiv 1$.

Exercise 16.16. Show that $f_n(x) := n\chi_{(0,1/n)}(x)$ converges a.e. to 0 in $[0,1]$ but $f_n \not\rightharpoonup 0$ in any $L^p([0,1])$.

Answer. For $p>1$ we have $\|f_n\|_{L^p(0,1)} \xrightarrow{n\to+\infty} +\infty$, while if f_n converged weakly we would have $\sup_n \|f_n\|_{L^p(0,1)} < \infty$. For $p=1$ we have $f_n(x) = n\chi_{(0,1)}(nx)$ and we know already by Exercise 11.24 that f_n does not converge weakly.

Example 16.17. We have $(c_0(\mathbb{N}))' = \ell^1(\mathbb{N})$.
Indeed, First of all $c_0' \supseteq \ell_1$. Given $\phi \in c_0'$ we can define u by $u_n = \phi(\delta_n)$, where $(\delta_n)_m = \delta_{n,m}$ the Kronecker delta. Now, if $u \notin \ell^1$, for any M $\exists$ N such that

$$\begin{aligned} M &\le \sum_{n=1}^{N}|u_n| = \sum_{n=1}^{N}\mathrm{sign(u_n)u_n} = \phi\left(\sum_{\mathrm{n=1}}^{\mathrm{N}}\mathrm{sign(u_n)\delta_n}\right) \\ &= \left\langle \sum_{n=1}^{N}\mathrm{sign(u_n)\delta_n}, \phi\right\rangle_{c_0\times c_0'} \le \left\|\sum_{n=1}^{N}\mathrm{sign(u_n)\delta_n}\right\|_{\ell^\infty}\|\phi\|_{c_0'} = \|\phi\|_{c_0'}. \end{aligned}$$

Obviously this is impossible, by the arbitrariness of M.

Notice that the map $\phi \in (c_0(\mathbb{N}))' \to \{\phi(\delta_n)\}_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$ is an isometric isomorphism.

Example 16.18. Recall that the Hahn–Banach Theorem, see Corollary 6.2, implies that for any $\phi \in (c_0(\mathbb{N}))' = \ell^1(\mathbb{N})$ there is an extension in $(\ell^\infty(\mathbb{N}))'$ with the same norm. It turns out that this extension is unique. Indeed, suppose $\|\phi\|_{\ell^1(\mathbb{N})} = 1$, where

$$\phi(x_1,x_2,...) = \sum_{j=1}^{\infty}\phi_j x_j.$$

Notice that this defines also an element in $(\ell^\infty(\mathbb{N}))'$. Now suppose that there is another extension $\psi \in (\ell^\infty(\mathbb{N}))'$ different from ϕ. It is not restrictive to assume there exists a unitary element $\mathbf{x} = \{x_n\}$ in $\ell^\infty(\mathbb{N})$ where

$$\gamma := \psi(\mathbf{x}) - \phi(\mathbf{x}) > 0.$$

Let N such that $\sum_{j=1}^{N}|\phi_j| > 1-\gamma/2$. By linearity, and by the fact that the functionals coincide in $c_0(\mathbb{N})$, for $\mathbf{x}_N = (1-\chi[0,N])\mathbf{x}$ we have

$$\gamma = \psi(\mathbf{x}_N) - \phi(\mathbf{x}_N).$$

Now

$$|\phi(\mathbf{x}_N)| \le \sum_{j=N+1}^{\infty}|\phi_j| < \gamma/2.$$

Then $\psi(\mathbf{x}_N) > \gamma/2$. Furthermore, for $\mathbf{z} = (\mathrm{sign}\phi_1, ..., \mathrm{sign}\phi_\mathrm{N}, 0, 0, ...)$

$$\psi(\mathbf{z}) = \phi(\mathrm{sign}\phi_1, ..., \mathrm{sign}\phi_\mathrm{N}, 0, 0, ...) = \sum_{\mathrm{j}=1}^{\mathrm{N}} |\phi_\mathrm{j}| > 1 - \gamma/2.$$

So

$$\psi(\mathbf{x}_N + \mathbf{z}) = \psi(\mathbf{x}_N) + \psi(\mathbf{z}) > \gamma/2 + 1 - \gamma/2 = 1.$$

But since $\|\mathbf{x}_N + \mathbf{z}\|_{\ell^\infty} = 1$, we contradict $\|\psi\|_{(\ell^\infty(\mathbb{N}))'} = 1$.

The following important theorem holds true.

Theorem 16.19 (Reisz Representation)**.** *Let X be a locally compact Hausdorff space and consider $C_c^0(X, \mathbb{R})$. Then $\big(C_c^0(X, \mathbb{R})\big)'$ is isomorphic to the space of bounded Borel measures (without sign) which are regular, see Remark 16.20. Furthermore the relation between $\Phi \in \big(C_c^0(X, \mathbb{R})\big)'$ and measure μ is given by*

$$\Phi(f) = \int_X f d\mu \text{ for any } f \in C_c^0(X, \mathbb{R}). \tag{16.10}$$

Remark 16.20. A Borel measure is regular if its absolute value measure

$$|\mu|(E) = \tag{16.11}$$
$$\sup\{\sum |\mu(E_n)| : \text{ over all disjoint finite or countable unions } E = \bigcup E_n \text{ with measurable sets}\}, \tag{16.12}$$

is regular.

Proof of Theorem 16.19. We skip the discussion of uniqueness, which is easier, and we discuss existence. We can assume that

$$|\Phi\|_{(C_c^0(X,\mathbb{R}))'} = 1. \tag{16.13}$$

We claim that

$$\begin{aligned}&\text{there exists a positive linear map } \Lambda : C_c^0(X, \mathbb{R}) \to \mathbb{R} \text{ such that } |\Phi(f)| \le \Lambda|f| \le \|f\|_{C_c^0(X,\mathbb{R})} \\ &\text{for all } f \in C_c^0(X, \mathbb{R})\end{aligned} \tag{16.14}$$

Let now λ be the measure associated to Λ by Theorem 1.22. Notice that we have

$$\lambda(X) = \sup\{\Lambda f : 0 \le f \le 1 \text{ with } f \in C_c^0(X, \mathbb{R})\} \le 1. \tag{16.15}$$

Then

$$|\Phi(f)| \le \Lambda|f| = \int_X |f| d\lambda.$$

Then by Hahn Banach there exists an extension $\Phi \in (L^1(X, d\lambda)'$ with norm 1. So there exists $g \in L^\infty(X, d\lambda)$ such that

$$\Phi(f) = \int_X fg d\lambda. \tag{16.16}$$

with $\|g\|_{L^\infty} = 1$.
We set as our measure $d\mu := g d\lambda$. Notice that $d|\mu| = |g| d\lambda$.
We have

$$1 = \|\Phi\|_{(C_c^0(X,\mathbb{R}))'} = \sup\{|\Phi(f)| : \|f\|_{C_c^0(X,\mathbb{R})} = 1\} \le \int_X |g| d\lambda = |\mu|(X).$$

On the other hand, (16.15) and $\|g\|_{L^\infty} = 1$ and the latter, imply $|g| = 1$ a.e. , so $d|\mu| = d\lambda$ and $|\mu|(X) = \lambda(X)$.

We now turn to the proof of Claim (16.14). For $C_c^+(X)$ the positive elements of $C_c^0(X, \mathbb{R})$, let

$$\Lambda f = \sup\{|\Phi(h)| : h \in C_c^0(X, \mathbb{R}) \text{ such that } |h| \le f\}. \tag{16.17}$$

Then $\Lambda f \ge 0$, (16.15) is satisfied, Λ is order preserving and $\Lambda c f = c \Lambda f$ for $c \ge 0$. Now we need to prove

$$\Lambda f + \Lambda g = \Lambda(f + g) \text{ for } f, g \in C_c^+(X). \tag{16.18}$$

Let $h_1, h_2 \in C_c^0(X, \mathbb{R})$ be such that $|h_1| \le f$ and $|h_2| \le g$ with

$$\Lambda f \le |\Phi(h_1)| + \epsilon \text{ and } \Lambda g \le |\Phi(h_2)| + \epsilon.$$

Let α_j be unitary complex numbers such that $|\Phi(h_j)| = \alpha_j \Phi(h_j)$. Then

$$\begin{aligned}&\Lambda f + \Lambda g \le |\Phi(h_1)| + |\Phi(h_2)| + 2\epsilon = \alpha_1 \Phi(h_1) + \alpha_2 \Phi(h_2) + 2\epsilon \\ &= \Phi(\alpha_1 h_1 + \alpha_2 h_2)) + 2\epsilon \le \Lambda(|h_1| + |h_2|) + 2\epsilon \le \Lambda(f + g) + 2\epsilon.\end{aligned}$$

Hence we have proved $\le$ in (16.18). Let now $|h| \le f + g$, call $V := \{x : f(x) + g(x) > 0\}$ and set

$$\begin{aligned}&h_1(x) := \frac{f(x)h(x)}{f(x) + g(x)} \text{ and } h_2(x) := \frac{g(x)h(x)}{f(x) + g(x)} \text{ in } V \\ &h_1(x) := 0 \text{ and } h_2(x) := 0 \text{ outside } V.\end{aligned}$$

Inside V the functions h_j are continuous. For $x_0 \notin V$, we have $h(x_0) = 0$. Furthermore we have $0 \le h_1(x) \le h(x)$ everywhere, so $\lim_{x \to x_0} h_j(x) = \lim_{x \to x_0} h_j(x) = 0$ and we get continuity also for $x_0 \notin V$. Then,

$$|\Phi(h)| = |\Phi(h_1 + h_2)| \le |\Phi(h_1)| + |\Phi(h_2)| \le \Lambda(f + f) \text{ for any } h \in C_c^0(X, \mathbb{R}) \text{ be such that } |h| \le f + g.$$

This implies the inequality $\ge$, and so also the equality, in (16.18).

Having proved (16.18), by linearity it is possible to extend Λ.

$\square$

The following is discussed in Yoshida [15, p.118] .

Example 16.21. $(L^\infty(X,\mathcal{M},d\lambda))'$ is the space of maps $\mu:\mathcal{M}\to\mathbb{R}$ with the following three properties:

$$E_1\cap E_2=\emptyset\Rightarrow \mu(E_1\cup E_2)=\mu(E_1)+(E_2); \tag{16.19}$$

$$\sup_{E\in\mathcal{M}}|\mu(E)|<\infty \tag{16.20}$$

$$\lambda(E)=0\Rightarrow\mu(E)=0. \tag{16.21}$$

Here $\phi\in(L^\infty(X,\mathcal{M},d\lambda)'$ and set $\mu(E):=\phi(\chi_E)$ for any $E\in\mathcal{M}$ and it can be checked that properties (16.19)–(16.21) are true, see [15]. Viceversa, given μ with the above properties, for $f\in L^\infty(X,\mathcal{M},d\lambda)$ it is possible to define $\phi\in(L^\infty(X,\mathcal{M},d\lambda)'$ by setting

$$\phi(f)=\lim_{n\to+\infty}\sum_{j=1}^n a_{jn}\mu(E_{jn})\text{ where }\lim_{n\to+\infty}\|f-\sum_{j=1}^n a_{jn}\mu(E_{jn})\|_{L^\infty}\xrightarrow{n\to+\infty}0,$$

see Example 14.5 on the density of simple functions in $L^\infty(X,\mathcal{M},d\lambda)$.

Theorem 16.22 (Young's convolution inequality)**.** *Let $f\in L^p(\mathbb{R}^d)$, $g\in L^q(\mathbb{R}^d)$ for $p,q\in[1,\infty]$. Set*

$$f*g(x):=\int_{\mathbb{R}^d}f(x-y)g(y)dy. \tag{16.22}$$

Then

$$\|f*g\|_{L^r(\mathbb{R}^d)}\le\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}\text{ for }\frac{1}{r}+1=\frac{1}{p}+\frac{1}{q}. \tag{16.23}$$

Proof. We consider the trilinear form

$$I(f,g,h)=\int f(y)g(x-y)h(x)dxdy, \tag{16.24}$$

for h in an appropriate dense subspace of $L^{r'}(\mathbb{R}^d)$, f in an appropriate dense subspace of $L^p(\mathbb{R}^d)$ and g in an appropriate dense subspace of $L^q(\mathbb{R}^d)$. It is enough to prove it is bounded in a dense set, to conclude that it automatically extends, uniquely, in a bounded trilinear form in the whole spaces, see Exercise 5.13. Notice that this, for similar reasons, will imply that (16.22) extends to a bounded bilinear map $L^p(\mathbb{R}^d)\times L^q(\mathbb{R}^d)\to L^r(\mathbb{R}^d)$.
To bound (16.24), it is enough to show if we assume $f\ge0$, $g\ge0$, $h\ge0$, $\|g\|_{L^q}=\|f\|_{L^p}=\|h\|_{L^{r'}}=1$, that

$$I(f,g,h)\le1. \tag{16.25}$$

The condition $\frac{1}{r}+1=\frac{1}{p}+\frac{1}{q}$ is the same as $2=\frac{1}{r'}+\frac{1}{p}+\frac{1}{q}$. So we have

$$\left(2-\frac{1}{p}-\frac{1}{q}\right)r'=1\quad,\quad\left(2-\frac{1}{p}-\frac{1}{r'}\right)q=1,$$
$$\left(2-\frac{1}{r'}-\frac{1}{q}\right)p=1,$$

which obviously is the same of as

$$\begin{aligned}
\left(1-\frac{1}{p}\right)r' + \left(1-\frac{1}{q}\right)r' = 1\\
\left(1-\frac{1}{p}\right)q + \left(1-\frac{1}{r'}\right)q = 1\\
\left(1-\frac{1}{r'}\right)p + \left(1-\frac{1}{q}\right)p = 1.
\end{aligned}$$

Hence

$$I(f,g,h) = \int \left(f^p(y)g^q(x-y)\right)^{1-\frac{1}{r'}} \left(f^p(y)h^{r'}(x)\right)^{1-\frac{1}{q}} \left(g^q(x-y)h^{r'}(x)\right)^{1-\frac{1}{p}} dxdy.$$

Using $\frac{1}{r}+\frac{1}{p'}+\frac{1}{q'}=1$, by Hölder inequality we obtain

$$I(f,g,h) \le \left(\int f^p(y)g^q(x-y)dxdy\right)^{\frac{1}{r}} \left(\int f^p(y)h^{r'}(x)dxdy\right)^{\frac{1}{q'}} \left(\int g^q(x-y)h^{r'}(x)dxdy\right)^{\frac{1}{p'}}.$$

From this we obtain the Young's convolution inequality (16.23). $\square$

Proposition 16.23. *Let $f \in C^k_c(\mathbb{R}^d)$ and let $g \in L^1_{loc}(\mathbb{R}^d)$. Then $f * g \in C^k(\mathbb{R}^d)$ with $\nabla^j(f*g) = (\nabla^j f)*g$ for $j \le k$.*

Proof. For any fixed x the map $F(x,y) = f(x-y)g(y)$ is in L^1_y. For $x_n \to x$, then there is a compact set K such that $F(x,y) = \chi_K(y)F(x,y)$, $F(x_n,y) = \chi_K(y)F(x_n,y)$. We have pointwise $F(x_n,y) \to F(x,y)$ for all y and $|F(x_n,y)| \le \chi_K(y)|f|_\infty|g(y)|$. Then we can apply dominated convergence and conclude $\lim \int F(x_n,y)dy = \int F(x,y)dy$. This sets case $k=0$. For the general case it is enough to prove the case $k=1$ and then proceed by induction. We have

$$\begin{aligned}
f(x+h-y) - f(x-y) - h\cdot\nabla f(x-y) = h\cdot I(x-y,h) \text{ , where}\\
I(x-y,h) := \int_0^1 \left[\nabla f(x+sh-y) - \nabla f(x-y)\right] ds.
\end{aligned}$$

Notice that $\nabla f \in C^0_c(\mathbb{R}^d,\mathbb{R}^d)$ implies that ∇f is uniformly continuous. This implies that $|I(z,h)| \le o(1)$, where $o(1)$ is a function dependent only on h with $o(1) \xrightarrow{h\to 0} 0$. Then

$$|f(x+h-y) - f(x-y) - h\cdot\nabla f(x-y)| \le |h|o(1).$$

For fixed x in some bounded set, there is a compact set K such that for $|h|\le 1$

$$|f(x+h-y) - f(x-y) - h\cdot\nabla f(x-y)| \le |h|o(1)\chi_K(y).$$

Then

$$|f*g(x+h)-f*g(x)-h\cdot\nabla f*g(x)|\le|h|o(1)\int_K|g(y)|dy$$

and so $f*g$ is differentiable in x with gradient $\nabla f*g$.

$\square$

Theorem 16.24. *Let $\rho\in L^1(\mathbb{R}^d)$ be s.t. $\int\rho(x)dx=1$. Set $\rho_\epsilon(x):=\epsilon^{-d}\rho(x/\epsilon)$. Then for any $f\in L^p(\mathbb{R}^d)$ with $1\le p<\infty$ we have*

$$\lim_{\epsilon\searrow0}\rho_\epsilon*f=f \text{ in } L^p(\mathbb{R}^d). \tag{16.26}$$

In particular we have, see (7.23),

$$\lim_{t\searrow0}e^{t\triangle}f=f \text{ in } L^p(\mathbb{R}^d). \tag{16.27}$$

Proof. Clearly (16.27) is a special case of (16.26) setting $\epsilon=\sqrt{t}$ and $\rho(x)=(4\pi)^{-\frac{d}{2}}e^{-\frac{|x|^2}{4}}$. To prove (16.26) we start with $f\in C^0_c(\mathbb{R}^d)$. In this case

$$\rho_\epsilon*f(x)-f(x)=\int_{\mathbb{R}^d}(f(x-\epsilon y)-f(x))\rho(y)dy$$

so that, by Minkowski inequality and for $\Delta(y):=\|f(\cdot-y)-f(\cdot)\|_{L^p}$, we have

$$\|\rho_\epsilon*f(x)-f(x)\|_{L^p}\le\int|\rho(y)|\Delta(\epsilon\ y)dy.$$

Now we have $\lim_{y\to0}\Delta(y)=0$ and $\Delta(y)\le2\|f\|_{L^p}$. So, by dominated convergence we get

$$\lim_{\epsilon\searrow0}\|\rho_\epsilon*f(x)-f(x)\|_{L^p}=\lim_{\epsilon\searrow0}\int|\rho(y)|\Delta(\epsilon\ y)dy=0.$$

So this proves (16.26) for $f\in C^0_c(\mathbb{R}^d)$. The general case is proved by a density argument. $\square$

Exercise 16.25. Show that the statement in Corollary 16.10 would be wrong for $p=\infty$.

Exercise 16.26. Show that the statement in Corollary 16.10 is correct with $p=\infty$ in (16.27)–(16.26) when taking $f\in C^0_0(\mathbb{R}^d)$.

Exercise 16.27. Show that if $f,g\in C^0_c(\mathbb{R}^d)$, then

$$\text{supp } f*g\subseteq\overline{\text{supp } f+\text{supp } g}. \tag{16.28}$$

Proposition 16.28. *For any open set $\Omega\subseteq\mathbb{R}^d$, $C^\infty_c(\Omega)$ is dense in $L^p(\Omega)$ for any $1\le p<\infty$.*

Proof. Let us start with $\Omega = \mathbb{R}^d$. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ with $\int \phi = 1$ and $R > 0$ such that $D_{\mathbb{R}^d}(0, R) \supset \text{supp}\, \phi$. Consider $\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(x/\epsilon)$. Notice that supp $\phi_\epsilon \subseteq D_{\mathbb{R}^d}(0, \epsilon R)$. Then for any $g \in C_c^0(\mathbb{R}^n)$ with $K := \text{supp}\, g$ we have $\phi_\epsilon * g \to g$ in $L^p(\mathbb{R}^d)$ and furthermore supp $(\phi_\epsilon * g) \subseteq \overline{D_{\mathbb{R}^d}(0, \epsilon R) + K}$ is compact. By $\overline{C_c^0(\mathbb{R}^d)} = L^p(\mathbb{R}^d)$ we get the desired result for $\Omega = \mathbb{R}^d$.

For more general Ω, and for ϕ and g as above, with $K \subset \Omega$, then $\text{dist}(K, \partial\Omega) =: \gamma > 0$. Then, for $\epsilon \in (0, \gamma/R)$, supp $(\phi_\epsilon * g) \subset \Omega$. Hence also in this case we have proved $\overline{C_c^\infty(\Omega)} \supset C_c^0(\Omega)$ and, consequently, $\overline{C_c^\infty(\Omega)} = L^p(\Omega)$.

□

Exercise 16.29. Consider the group actions

$$\mathbb{R}^d \times L^p(\mathbb{R}^d) \ni (y, f) \to \tau_y f := f(\cdot - y) \in L^p(\mathbb{R}^d)$$
$$\mathbb{R}^d \times L^p(\mathbb{R}^d) \ni (\lambda, f) \to \delta_{p,\lambda} f := \lambda^{\frac{d}{p}} f(\lambda \cdot) \in L^p(\mathbb{R}^d).$$

a Show that for $1 \le p < \infty$ we have $\tau_y f \xrightarrow{y \to 0} f$ and $\delta_{p,\lambda} f \xrightarrow{\lambda \to 1} f$ for any $f \in L^p(\mathbb{R}^d)$.

b Show that claim **a** if false for $p = \infty$.

c Show that for $1 \le p < \infty$ it is not true that $\tau_y \xrightarrow{y \to 0}$Identity in $\mathcal{L}(L^p(\mathbb{R}^d))$ and similarly that it is not true that $\delta_{p,\lambda} \xrightarrow{\lambda \to 1}$Identity in $\mathcal{L}(L^p(\mathbb{R}^d))$.

Remark 16.30. Notice that Exercise 16.29 is closely related to Remark 7.27. Notice for example, that $(y, f) \to \tau_y f := f(\cdot - y)$ is really the group $e^{-y \cdot \nabla} f$, and similarly, $\delta_{e^t} f = e^{t\left(\frac{d}{p} + x \cdot \nabla\right)} f$. In other words, associated to these group actions, are certain differential operators.

Exercise 16.31. Let $k \in L^q(\mathbb{R}^d)$ and consider the convolution operator $T : L^p(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$, where $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$, defined by $Tf = k * f$. Show that this operator commutes with translations, that is

$$\tau_y T = T \tau_y \text{ for any } y \in \mathbb{R}^d. \tag{16.29}$$

Theorem 16.32 (Kolmogorov, Riesz, Frechét)**.** *Let $\mathcal{F} \subset L^p(\mathbb{R}^d)$ for $p < \infty$ be bounded and s.t. the following property is true:*

$$\text{for any } \epsilon > 0 \text{ there is } \delta(\epsilon) > 0 \text{ s.t. } |h| < \delta(\epsilon) \Rightarrow \|\tau_h f - f\|_{L^p(\mathbb{R})} \le \epsilon \text{ for all } f \in \mathcal{F}. \tag{16.30}$$

Then for any open bounded Ω in $\mathbb{R}^d$ the restriction $\mathcal{F}|_\Omega$ is relatively compact in $L^p(\Omega)$.

Proof. We will prove that

$$\forall\, \epsilon > 0,\ \mathcal{F}|_\Omega \text{ is contained in the union of finitely many balls of radius } \epsilon \text{ in } L^p(\Omega). \tag{16.31}$$

The proof is related to Ascoli's Theorem. We first claim that

$$\text{for any } \varepsilon > 0 \text{ there exists } \omega \subset\subset \Omega \text{ s.t. } \|f\|_{L^p(\Omega\backslash\omega)} \le \frac{\epsilon}{3} \text{ for all } f \in \mathcal{F}. \tag{16.32}$$

We skip the proof of (16.32) for the moment. Notice now that for any $a, b \in \mathbb{R}_+$, if we set

$$T(a,b) = \{f \in C^1(\mathbb{R}^d) : \quad \|f\|_{L^\infty(\mathbb{R}^d)} \le a \text{ and } \|\nabla f\|_{L^\infty(\mathbb{R}^d)} \le b\},$$

then $T(a,b)|_\omega$ is relatively compact in $C^0(\omega, \mathbb{R})$ by Ascoli's Theorem. Let us consider now a standard sequence of mollifiers $\rho_n(x) = n^d \rho(nx)$, with $\rho \in C_c^\infty(D_{\mathbb{R}^d}(0,1),[0,1])$ a function of integral 1. Then using (16.30), for $n > 1/\delta\left(\frac{\epsilon}{4}\right)$ we have

$$\|\rho_n * f - f\|_{L^p(\mathbb{R}^d)} = \|\int_{\mathbb{R}^d} \rho_n(y)(f(x-y) - f(x))dy\|_{L^p(\mathbb{R}^d)} \le \int_{\mathbb{R}^d} \rho_n(y)\|\tau_{-y}f - f\|_{L^p(\mathbb{R}^d)}dy$$
$$< \int_{\mathbb{R}^d} \rho_n(y)\frac{\epsilon}{4} = \frac{\epsilon}{4},$$

since $\rho_n(y) \ne 0$ only if $|y| < 1/n < \delta\left(\frac{\epsilon}{4}\right)$.

Fix now $n \in \mathbb{N}$. We have for any $x \in \mathbb{R}^d$

$$|\rho_n * f(x)| \le \int_{\mathbb{R}^d} \rho_n(x-y)|f(y)|dy \le \|\rho_n\|_{L^{p'}(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)} \le a_n \text{ for all } f \in \mathcal{F}. \tag{16.33}$$

Similarly

$$|\nabla\rho_n * f(x)| \le \int_{\mathbb{R}^d} |\nabla\rho_n(x-y)||f(y)|dy \le \|\nabla\rho_n\|_{L^{p'}(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)} \le b_n \text{ for all } f \in \mathcal{F}$$

So we have a sequence $\{(a_n, b_n)\}$ in $\mathbb{R}_+^2$ s.t. for any fixed n we have $\{\rho_n * f : f \in \mathcal{F}\} \subset T(a_n, b_n)$. For any n the latter set, being relatively compact in $C^0(\omega,\mathbb{R}) \subset L^\infty(\omega) \subset L^p(\omega)$, is contained in a finite union of balls or radius $\frac{\epsilon}{3}$ in $L^p(\omega)$. Fix $n_0 > 1/\delta\left(\frac{\epsilon}{3}\right)$. Let $T(a_{n_0}, b_{n_0}) \subset \cup_{j=1}^N D_{L^p(\omega)}(u_j, \epsilon/3)$. Then we claim

$$\mathcal{F}|_\Omega \subset \cup_{j=1}^N D_{L^p(\Omega)}(u_j, \epsilon) \tag{16.34}$$

where $u_j|_{\Omega\backslash\omega} := 0$ since above we can take $u_j \in C^0(\omega, \mathbb{R})$ with supp $(u_j) \subseteq \omega$. Indeed, let $f \in \mathcal{F}$. Then there is u_j s.t. $\|\rho_{n_0} * f - u_j\|_{L^p(\omega)} < \epsilon/3$. This implies

$$\|f - u_j\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega\backslash\omega)} + \|f - u_j\|_{L^p(\omega)} \le \|f\|_{L^p(\Omega\backslash\omega)} + \|\rho_{n_0} * f - u_j\|_{L^p(\omega)} + \|f - \rho_{n_0} * f\|_{L^p(\mathbb{R}^d)} < \epsilon.$$

Hence (16.34) is proved, and so also (16.31). However we need yet to prove (16.32). We write

$$\|f\|_{L^p(\Omega\backslash\omega)} \le \|f - \rho_{n_0} * f\|_{L^p(\mathbb{R}^d)} + \|\rho_{n_0} * f\|_{L^p(\Omega\backslash\omega)} \le \frac{\epsilon}{4} + \|\rho_{n_0} * f\|_{L^\infty(\mathbb{R}^d)}|\Omega\backslash\omega|^{\frac{1}{p}}$$
$$\le \frac{\epsilon}{4} + a_{n_0}|\Omega\backslash\omega|^{\frac{1}{p}} \le \frac{\epsilon}{3} \text{ for } |\Omega\backslash\omega| \text{ sufficiently small.}$$

□

Example 16.33 (An application of the Reisz Representation Theorem 16.19). We show that if $h(z)$ is harmonic in $U := D_{\mathbb{C}}(0,1)$, that is if $\triangle h = 0$ in U, and if

$$\sup_{0 \le r < 1} \int_{-\pi}^{\pi} |h(re^{\mathrm{i}\theta})| d\theta = M < \infty,$$

then there is a complex valued measure μ on $\mathbb{T} := \partial U$ such that for $r < 1$

$$h(re^{i\theta}) = P(\mu)(re^{\mathrm{i}\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) d\mu(e^{\mathrm{i}t})$$

with

$$P_r(\theta - t) = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}.$$

To check this, for any $f \in C^0(\mathbb{T})$ and $0 \le s < 1$ we set

$$\Lambda_s(f) := \int_{-\pi}^{\pi} h(se^{\mathrm{i}t}) f(e^{\mathrm{i}t}) dt.$$

We have that $\|\Lambda_s\|_{(C(\mathbb{T}))'} \le M$. By $\sigma((C(\mathbb{T}))', C(\mathbb{T}))$ compactness of the unit ball in $(C(\mathbb{T}))'$ (that is Banach–Alaouglu), there exists a sequence $s_n \nearrow 1$ and a Λ such that $\Lambda_{s_n} \rightharpoonup \Lambda$ in $(C(\mathbb{T}))'$ for the $\sigma((C(\mathbb{T}))', C(\mathbb{T}))$ topology, that is

$$\lim_{n \to +\infty} \Lambda_{s_n} f = \Lambda f \, \forall \, f \in C^0(\mathbb{T}).$$

As a consequence of Theorem 16.19, there is a complex Borel measure μ with

$$\lim_{n \to +\infty} \int_{-\pi}^{\pi} h(s_n e^{\mathrm{i}t}) f(e^{\mathrm{i}t}) dt = \int_{-\pi}^{\pi} f(e^{\mathrm{i}t}) d\mu(e^{\mathrm{i}t}).$$

Now, by Example 6.10, for $r < 1$ we have

$$h(rs_n e^{\mathrm{i}\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) h(s_n e^{\mathrm{i}t}) dt.$$

For $s_n \to 1$ in the latter we get, applying previous formula for $f(e^{\mathrm{i}t}) = P_r(\theta - t)$,

$$h(re^{\mathrm{i}\theta}) = \lim h(rs_n e^{\mathrm{i}\theta}) = \lim \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) h(s_n e^{\mathrm{i}t}) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) d\mu(e^{\mathrm{i}t}).$$

Exercise 16.34. Prove the following, which we will use later: for $f \in L^1(\mathbb{T})$, we have

$$\lim_{n \to +\infty} \int_{\mathbb{T}} |\sin(nx) f(x)| dx = \frac{2}{\pi} \int_{\mathbb{T}} |f(x)| dx \tag{16.35}$$

Answer. Let us start with $f = 1$. Then, using a scale change and the 2π periodicity of $\sin x$ and the π periodicity of $|\sin x|$,

$$\int_0^{2\pi} |\sin(nx)|dx = n^{-1}\int_0^{2\pi n} |\sin(x)|dx = n^{-1}n\int_0^{2\pi} |\sin(x)|dx = 2\int_0^{\pi} \sin(x)dx = 4$$
$$= \frac{4}{2\pi}\|1\|_{L^1(\mathbb{T})}.$$

More generally, let $f \in L^1(\mathbb{T})$. By density it is enough to focus on simple functions

$$f = \sum_{j=1}^{N} \lambda_j \chi_{[2\pi a_j, 2\pi b_j]},$$

where the intervals $[2\pi a_j, 2\pi b_j]$ are pairwise disjoint. Then

$$\int_{\mathbb{T}} |\sin(nx)f(x)|dx = \sum_{j=1}^{N} |\lambda_j| \int_{2\pi a_j}^{2\pi b_j} |\sin(nx)|dx = \sum_{j=1}^{N} |\lambda_j| n^{-1} \int_{2n\pi a_j}^{2n\pi b_j} |\sin(x)|dx. \quad (16.36)$$

Now, for $\lfloor t \rfloor \in \mathbb{Z}$ the integral part of $t \in \mathbb{R}$, defined by $\lfloor t \rfloor \le t < \lfloor t \rfloor + 1$, we have

$$\int_{2n\pi a_j}^{2n\pi b_j} |\sin(x)|dx = \int_{2\pi\lfloor na_j \rfloor}^{2\pi\lfloor nb_j \rfloor} |\sin(x)|dx - \int_{2\pi\lfloor na_j \rfloor}^{2\pi na_j} |\sin(x)|dx + \int_{2\pi\lfloor nb_j \rfloor}^{2\pi nb_j} |\sin(x)|dx.$$

Then

$$\left| \int_{2n\pi a_j}^{2n\pi b_j} |\sin(x)|dx - \int_{2\pi\lfloor na_j \rfloor}^{2\pi\lfloor nb_j \rfloor} |\sin(x)|dx \right| \le 2\pi\left(na_j - \lfloor na_j \rfloor\right) + 2\pi\left(nb_j - \lfloor nb_j \rfloor\right)$$
$$< 2\pi + 2\pi = 4\pi.$$

Going back to (16.36), we conclude that

$$\int_{\mathbb{T}} |\sin(nx)f(x)|dx = \sum_{j=1}^{N} |\lambda_j| n^{-1} \int_{2\pi\lfloor na_j \rfloor}^{2\pi\lfloor nb_j \rfloor} |\sin(x)|dx + o(1)$$
$$= o(1) + \sum_{j=1}^{N} |\lambda_j| n^{-1} 4\left(\lfloor nb_j \rfloor - \lfloor na_j \rfloor\right)$$
$$= o(1) + \sum_{j=1}^{N} |\lambda_j| n^{-1} 4\left[nb_j - na_j - (nb_j - \lfloor nb_j \rfloor) + (na_j - \lfloor na_j \rfloor)\right]$$
$$= o(1) + \sum_{j=1}^{N} |\lambda_j| 4\left(b_j - a_j\right) \xrightarrow{n\to+\infty} \sum_{j=1}^{N} |\lambda_j| 4\left(b_j - a_j\right) = \frac{4}{2\pi}\|f\|_{L^1(\mathbb{T})}.$$

A different take of Exercise 16.34 is follows in the two next exercises.

Exercise 16.35. Show that for any $f \in L^1(\mathbb{T})$ we have $\sigma_n f \xrightarrow{n\to+\infty} f$ for the Fejer series (7.14)

The following implies Exercise 16.34.

Exercise 16.36. Show that for any $f \in L^1(\mathbb{T})$ we have that the sequence $f_n(x) := f(nx)$ we have $f_n \rightharpoonup \widehat{f}(0)$ in $L^1(\mathbb{T})$. Hint, treat first the case of trigonometric polynomials, and then use the approximation in Exercise 16.35 to obtain the result for all $f \in L^1(\mathbb{T})$.

Exercise 16.37. Show that $C := \{f \in L^1(0,1) : \int_0^1 |f|^2 dx \le 1\}$ is a closed subset of $L^1(0,1)$. Show that $\mathring{C} = \emptyset$ in $L^1(0,1)$.

Exercise 16.38. Show that $C := \{f \in L^1(0,1) : \int_0^1 |f|^2 dx \le 1\}$ is a closed subset of $L^1(0,1)$. Show that $\mathring{C} = \emptyset$ in $L^1(0,1)$.

Exercise 16.39. Show that $Tf(x) = x^{-1}\int_0^x f(t)dt$ defines an unbounded linear operator in $L^1(0,1)$.

Example 16.40. Notice that $Tf(x) = x^{-1}\int_0^x f(t)dt$ defines a bounded linear operator in $L^p(0,1)$ for all $1 < p \le +\infty$, and this is part of the famous Hardy inequality. Case $p = \infty$ is trivial. The general case can be seen in a variety of ways. One, is to say that $|Tf| \le 2M(|f|)$ with M the Hardy–Littlewood Maximal function (see next semester), and then use $\|Tf\|_{L^p(0,1)} \le \|M(|f|)\|_{L^p(0,1)} \le C_p \|f\|_{L^p(0,1)}$.
Another possibility is the following direct computation:

$$\begin{aligned}\|x^{-1}\int_0^x f(t)dt\|_{L^p(0,1)} = \|\int_0^1 f(tx)dt\|_{L^p(0,1)} &\le \int_0^1 \|f(\cdot t)\|_{L^p(0,1)} dt \text{ (by Minkowski inequality)} \\ &\le \int_0^1 \|f(\cdot)\|_{L^p(0,1)} t^{-\frac{1}{p}} dt = \frac{1}{1-\frac{1}{p}} \|f\|_{L^p(0,1)}.\end{aligned}$$

Exercise 16.41. Consider a sequence $x. = \{x_n\}_{n\in\mathbb{N}}$. Consider the operator

$$(Tx.)_n = \begin{cases} 0 \text{ if } n = 1 \\ x_{n-1} \text{ if } n \ge 2 \end{cases}. \tag{16.37}$$

a Show that it defines a bounded operator in $\ell^\infty(\mathbb{N})$. Show that $\sigma(T) = \{z \in \mathbb{C} : |z| \le 1\}$.

b Show exactly the same things in $\ell^p(\mathbb{N})$ also for $1 \le p < \infty$.

17 Hilbert spaces

Definition 17.1. A Pre–Hilbert space on $\mathbb{R}$ consists of a vector space H on $\mathbb{R}$ with a symmetric bilinear form $(u, v)_H$, positive, that is $(u, u)_H \ge 0$, and strictly positive, that is $(u, u)_H = 0 \Rightarrow u = 0$. Then $\|u\|_H := \sqrt{(u, u)_H}$ defines a norm, and the space is said Hilbert, if for this norm it is complete.

Definition 17.2. Let X be a vector space on $\mathbb{C}$. A sesquilinear form is a map $B : X \times X \to \mathbb{C}$ such that:

a $B(\lambda x + \mu y, z) = \lambda B(x, z) + \mu B(y, z)$;

b $B(z, \lambda x + \mu y) = \overline{\lambda} B(z, x) + \overline{\mu} B(z, y)$.

A sesquiliner for is said to be Hermitian if additionally:

c $B(x, y) = \overline{B(y, x)}$.

It is positive if

d $B(x, x) \geq 0$ for all $x \in X$.

Nondegenerate if

e if $B(x, x) = 0 \Longrightarrow x = 0$.

Definition 17.3. A Pre–Hilbert space on $\mathbb{C}$ consists of a vector space H on $\mathbb{C}$ with a sesquilinear for $(\cdot, \cdot)_H : H^2 \to \mathbb{C}$ satisfying conditions **a**–**e** in Definition 17.2. $\|u\|_H := \sqrt{(u, u)_H}$ defines a norm, and the space is said Hilbert, if for this norm it is complete.

Example 17.4. Consider

$$\begin{aligned}(f, g)_{L^2(X, d\mu)} &= \int_X f(x)\overline{g(x)} d\mu \text{ and, in particular} \\ (\mathbf{a}, \mathbf{b})_{\ell^2(\mathbb{Z}^d)} &= \sum_{\mathbf{n} \in \mathbb{Z}^d} a_{\mathbf{n}} \overline{b_{\mathbf{n}}}.\end{aligned}$$

They make $L^2(X, d\mu)$ and, in particular, $\ell^2(\mathbb{Z}^d)$, into Hilbert spaces,

Remark 17.5. It is possible to complexify $H, (\cdot, \cdot)_H$ like in Remark 2.13 .

The parallelogram identity is

$$\left\| \frac{a+b}{2} \right\|_H^2 + \left\| \frac{a-b}{2} \right\|_H^2 = \frac{1}{2}(\|a\|_H^2 + \|b\|_H^2), \tag{17.1}$$

and can be obtained by expanding the left hand side, and observing that the mixed terms cancel out. We claim now the Cauchy Schwartz inequality $|(a, b)_H| \leq \|a\|_H \|b\|_H$. Obviously

$$2(a, b)_H + 2(b, a)_H = \|a + b\|_H^2 - \|a - b\|_H^2 \leq 2(\|a\|_H^2 + \|b\|_H^2),$$

so that

$$2 \operatorname{Re}(a, b)_H \leq \|a\|_H^2 + \|b\|_H^2.$$

But

$$2\,\mathrm{Re}(a,b)_H = 2\,\mathrm{Re}\left(\lambda a, \frac{1}{\lambda} b\right)_H \leq \lambda^2 \|a\|_H^2 + \frac{1}{\lambda^2}\|b\|_H^2 \text{ for all } \lambda > 0$$

so $\mathrm{Re}(a,b)_H \leq \|a\|_H\,\|b\|_H$ by taking λ such that $\lambda^2\|a\|_H^2 = \frac{1}{\lambda^2}\|b\|_H^2$, that is $\lambda^2 = \frac{\|b\|_H}{\|a\|_H}$. Notice that

$$\begin{aligned}\|a+b\|_H^2 = (a+b,a+b)_H &= \|a\|_H^2 + \|b\|_H^2 + 2\,\mathrm{Re}(a,b)_H \\ &\leq \|a\|_H^2 + \|b\|_H^2 + 2\|a\|_H\|b\|_H = (\|a\|_H + \|b\|_H)^2.\end{aligned}$$

This proves Minkowski inequality

$$\|a+b\|_H \leq \|a\|_H + \|b\|_H\,.$$

Proposition 17.6. *H Hilbert implies H uniformly convex.*

Recall that H uniformly convex means that for any $\epsilon > 0$ there is $\delta > 0$ such that for any $\|a\|_H \leq 1$, $\|b\|_H \leq 1$ and $\|a-b\|_H > \epsilon$ we have $\left\|\frac{a+b}{2}\right\|_H < 1-\delta$. Now, using the parallelogram identity we get

$$\left\|\frac{a+b}{2}\right\|_H^2 = \frac{1}{2}(\|a\|_H^2 + \|b\|_H^2) - \left\|\frac{a-b}{2}\right\|_H^2 \leq 1 - \frac{\epsilon^2}{4}.$$

and so

$$\left\|\frac{a+b}{2}\right\|_H \leq 1 - (1 - \sqrt{1 - \frac{\epsilon^2}{4}}).$$

Theorem 17.7 (Projection on a closed convex set)**.** *Let $K \subseteq H$ be a closed nonempty convex set and fix $f \in H$. Then there is a unique $u \in K$ such that $\|f-u\|_H \leq \|f-v\|_H$ for all $v \in K$. u is also characterized by*

$$\mathrm{Re}(f-u, v-u)_H \leq 0\,\forall v \in K.$$

Proof. Since the map $\phi(x) = \|x-f\|_H$ is continuous and convex with $\lim \phi(x) = +\infty$ as $\|x\|_H \to \infty$, the existence of a minimizer in K follows from the fact that H is reflexive. However a more direct proof of existence of a minimizer is the following one. We consider a sequence $x_n \in K$ such that $d_n := \|x_n - f\|_H \to d := \min_{x\in K}\|x-f\|_H$. Now by the parallelogram identity applied to $a = f - x_n$ and $b = f - x_m$ we get

$$\left\|f - \frac{x_n + x_m}{2}\right\|_H^2 + \left\|\frac{x_n - x_m}{2}\right\|_H^2 = \frac{1}{2}(d_n^2 + d_m^2).$$

By convexity $\frac{x_n+x_m}{2} \in K$ and so

$$\left\|\frac{x_n - x_m}{2}\right\|_H^2 \leq \frac{1}{2}(d_n^2 + d_m^2) - d^2 \Rightarrow \lim_{m,n\to\infty}\|x_n - x_m\|_H = 0.$$

So x_n is Cauchy and converges to some $u \in K$.

Next step is to show that the characterization holds, that is, u is a minimizer if and only if $\operatorname{Re}(f-u, v-u)_H \leq 0 \, \forall v \in K$. If for a moment we accept this equivalence, then we can see that u is the only minimizer as follows. If we had two minimizers u_1 and u_2, then

$$\begin{aligned} \operatorname{Re}(f-u_1, v-u_1)_H \leq 0 \, \forall v \in K \\ \operatorname{Re}(f-u_2, v-u_2)_H \leq 0 \, \forall v \in K. \end{aligned}$$

In particular

$$\begin{aligned} 0 &\geq \operatorname{Re}(f-u_1, u_2-u_1)_H + \operatorname{Re}(f-u_2, u_1-u_2)_H \\ &= \operatorname{Re}(f-u_1, u_2-u_1)_H - \operatorname{Re}(f-u_2, u_2-u_1)_H = 2\|u_2-u_1\|_H^2 \end{aligned}$$

and hence $u_1 = u_2$. Now let us show the second characterization of the minimizer. Assume u is a minimizer and pick $v \in K$ and consider for $t \in [0,1]$

$$\|f-u-t(v-u)\|_H^2 = \|f-u\|_H^2 - 2t \operatorname{Re}(f-u, v-u)_H + t^2\|v-u\|_H^2.$$

For $t=0$ to be an absolute minimum we need $\operatorname{Re}(f-u, v-u)_H \leq 0$, so u minimizer implies $\operatorname{Re}(f-u, v-u)_H \leq 0 \, \forall v \in K$. Viceversa, assuming this latter property, for any $v \in K$

$$\begin{aligned} &\|u-f\|_H^2 - \|v-f\|_H^2 = \|u\|_H^2 + 2(v-u, f)_H - \|v\|_H^2 \\ &= 2\operatorname{Re}(v-u, f-u)_H + 2\operatorname{Re}(v-u, u)_H + \|u\|_H^2 - \|v\|_H^2 \\ &= 2\operatorname{Re}(v-u, f-u)_H - \|u-v\|_H^2 \leq -\|u-v\|_H^2 \end{aligned}$$

In particular $\|u-f\|_H < \|v-f\|_H$ unless $u = v$. □

Proposition 17.8. *Let $K \subseteq H$ be a closed nonempty convex set and for any $f \in H$ let $P_K f \in K$ the corresponding projection in K. Then P_K is a contraction:*

$$\|P_K f - P_K g\|_H \leq \|f-g\|_H.$$

Proof. Let $u = P_K f$ and $v = P_K g$. Then

$$\begin{aligned} \operatorname{Re}(f-u, w-u)_H \leq 0 \, \forall w \in K \\ \operatorname{Re}(g-v, w-v)_H \leq 0 \, \forall w \in K. \end{aligned}$$

Then $\operatorname{Re}(f-u, v-u)_H \leq 0$ and $\operatorname{Re}(g-v, u-v)_H \leq 0$ and, adding up,

$$\begin{aligned} 0 &\geq \operatorname{Re}(f-u, v-u)_H + \operatorname{Re}(g-v, u-v)_H = \operatorname{Re}(f-u, v-u)_H - \operatorname{Re}(g-v, v-u)_H \\ &= \operatorname{Re}(f-g, v-u)_H + \operatorname{Re}(v-u, v-u)_H \end{aligned}$$

So $\|v-u\|_H^2 \leq \operatorname{Re}(f-g, u-v)_H \leq \|f-g\|_H \, \|v-u\|_H$ and so $\|v-u\|_H \leq \|f-g\|_H$. □

Corollary 17.9. *Let $K \subseteq H$ be a closed vector subspace. Then $u = P_K f \in K$ is characterized by $(f-u, v)_H = 0$ for all $v \in K$. Furthermore, P_K is a bounded linear operator.*

Proof. The characterization $\mathrm{Re}(f-u, v-u)_H \leq 0$ for all $v \in K$ and so by $K = K - u$, $\mathrm{Re}(f-u, v)_H \leq 0$ for all $v \in K$ and the fact that $v \in K$ implies $-v \in K$, yield $\mathrm{Re}(f - u, v)_H = 0$ for all $v \in K$, and in fact also $(f-u, v)_H = 0$ for all $v \in K$. If $(P_K u - u, w)_H = 0$ and $(P_K v - v, w)_H = 0$ for all $w \in K$, then

$$\lambda(P_K u - u, w)_H + \mu(P_K v - v, w)_H = (\lambda P_K u + \mu P_K v - (\lambda u + \mu v), w)_H = 0\, \forall w \in K.$$

But this means $P_K(\lambda u + \mu v) = \lambda P_K u + \mu P_K v$ so P_K is linear. We know P_K is continuous. □

Theorem 17.10 (Riesz Frechet)**.** *Let $f \in H'$. Then there is $y \in H$ such that $\langle f, x\rangle_{H' \times H} = (x, y)_H$ for all $x \in H$. Furthermore, $\|f\|_{H'} = \|y\|_H$.*

Proof. The map $T : H \to H'$ defined by $y \to (\cdot, y)_H$ is continuous. By $|\langle Ty, x\rangle_{H' \times H}| = |(x, y)_H| \leq \|y\|_H \|x\|_H$ we get $\|Ty\|_{H'} \leq \|y\|_H$, and by $\|y\|_H^2 = \langle Ty, y\rangle_{H' \times H} \leq \|Ty\|_{H'} \|y\|_H$ we get $\|Ty\|_{H'} \geq \|y\|_H$. So, in particular, T is an isometry and $T(H)$ is closed in H'. If $T(H) \neq H'$, there is by Hahn Banach a $h \in H''$ such that $\langle Ty, h\rangle_{H' \times H''} = 0$ for all $y \in H$. But H is reflexive, so $h = Jx$ for some $x \in H$ and $\langle Ty, h\rangle_{H' \times H} = \langle Ty, Jx\rangle_{H' \times H''} = (x, y)_H = 0$. Picking $y = x$ we get $x = 0$, and so also $h = 0$. □

Definition 17.11. A subset $S \subset H$ is called orthonormal if $\|x\|_H = 1$ for all $x \in H$ and $(x, y)_H = 0$ for any pair $x \neq y$ of elements in S.

Theorem 17.12. *Let $S \subset H$ be orthonormal. Then the following hold.*

1 For any $u \in H$ we have

$$\sum_{s \in S} |(u, s)_H|^2 \leq \|u\|_H^2 \text{ (Bessel Inequality).} \tag{17.2}$$

2 Let V_S be the closure of the subspace of H spanned by S. The following are equivalent:

a) $u \in V_S$;

b) $\sum_{s \in S} |(u, s)_H|^2 = \|u\|_H^2$;

c) The series $\sum_{s \in S} (u, s)_H s$ is convergent in H with limit u.

3 For any $u \in H$ the series $\sum_{s \in S} (u, s)s$ is convergent in V_S with limit $P_{V_S} u$ and we have

$$\sum_{s \in S} |(u, s)|^2 = \|P_{V_S} u\|_H^2 \text{ (Parseval Identity).} \tag{17.3}$$

Proof. Let, to begin with, S be at most numerable. We will suppose S is exactly numerable and we will write the elements of S as $\{s_j\}_{j\in\mathbb{N}}$. Consider $s_1, ..., s_n$ and let

$$S_n u := \sum_{j=1}^{n} (u, s_j)_H s_j. \tag{17.4}$$

Then

$$\|S_n u\|_H^2 = \sum_{j=1}^{n} |(u, s_j)_H|^2. \tag{17.5}$$

and

$$\|u - S_n u\|_H^2 = \|u\|_H^2 - 2\,\mathrm{Re}(u, S_n u)_H + \|S_n u\|_H^2 = \|u\|_H^2 - \|S_n u\|_H^2 \tag{17.6}$$

by $(u, S_n u)_H = \sum_{j=1}^{n} |(u, s_j)_H|^2$, which follows from $(u, u - S_n u)_H = 0$. Hence $\|S_n u\|_H \le \|u\|_H$. Then we conclude

$$\sum_{j=1}^{\infty} |(u, s_j)_H|^2 \le \|u\|_H^2, \tag{17.7}$$

in the case S countable. Obviously, also the case S finite set is proved.
Let us assume now that S is infinite with cardinality strictly larger than card($\mathbb{N}$). Let $\widehat{S} = \{s \in S : (u, s)_H \neq 0\}$. If card($\widehat{S}$) $\le$card($\mathbb{N}$) there is nothing more to prove. Let card($\widehat{S}$) $>$card($\mathbb{N}$). Then it is not restrictive to assume $\widehat{S} = S$.
For any $m \in \mathbb{N}$ let $S(m) = \{s \in S : |(u, s)_H| > 1/m\}$. It is immediately clear that $S(m)$ is a finite set, since otherwise we could consider a sequence of distinct terms $\{s_j\}_{j\in\mathbb{N}}$ which from (17.7) satisfies

$$+\infty = \sum_{j=1}^{\infty} \frac{1}{m^2} \le \sum_{j=1}^{\infty} |(u, s_j)_H|^2 \le \|u\|_H^2 < \infty,$$

which is obviously absurd. But from $S = \cup_{m\in\mathbb{N}} S(m)$ and card($S(m)$) $< \infty$ for any m imply card(S) $\le$card($\mathbb{N}$) yielding a contradiction. This completes the proof of the 1st claim of Theorem 17.12.
Let us turn to the 2nd claim. Let $u \in V_S$. For any $\epsilon > 0$ there exists $s_{\sigma_1}, ..., s_{\sigma_k} \in S$ and $\lambda_1, ..., \lambda_k \in K$ such that $\|u - \sum_{l=1}^{k} \lambda_j s_{\sigma_l}\|_H < \epsilon$. Collecting all these s_{σ_l} for a sequence $\epsilon \searrow 0$, we see that an at most countable subset S' of S remains defined, such that $u \in V_{S'}$. So, it is not restrictive to assume that the initial S is at most countable. Then we can write $S = \{s_j\}_{j\in J}$, with J either finite or countable. For definiteness, let $J = \mathbb{N}$. Then $u \in V_S$ implies that for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ and $\lambda_1, ..., \lambda_n \in K$ such that

$\|u - \sum_{j=1}^{n} \lambda_j s_j\|_H < \epsilon$. We have, for $S_n u := \sum_{j=1}^{n} (u, s_j)_H s_j$,

$$\left\| u - \sum_{j=1}^{n} \lambda_j s_j \right\|_H^2 = \left\| u - \sum_{j=1}^{n} (u, s_j)_H s_j + \sum_{j=1}^{n} \left((u, s_j)_H - \lambda_j \right) s_j \right\|_H^2$$
$$= \|u - S_n u\|_H^2 + \sum_{j=1}^{n} |(u, s_j)_H - \lambda_j|^2 \geq \|u - S_n u\|_H^2 .$$

which shows that

$$\|u - S_n u\|_H \leq \left\| u - \sum_{j=1}^{n} \lambda_j s_j \right\|_H < \epsilon.$$

Notice that the above implies also

$$\|u - S_m u\|_H \leq \left\| u - \sum_{j=1}^{m} \lambda_j s_j \right\|_H < \epsilon \text{ for all } m > n$$

just by setting $\lambda_j = 0$ for $n < j \leq m$. Then

$$\|u - S_n u\|_H \xrightarrow{n \to +\infty} 0 \Longleftrightarrow u = \sum_{j=1}^{\infty} (u, s_j)_H s_j.$$

It follows that, c) must be true. Obviously c) implies a). Next, if c) is true, from $S_n u \xrightarrow{n \to +\infty} u$ in H we have $\|S_n u\|_H^2 \xrightarrow{n \to +\infty} \|u\|_H^2$. So, since

$$\|S_n u\|_H^2 = \sum_{j=1}^{n} |(u, s_j)_H|^2 \text{ we get } \sum_{j=1}^{\infty} |(u, s_j)_H|^2 = \|u\|_H^2,$$

hence proving c)$\Longrightarrow$b).
Now, let us assume b). By (17.5) and (17.6) we have

$$\|u - S_n u\|_H^2 = \|u\|_H^2 - \|S_n u\|_H^2 = \|u\|_H^2 - \sum_{j=1}^{n} |(u, s_j)_H|^2 \xrightarrow{n \to +\infty} 0,$$

where the limit holds by b), obviously proving $S_n u \xrightarrow{n \to +\infty} u$ in H, and thus c).

Notice that for $u \in H$ with have $P_{V_S} u \in V_S$ and $(u, s)_H = (P_{V_S} u, s)_H$ for any $s \in S$. So in particular c) is true for $P_{V_S} u$ yielding

$$P_{V_S} u = \sum_{s \in S} (P_{V_S} u, s)_H s \text{ in } H$$

(17.3) follows by b). The proof of Theorem 17.12 is completed. □

Definition 17.13. Given a Hilbert space H, an orthonormal basis is an orthonormal subset $S \subset H$ such that $V_S = H$.

Theorem 17.14. *Every Hilbert space H admits an orthonormal basis.*

Proof. It can be proved using Zorn's Lemma. In fact, consider

$$\mathfrak{S} := \{S : S \text{ is an orthonormal subset of } H\},$$

with the order relation $\subseteq$. Notice that $\mathfrak{S}$ is inductive, that is given a totally ordered set $\mathfrak{Q} \subseteq \mathfrak{S}$, then $\widetilde{S} = \cup_{S \in \mathfrak{Q}} S$ is an upper bound of $\mathfrak{Q}$. By Zorn's Lemma, there is a maximal element $S \in \mathfrak{S}$. If $V_S \subsetneqq H$, let $H \ni u \notin V_S$. Then setting, $v = \frac{u - P_{V_S} u}{\|u - P_{V_S} u\|_H}$ we have $S_1 := \{v\} \cup S \supsetneqq S$ is an orthonormal set strictly larger than S, which is absurd. So $V_S = H$. $\square$

Example 17.15. Consider the set $S := \left\{ \dfrac{e^{\mathrm{i}\ell \cdot x}}{(2\pi)^{\frac{d}{2}}} : \ell \in \mathbb{Z}^d \right\}$. It is easy to conclude that it is an orthonormal subset in $L^2(\mathbb{T}^d)$. We claim it is an orthonormal basis. To see this, notice from $L^2(\mathbb{T}^d) = \overline{L^2(\mathbb{T})^{\bigotimes_{\mathbb{C}} d}}$ that it is enough to prove this for $d = 1$. Since $\overline{C^0(\mathbb{T})} = L^2(\mathbb{T})$, it is enough to prove that $C^0(\mathbb{T}) \subseteq V_S$. Recall from (7.21) that for any $f \in C^0(\mathbb{T})$ we have $\sigma_n f \xrightarrow{n \to +\infty} f$ in $C^0(\mathbb{T})$, for the Féjer sequence $\sigma_n f$: obviously, this implies convergence also in the weaker topology of $L^2(\mathbb{T})$. On the other hand, any $\sigma_n f$ is a trigonometric polynomial and so we have $\sigma_n f \in V_S$. Hence $f \in V_S$ for any $f \in C^0(\mathbb{T})$.

Having proved that S is an orthonormal basis of $L^2(\mathbb{T}^d)$, we have from Parseval Identity

$$\sum_{\ell \in \mathbb{Z}^d} \left| \left(u, \frac{e^{\mathrm{i}\ell \cdot x}}{(2\pi)^{\frac{d}{2}}} \right) \right|^2 = \sum_{\ell \in \mathbb{Z}^d} \left| \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{T}^d} u(x) e^{-\mathrm{i}\ell \cdot x} dx \right|^2 = (2\pi)^d \sum_{\ell \in \mathbb{Z}^d} |\widehat{u}(\ell)|^2 = \|u\|^2_{L^2(\mathbb{T}^d)} \tag{17.8}$$

and in $L^2(\mathbb{T}^d)$ we have $u = \displaystyle\sum_{\ell \in \mathbb{Z}^d} \widehat{u}(\ell) \frac{e^{\mathrm{i}\ell \cdot x}}{(2\pi)^{\frac{d}{2}}}$.

Notice that we got an 1–1 map $L^2(\mathbb{T}^d) \hookrightarrow \ell^2(\mathbb{Z}^d)$ which is an isometry with the image, and since this image is dense in $\ell^2(\mathbb{Z}^d)$ (since it contains all the trigonometric polynomials) and is complete (being isometric to the Hilbert space $L^2(\mathbb{T}^d)$) we have an isomorphism $L^2(\mathbb{T}^d) \ni f \to \widehat{f} \in \ell^2(\mathbb{Z}^d)$.

Exercise 17.16. Show that, if $\{e_n\}$ is an orthonormal basis of a separable Hilbert space H, we have $e_n \rightharpoonup 0$.

Exercise 17.17. Show that, if $\{e_n\}$ is an orthonormal sequence in a Hilbert space H, we have $e_n \rightharpoonup 0$.

Remark 17.18. Notice, by Exercise 11.28, that there is no sequence f_n in $\ell^1(\mathbb{N})$ with $\|f_n\|_{\ell^1(\mathbb{N})} = 1$ and $f_n \rightharpoonup 0$.

Lemma 17.19. *Consider $f, g \in L^1(\mathbb{T}^d)$. Then we have*

$$\widehat{f * g}(\mathbf{n}) = (2\pi)^d \widehat{f}(\mathbf{n})\widehat{g}(\mathbf{n}). \tag{17.9}$$

Proof. We have

$$\widehat{f * g}(\mathbf{n}) = (2\pi)^{-d} \int_{\mathbb{T}^d} e^{-\mathrm{i}\mathbf{n}\cdot x} f * g(x) dx = (2\pi)^{-d} \int_{\mathbb{T}^d} dx e^{-\mathrm{i}\mathbf{n}\cdot x} \int_{\mathbb{T}^d} f(x-y)g(y)dy$$
$$= (2\pi)^{-d} \int_{\mathbb{T}^d \times T^d} e^{-\mathrm{i}\mathbf{n}\cdot(x-y)} e^{-\mathrm{i}\mathbf{n}\cdot y} f(x-y)g(y)dxdy = (2\pi)^d \widehat{f}(\mathbf{n})\widehat{g}(\mathbf{n}).$$

□

Exercise 17.20. Consider a $\rho \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ s.t. $\int \rho(x)dx = 1$ and set $\rho_\epsilon(x) := \epsilon^{-d}\rho(x/\epsilon)$. Show that in the space

$$C_0^0(\mathbb{R}^d) := \{f \in C^0(\mathbb{R}^d, \mathbb{R}) : \lim_{x\to\infty} f(x) = 0\} \subseteq L^\infty(\mathbb{R}^d)$$

we have $\rho_\epsilon * f \xrightarrow{\epsilon\to 0^+} f$.

Exercise 17.21. Show that it is not true that $\rho_\epsilon * f \xrightarrow{\epsilon\to 0^+} f$ for all f in the space $BC^0(\mathbb{R}^d) := C^0(\mathbb{R}^d, \mathbb{R}) \cap L^\infty(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$.

This exercise is better understood using the Fourier transform. Consider the formula

$$\rho_\epsilon * f(x) - f(x) = \int_{\mathbb{R}^d} \epsilon^{-d}\rho(y/\epsilon)\left(f(x-y) - f(x)\right) dy$$

and, for $\mathbf{n} \in \mathbb{Z}^d$, let us apply this formula to

$$\rho_\epsilon * \left(e^{\mathrm{i}\mathbf{n}\cdot\sqcup} f(\sqcup)\right)(x) - e^{\mathrm{i}\mathbf{n}\cdot x} f(x) = e^{\mathrm{i}\mathbf{n}\cdot x} \int_{\mathbb{R}^d} \epsilon^{-d}\rho(y/\epsilon)\left(e^{-\mathrm{i}\mathbf{n}\cdot y} f(x-y) - f(x)\right) dy$$

Now, we can suppose that $\rho \in C_c^\infty((-\pi,\pi)^d, \mathbb{R})$, so that in the right we have a Fourier coefficient. By the Riemann–Lebesgue lemma we have, for fixed ϵ,

$$\left|\rho_\epsilon * \left(e^{\mathrm{i}\mathbf{n}\cdot\sqcup} f(\sqcup)\right)(x) - e^{\mathrm{i}\mathbf{n}\cdot x} f(x)\right|$$
$$= \left|\int_{\mathbb{R}^d} \epsilon^{-d}\rho(y/\epsilon)\left(e^{-\mathrm{i}\mathbf{n}\cdot y} f(x-y) - f(x)\right) dy\right| \xrightarrow{\mathbf{n}\to+\infty} |f(x)|.$$

Now consider $\varphi \in C_c^\infty((-1/2, 1/2)^d, \mathbb{R})$, for simplicity even, and consider the function

$$f(x) = \sum_{n\in\mathbb{Z}} e^{inx_1}\varphi(x - ne_1), \text{ with } e_1 = (1, 0, ..., 0) \text{ and with } x_1 = x \cdot e_1.$$

Then

$$\rho_\epsilon * f(x) - f(x) = \sum_{n\in\mathbb{Z}} e^{inx_1} \int_{\mathbb{R}^d} \epsilon^{-d}\rho(y/\epsilon)\left(e^{-iny_1}\varphi(x - y - ne_1) - \varphi(x - ne_1)\right) dy$$

and fixing $m \in \mathbb{Z}$ we get

$$\rho_\epsilon * f(me_1) - f(me_1) = \sum_{n\in\mathbb{Z}} e^{inm} \int_{\mathbb{R}^d} \epsilon^{-d}\rho(y/\epsilon) \left(e^{-iny_1}\varphi((m-n)e_1 - y) - \varphi((m-n)e_1)\right) dy$$

$$= e^{\mathrm{i}m^2} \int_{\mathbb{R}^d} \epsilon^{-d}\rho(y/\epsilon) \left(e^{-\mathrm{i}ny_1}\varphi(-y) - \varphi(0)\right) dy = e^{\mathrm{i}m^2} \int_{\mathbb{R}^d} \epsilon^{-d}\rho(y/\epsilon) e^{-\mathrm{i}my_1}\varphi(-y)dy - e^{\mathrm{i}m^2}\varphi(0)$$

where we used the fact that

$$\begin{aligned}
&\text{supp } e^{-ine_1\cdot\sqcup}\varphi(-\sqcup - (n-m)e_1) = (n-m)e_1 + \text{supp } \varphi \subset (n-m)e_1 + (-1/2, 1/2)^d \\
&\text{supp } \rho_\epsilon = \epsilon\text{supp } \rho \subset D_{\mathbb{R}^d}(0,\epsilon) \text{ and} \\
&(n-m)e_1 + (-1/2,1/2)^d \cap D_{\mathbb{R}^d}(0,\epsilon) = \emptyset \text{ if } n \neq m \text{ and } 0 < \epsilon < 1/2,
\end{aligned}$$

since $x \in (n-m)e_1 + (-1/2,1/2)^d$ implies $|x| \geq |n-m| - 1/2$.
Now again, by Riemann–Lebesgue we get, for any given $0 < \epsilon < 1/2$,

$$|\rho_\epsilon * f(me_1) - f(me_1)| \xrightarrow{m\to+\infty} |\varphi(0)|.$$

For $\varphi(0) \neq 0$ this implies that $\|\rho_\epsilon * f - f\|_{L^\infty(\mathbb{R}^d)} \geq \sup_{m\in\mathbb{Z}} |\rho_\epsilon * f(me_1) - f(me_1)| \geq |\varphi(0)| > 0$ and so $\|\rho_\epsilon * f - f\|_{L^\infty(\mathbb{R}^d)} \not\xrightarrow{\epsilon\to 0^+} 0$.

Exercise 17.22. Find the spectrum $\sigma(T)$ of the operator $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ given by $(T\mathbf{x})_k = x_{k-1}$ for all $k \in \mathbb{Z}$, where $\mathbf{x} = (x_k)_{k\in\mathbb{Z}}$.

Answer. Notice that

$$(2\pi)^{-1} \sum_{k\in\mathbb{Z}} (T\widehat{f})_k e^{\mathrm{i}kx} = (2\pi)^{-1} \sum_{k\in\mathbb{Z}} \widehat{f}(k-1) e^{\mathrm{i}kx} = e^{\mathrm{i}x} f(x),$$

so that we conclude that

$$T\widehat{f} = \widehat{e^{\mathrm{i}x} f}.$$

This means that the $\sigma(T)$ coincides with the spectrum of the multiplier operator $f \to e^{\mathrm{i}x} f$ inside $L^2([-\pi,\pi])$. The latter spectrum is $\{e^{\mathrm{i}x} : x \in [-\pi,\pi]\}$ =the unitary circle centered at the origin. □

Example 17.23. The operator defined in $\ell^2(\mathbb{Z}^d)$ by

$$\triangle u(\mathbf{n}) = \sum_{\substack{\mathbf{m}\in\mathbb{Z}^d \\ |\mathbf{n}-\mathbf{m}|=1}} u(\mathbf{m}) - 2du(\mathbf{n}) \text{ (where } |\mathbf{n}-\mathbf{m}| = \sum_{j=1}^d |n_j - m_j|) \tag{17.10}$$

is a discrete version of the Laplacian (the finite differences Laplacian). It is a bounded operator in $\ell^2(\mathbb{Z}^d)$.Keeping in mind the isomorphism $L^2(\mathbb{T}^d) \to \ell^2(\mathbb{Z}^d)$, see Example 17.15, we have

$$
\begin{aligned}
(2\pi)^{-d} \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{\mathrm{i}\mathbf{n}\cdot x} \triangle \widehat{f}(\mathbf{n}) &= (2\pi)^{-d} \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{\mathrm{i}\mathbf{n}\cdot x} \left(\sum_{\substack{\mathbf{m}\in\mathbb{Z}^d \\ |\mathbf{n}-\mathbf{m}|=1}} \widehat{f}(\mathbf{m}) - \widehat{f}(\mathbf{n}) 2d \right) \\
&= (2\pi)^{-d} \sum_{\mathbf{m}\in\mathbb{Z}^d} e^{\mathrm{i}\mathbf{m}\cdot x} \widehat{f}(\mathbf{m}) \left(\sum_{\substack{\mathbf{n}\in\mathbb{Z}^d \\ |\mathbf{n}-\mathbf{m}|=1}} e^{\mathrm{i}(\mathbf{n}-\mathbf{m})\cdot x} - 2d \right) \\
&= (2\pi)^{-d} \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{\mathrm{i}\mathbf{n}\cdot x} \widehat{f}(\mathbf{n}) \left(\sum_{\substack{\mathbf{m}\in\mathbb{Z}^d \\ |\mathbf{m}|=1}} \cos(\mathbf{m}\cdot x) - 2d \right) \\
&= \phi(x) f(x) \text{ , where } \phi(x) := \sum_{\substack{\mathbf{m}\in\mathbb{Z}^d \\ |\mathbf{m}|=1}} \cos(\mathbf{m}\cdot x) - 2d = 2\sum_{j=1}^{d} \psi(x_j)
\end{aligned}
$$

where $\psi(x_j) = \cos(x_j) - 1$. Here $\phi(\mathbb{T}^d) = [-4d, 0]$. Notice that we have shown

$$\triangle \widehat{f}(\mathbf{n}) = \widehat{\phi f}(\mathbf{n}).$$

Up to a conjugation by an isomorphism, the map (17.10) is equal to the multiplier operator $f \to \phi f$. These two operators have the same spectrum and so, recalling Exercise 5.19, we have $\sigma(\triangle) = [-4d, 0]$. Notice that there are no eigenvalues.

Remark 17.24 (Schmidt's Orthogonalization). Given a finite or countable sequence $\{f_j\}$ sequence of linearly independent elements of a pre–Hilbert space H, then there exists an orthonormal set S spanning the same linear space of $\{f_j\}$

Indeed, setting $h_1 = f_1$ and $g_1 = h_1/\|h_1\|_H$ and by recurrence

$$h_n = f_n - \sum_{j=1}^{n-1} (f_n, g_j) g_j \text{ and } g_n = h_n/\|h_n\|_H,$$

It is easy to see by induction that for any n, $\mathrm{Span}\{f_1, ..., f_n\}$ $=\mathrm{Span}\{g_1, ..., g_n\}$ and that $\{g_1, ..., g_n\}$ is an orthonormal set. The statement follows.

Example 17.25. By the Weierstrass Approximation Theorem we know that the span of $\{1, t, t^2, t^3, ...\}$ is dense in $C^0([a, b], \mathbb{R})$ for any closed interval $[a, b]$, and so in particular is also dense in $L^2([a, b], \alpha(t)dt)$, for $\alpha(t) \in L^1([a, b])$. If we consider Schmidt's Orthogonalization $\{P_0(t), P_1(t), P_2(t), P_3(t), ...\}$ we obtain the *Tchebyschev system of orthogonal polynomials* in $L^2([a, b])$.

Notice that if take away any finite number N of elements from $\{1, t, t^2, t^3, ...\}$, then its span is not dense in $L^2([a, b], dt)$ and in fact its closure has codimension equal to N. This is conspicuously different to what happens in $C^0([a, b])$, see Example 6.21.

Exercise 17.26. Show that if $x_n \rightharpoonup x$ in a Hilbert space H and $\|x_n\|_H \xrightarrow{n\to+\infty} \|x\|_H$, then $x_n \xrightarrow{n\to+\infty} x$ strongly in H.

Remark 17.27. The above statement continues to be true for uniformly convex Banach spaces, see Proposition 3.32 [3].

Exercise 17.28. Let H be a Hilbert space and $Y \subsetneqq H$ a proper, nontrivial closed subspace.

a Show in an elementary fashion, without resorting to Corollary 6.2, that for any $y' \in Y'$ there exists an extension $h' \in H'$ of y' with $\|h'\|_{H'} = \|y'\|_{Y'}$.

b How many such extensions exist?

Answer. Consider the orthogonal decomposition $H = Y \oplus Y^\perp$. Recall that we have isometric isomorphisms $H \ni x \to (\cdot, x)_H \in H'$, $Y \ni y \to (\cdot, y)_H \in Y'$ and $Y^\perp \ni y_\perp \to (\cdot, y_\perp)_H \in (Y^\perp)'$. So any functional $g \in Y'$ can be identified with a $y \in Y$, with $\|g\|_{Y'} = \|y\|_H$. Obviously $y \in H$ defines a functional $(\cdot, y)_H \in H'$ which is an extension of g and has the same norm of g.

The question is if there are other possible extensions, and the answer is no. Any such extension $f \in H'$ of $g = (\cdot, y)_H$ would have to be of the form $f = (\cdot, x)_H$ for some $x \in H$. It is elementary that with respect to the splitting $H = Y \oplus Y^\perp$ we would have $x = y + y_\perp$ with some $y_\perp \in Y^\perp$ and with the same $y \in Y$ of $g = (\cdot, y)_H$. It is elementary that

$$\|x\|_H = \sqrt{\|y\|_H^2 + \|y_\perp\|_H^2}.$$

So, from

$$\sqrt{\|y\|_H^2 + \|y_\perp\|_H^2} = \|x\|_H = \|f\|_{H'} = \|g\|_{Y'} = \|y\|_H.$$

we conclude that we must have $\|y_\perp\|_H = 0$. □

17.1 Operators in Hilbert spaces

Definition 17.29. Given a Hilbert space H, for any $T \in \mathcal{L}(H)$ it remains defined another operator $T^* \in \mathcal{L}(H)$ such that

$$(Tx, y)_H = (x, T^*y)_H \text{ for all } x, y \in H \tag{17.11}$$

T is called symmetric or selfadjoint if $T = T^*$.
T is called unitary if it is an isometric isomorphism.

Remark 17.30. Notice that in the very important case of *unbounded* operators, the two notions of symmetric and of selfadjoint operator do not coincide.

Exercise 17.31. Show that T^* is well defined, that $T^{**} = T$, that $\|T\|_{\mathcal{L}(H)} = \|T^*\|_{\mathcal{L}(H)}$ and that $\|T^*T\|_{\mathcal{L}(H)} = \|TT^*\|_{\mathcal{L}(H)} = \|T\|_{\mathcal{L}(H)}^2$.

Exercise 17.32. Show that if T is unitary, then $T^* = T^{-1}$.

Definition 17.33. Given a Hilbert space H, an operator $T \in \mathcal{L}(H)$ is positive if

$$(Tx, x)_H \geq 0 \text{ for all } x \in H. \tag{17.12}$$

We write $T \geq 0$.
Given $T, S \in \mathcal{L}(H)$, we write $T \geq S$ if $T - S \geq 0$.

Remark 17.34. It is easy to see using an appropriate *polarization* that if H is a Hilbert space on $\mathbb{C}$, then $A \in \mathcal{L}(H)$ with $A \geq 0$ implies $A = A^*$.

Exercise 17.35. Show that $T^*T \geq 0$ and $TT^* \geq 0$.

Lemma 17.36. *For $T \in \mathcal{L}(H)$ selfadjoint, consider the orthogonal decomposition*

$$H = \ker T \oplus \ker^{\perp} T. \tag{17.13}$$

Then the above decomposition is T–invariant and, furthermore, we have

$$\ker^{\perp} T = \overline{R(T)}. \tag{17.14}$$

Proof. The invariance is elementary and left as an exercise, while (17.14) follows from (6.40), after the identification $H = H' = H''$. This can also be seen in an elementary fashion and from scratch, from

$$(x, Ty)_H = (Tx, y)_H \text{ for all } x, y \in H,$$

where we see that if $x \in \ker T$, then the above is zero for any y, which tells us $\ker^{\perp} T \supseteq \overline{R(T)}$. Viceversa, if we had $\ker^{\perp} T \supsetneq \overline{R(T)}$, there would be a $z \in \ker^{\perp} T \backslash \overline{R(T)}$. Furthermore, we could take $z \in \overline{R(T)}^{\perp}$. Then we would get $z \in \ker T$ which would imply $0 = (z, z)_H = \|z\|_H^2$ which implies $z = 0$, yielding a contradiction.

□

Remark 17.37. The decomposition (17.15) extends to a general and not necessarily selfadjoint $T \in \mathcal{L}(H)$ as

$$H = N_g(T) \oplus (N_g(T^*))^{\perp}, \tag{17.15}$$

where $N_g(T)$ is the *generalized kernel*, see formula (5.20).

Lemma 17.38. *The McLaurin series of $\sqrt{1-z}$ is absolutely convergent for all $|z|_{\mathbb{C}} \leq 1$.*

Proof. The series is

$$\sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} z^n \text{ , where } \binom{\frac{1}{2}}{n} = \frac{\prod_{j=1}^{n} \left(\frac{1}{2} - (j-1)\right)}{n!},$$

and has radius of convergence 1, so that it is absolute convergent for $|z|_{\mathbb{C}} < 1$. Let us see case $|z|_{\mathbb{C}} = 1$. By direct inspection, we have $(-1)^n\binom{\frac{1}{2}}{n} < 0$ for all $n \geq 1$. Then, for any $N \in \mathbb{N}$,

$$
\begin{aligned}
&\sum_{n=0}^{N} \left|(-1)^n \binom{\frac{1}{2}}{n}\right| = 2 - \sum_{n=0}^{N}(-1)^n \binom{\frac{1}{2}}{n} = 2 - \lim_{x \to 1^-} \sum_{n=0}^{N}(-1)^n \binom{\frac{1}{2}}{n} x^n \\
&\leq 2 - \lim_{x \to 1^-} \sum_{n=0}^{\infty}(-1)^n \binom{\frac{1}{2}}{n} x^n = 2 - \lim_{x \to 1^-} \sqrt{1-x} = 2.
\end{aligned}
$$

This implies the following, which completes the proof,

$$
\sum_{n=0}^{\infty} \left|(-1)^n \binom{\frac{1}{2}}{n}\right| \leq 2.
$$

□

Theorem 17.39 (Square root of a positive operator)**.** *Let $A \in \mathcal{L}(H)$ with $A \geq 0$ and let $A = A^*$. Then there exists and is unique a $B \in \mathcal{L}(H)$ with $B \geq 0$ and selfadjoint such that $B^2 = A$.*

Proof. First of all, it is not restrictive to assume $\|A\|_{\mathcal{L}(H)} \leq 1$. Next, we define

$$
(1-A)^{\frac{1}{2}} = \sum_{n=0}^{\infty}(-1)^n \binom{\frac{1}{2}}{n} A^n, \tag{17.16}
$$

where the series is convergent in $\mathcal{L}(H)$. We skip the proof that $\left((1-A)^{\frac{1}{2}}\right)^2 = 1 - A$. It is straightforward, using the series (17.16), that $(1-A)^{\frac{1}{2}}$ is selfadjoint. Similarly straightforward is the fact that $(1-A)^{\frac{1}{2}}$ commutes with A. Then we can write

$$
A(1-A) = (1-A)^{\frac{1}{2}} A (1-A)^{\frac{1}{2}}.
$$

We next claim [2]

$$
\|1 - A\|_{\mathcal{L}(H)} \leq 1 \tag{17.17}
$$

This follows from

$$
\begin{aligned}
&\|(1-A)x\|_H^2 = ((1-A)x, (1-A)x)_H = (x, (1-A)x)_H - (A(1-A)x, x)_H \\
&= (x, (1-A)x)_H - \left((1-A)^{\frac{1}{2}} A (1-A)^{\frac{1}{2}} x, x\right)_H \\
&= (x, (1-A)x)_H - \left(A(1-A)^{\frac{1}{2}} x, (1-A)^{\frac{1}{2}} x\right)_H \leq (x, (1-A)x)_H \leq \|(1-A)x\|_H \|x\|_H,
\end{aligned}
$$

[2] Operators which satisfy (17.17) are called *accretive* operators, which is an alternative to the notion of positive operator in Definition 17.33. Notice that (17.17) makes sense in a general Banach space.

where we used the fact that the operators are selfadjoint and A is positive. Claim (17.17) follows immediately.
Notice that by $\|A\|_{\mathcal{L}(H)} \le 1$ we have

$$((1-A)x, x)_H = \|x\|_H^2 - (Ax, x)_H \ge 0$$

and so $1 - A \ge 0$.
We also claim that $\sqrt{1-A} \ge 0$. Indeed, using $0 \le (A^n x, x)_H \le \|x\|_H^2$, we have

$$\begin{aligned}\left(\sqrt{1-A}x, x\right)_H &= \|x\|_H^2 + \sum_{n=1}^{N}(-1)^n \binom{\frac{1}{2}}{n} (A^n x, x)_H \\ &\ge \|x\|_H^2 + \|x\|_H^2 \sum_{n=1}^{N}(-1)^n \binom{\frac{1}{2}}{n} = \|x\|_H^2 \left(2 - \sum_{n=2}^{N}(-1)^n \binom{\frac{1}{2}}{n}\right) \ge 0.\end{aligned}$$

Thanks to (17.17), we can consider

$$A^{\frac{1}{2}} := (1-(1-A))^{\frac{1}{2}} = \sum_{n=0}^{\infty}(-1)^n \binom{\frac{1}{2}}{n}(1-A)^n \tag{17.18}$$

which has the desired properties.
Notice that

$$\ker A = \ker A^{\frac{1}{2}}. \tag{17.19}$$

Indeed, $\ker A \supseteq \ker A^{\frac{1}{2}}$ follows from

$$(Ax, x)_H = \|A^{\frac{1}{2}}x\|_H^2,$$

and $\ker A \subseteq \ker A^{\frac{1}{2}}$ follows from $A = \left(A^{\frac{1}{2}}\right)^2$. Notice that this implies that, for A positive and selfadjoint, $x \in \ker A$ if and only if $(Ax, x)_H = 0$.
To conclude the proof of Theorem 17.39 we need to check the uniqueness. Let $B \ge 0$ and selfadjoint satisfy $B^2 = A$. Notice that B commutes with $A = B^2$. We conclude that B commutes with the series (17.18), and so with $A^{\frac{1}{2}}$. Then we have

$$0 = A - A = B^2 - \left(A^{\frac{1}{2}}\right)^2 = \left(B - A^{\frac{1}{2}}\right)\left(B + A^{\frac{1}{2}}\right).$$

Then we conclude $B = A^{\frac{1}{2}}$ in $\overline{R\left(B + A^{\frac{1}{2}}\right)} = \ker^{\perp}\left(B + A^{\frac{1}{2}}\right)$.
So, since by (17.15) we have

$$H = \ker\left(B + A^{\frac{1}{2}}\right) \oplus \ker^{\perp}\left(B + A^{\frac{1}{2}}\right),$$

we need to check the behavior of $B - A^{\frac{1}{2}}$ in $\ker\left(B + A^{\frac{1}{2}}\right)$.
Since $B + A^{\frac{1}{2}} \geq 0$ and is selfadjoint, by a previous discussion we know that

$$x \in \ker\left(B + A^{\frac{1}{2}}\right) \Leftrightarrow \left(\left(B + A^{\frac{1}{2}}\right)x, x\right)_H = 0 \Leftrightarrow (Bx, x)_H = 0 = \left(A^{\frac{1}{2}}x, x\right)_H$$
$$\Leftrightarrow x \in \ker B \cap \ker A^{\frac{1}{2}}.$$

So, in $\ker\left(B + A^{\frac{1}{2}}\right)$ we have $B = A^{\frac{1}{2}} = 0$ and so, again and trivially, $B = A^{\frac{1}{2}}$. Hence $B = A^{\frac{1}{2}}$ in all H.

□

Exercise 17.40. Show that if $T \in \mathcal{L}(H)$ is such that $[T, A] = 0$, for A the operator in Theorem 17.39, then $[T, \sqrt{A}] = 0$

Theorem 17.41 (Polar decomposition of an operator)**.** *Any $A \in \mathcal{L}(H)$ can be written as $A = UR$ with R positive and selfadjoint and U unitary. There is a unique such R positive and self–adjoint operator, we denote it by $R = |A|$ and we call it absolute value of A.*

Proof. Let $R = \sqrt{A^*A}$. We have

$$\|Rx\|_H^2 = (Rx, Rx)_H = \left(R^2x, x\right)_H = (A^*Ax, x)_H = (Ax, Ax)_H = \|Ax\|_H^2 \text{ for any } x \in H,$$

that is $\|Rx\|_H = \|Ax\|_H$ for any $x \in H$. This implies

$$\ker R = \ker A. \tag{17.20}$$

Since $R^* = R$, by Lemma 17.36 we have the decomposition

$$H = \ker R \oplus \overline{R(R)}. \tag{17.21}$$

Set now

$$Ux := \begin{cases} Ax_1 \text{ if } x = Rx_1 \\ x \text{ if } x \in \ker R. \end{cases} \tag{17.22}$$

Notice that, by $\ker R = \ker A$, U is well defined in $\ker R \oplus R(R)$.
From $\|Ux\|_H = \|x\|_H$, it follows that U extends in an isometry on $H = \ker R \oplus \overline{R(R)}$. It is easy to check that U is an isomorphism (left as an exercise). Then we conclude $URx = Ax$ for any $x \in H$.

Now we need to show uniqueness of R. Let $A = U_1R_1$ be another polar decomposition. Then

$$R^2 = A^*A = R_1U_1^*U_1R_1 = R_1^2 \Longrightarrow R = R_1,$$

by the uniqueness of the positive square root of a positive self–adjoint operator.

□

Remark 17.42. We remark that we have shown that U splits

$$U : \ker R \oplus \overline{R(|A|)} \to \ker R \oplus \overline{R(A)}. \tag{17.23}$$

Exercise 17.43. Check whether or not the U in the factorization $A = U|A|$ is unique.

Exercise 17.44. Show that if $A, B, C \in \mathcal{L}(X)$ for X any topological vector space, then

$$[A, BC] = [A, B]C + B[A, C] \tag{17.24}$$
$$[AB, C] = [A, C]B + A[B, C]. \tag{17.25}$$

Show also

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0. \tag{17.26}$$

Exercise 17.45. Show that if A is self–adjoint then the U in the proof of Theorem 17.41 is self–adjoint.

Remark 17.46. The Spectral Theorem for self–adjoint operators (which is one of the most important theorems in Functional Analysis, and which will be treated in the next semester in the course named Functional Analysis) allows to define the operator $f(A)$ for any self–adjoint operator A and for any Borel function $f : \mathbb{R} \to \mathbb{C}$. Then $U = f(A)$, with

$$f(x) = \begin{cases} \frac{x}{|x|} \text{ if } x \neq 0 \\ 1 \text{ if } x = 0. \end{cases}$$

Notice that the absolute value operator $|A|$ in Theorem 17.41 is, for A self-adjoint, indeed $|A| = f(A)$, for $f(x) = |x|$. So the notation and the terminology are consistent.

17.2 Some remarks on Sobolev Spaces

Some of the most important Banach spaces are the Sobolev Spaces, which will be discussed in some length in the 2nd part of this course. They are based on the Lebesgue spaces L^p. The simplest ones, and the most important ones, are the ones based on L^2. We will discuss them only on the tori $\mathbb{T}^d$, where we will exploit the notion of Fourier Series and the isometric isomorphism $L^2(\mathbb{T}^d) \ni f \to \widehat{f} \in \ell^2(\mathbb{Z}^d)$ discussed in Example 17.15.

Definition 17.47. For $\xi \in \mathbb{R}^d$ we denote by $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$ the *Japanese bracket.*

For any $s \in \mathbb{R}$ we denote by $H^s(\mathbb{T}^d)$ the completion of the space of trigonometric polynomials

$$f(x) = \sum_{|\ell| \leq N} \widehat{f}(\ell) \frac{e^{\mathrm{i}\ell \cdot x}}{(2\pi)^{\frac{d}{2}}} \tag{17.27}$$

provided with the norm

$$\|f\|^2_{H^s(\mathbb{T}^d)} = \sum_{\ell \in \mathbb{Z}^d} \langle \ell \rangle^{2s} |\widehat{f}(\ell)|^2 = \|\langle \ell \rangle^s \widehat{f}(\ell)\|^2_{\ell^2(\mathbb{Z}^d)}. \tag{17.28}$$

Exercise 17.48. Prove that if $sp > d$ then $\| \langle \ell \rangle^{-s} \|_{\ell^p(\mathbb{Z}^d)} < \infty$.

Exercise 17.49. Show that if for $n \in \mathbb{N}$ we denote by $\mathcal{H}^n(\mathbb{T}^d)$ the completion of the space of trigonometric polynomials (17.27) provided with the norm

$$\|f\|^2_{\mathcal{H}^n(\mathbb{T}^d)} = \sum_{|\alpha| \le n} \|\partial_x^\alpha f\|_{L^2(\mathbb{T}^d)} \tag{17.29}$$

then the norms (17.28) and (17.29) are equivalent and the two spaces $H^n(\mathbb{T}^d)$ and $\mathcal{H}^n(\mathbb{T}^d)$ coincide.

Exercise 17.50. Show that $H^s(\mathbb{T}^d)$ has a natural structure of Hilbert space and write explicitly the inner product.

Exercise 17.51. Show that, for Λ^κ, for any $\kappa \in \mathbb{R}$, the operator defined by $\widehat{\Lambda^\kappa f}(\ell) := \langle \ell \rangle^\kappa \widehat{f}(\ell)$, then $\Lambda^{s-\tau} : H^s(\mathbb{T}^d) \to H^\tau(\mathbb{T}^d)$ is an isometry.

Example 17.52. One simple example of Sobolev's Embedding Theorem, which is a crucial theorem in Functional Analysis, discussed later in the 2nd part of this course, is the following: *if $s > d/2$ then there is an embedding $H^s(\mathbb{T}^d) \to C^0(\mathbb{T}^d)$.*
To see this embedding consider for trigonometric polynomials the identity (17.27). Then, taking absolute value of (17.27), we have

$$\begin{aligned}|f(x)| &\le \sum_{|\ell| \le N} |\widehat{f}(\ell)| \le \sum_{|\ell| \le N} \langle \ell \rangle^{-s} \langle \ell \rangle^{s} |\widehat{f}(\ell)| \le \left(\sum_{\ell \in \mathbb{Z}^d} \langle \ell \rangle^{-2s} \right)^{\frac{1}{2}} \left(\sum_{|\ell| \le N} \langle \ell \rangle^{2s} |\widehat{f}(\ell)|^2 \right)^{\frac{1}{2}} \\ &= \| \langle \ell \rangle^{-2s} \|_{\ell^2(\mathbb{Z}^d)} \|f\|_{H^s(\mathbb{T}^d)}.\end{aligned}$$

By density, this yields

$$\|f\|_{C^0(\mathbb{T}^d)} \le \| \langle \ell \rangle^{-2s} \|_{\ell^2(\mathbb{Z}^d)} \|f\|_{H^s(\mathbb{T}^d)} \text{ for any } f \in H^s(\mathbb{T}^d). \tag{17.30}$$

Exercise 17.53. Consider for some $1 \le m < n$ the embedding

$$\mathbb{T}^m \ni (x_1, ..., x_m) \to (x_1, ..., x_m, 0, .., 0, 0) \in \mathbb{T}^n.$$

Show that the restriction $C^\infty(\mathbb{T}^n, \mathbb{C}) \ni f \to f|_{\mathbb{T}^m} \in C^\infty(\mathbb{T}^m, \mathbb{C})$ extends into a bounded map

$$H^s(\mathbb{T}^n) \to H^{s'}(\mathbb{T}^m)$$

when $s > s' + \frac{n-m}{2}$.

Remark 17.54. Notice that restriction theorems like the one in Exercise 17.53 play a deep role in PDE's. For example, the celebrated Strichartz estimates, which for example for the group introduced in (7.29), tell that

$$\|e^{\mathrm{i}\triangle t} u_0\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \le C \|u_0\|_{L^2(\mathbb{R}^d)} \tag{17.31}$$

for all pairs (q,r) which are Schrödinger–*admissible*, that is

$$\frac{2}{q}+\frac{d}{r}=\frac{d}{2} \tag{17.32}$$

$$2\le r\le \frac{2d}{d-2}\ (2\le r\le\infty \text{ if } d=1,\ 2\le r<\infty \text{ if } d=2). \tag{17.33}$$

Strichartz proved the non sharp case, that is all cases except those with $r=2$ for $d\ge 3$, exactly as a restriction theorem on the paraboloid $\xi_0=\xi_1^2+...+\xi_d^2$ in Phase Space. The classical paper is Strichartz [12]. The best explanation for this, as for many other topics, is in Stein [11]. The endpoint case $r=2$ for $d\ge 3$ is another classical paper, this by Keel and Tao [6] (at the time of writing these notes, it is the most quoted paper of the 2006's Fields Medal laureate Terence Tao). Strichartz estimates is a very important topic and tool. A great expert is Damiano Foschi, at the nearby University of Ferrara.

Example 17.55. Recall from the Riesz Frechét Theorem 17.10 that, given a Hilbert space H, there is a natural isomorphism $H\to H'$ given by $u\to(u,\cdot)$. However, often it is natural not to identify H and H'. A case point are the spaces $H^s(\mathbb{T}^d)$ when $s\ne 0$, which are Sobolev spaces, that is, some of the spaces used in applications of Functional Analysis. If we consider two trigonometric polynomials, then we have

$$(f,g)_{L^2(\mathbb{T}^d)}=\int_{\mathbb{T}^d} f(x)\overline{g(x)}dx=\sum_{\ell\in\mathbb{Z}^d}\widehat{f}(\ell)\overline{\widehat{g}(\ell)}=\sum_{\ell\in\mathbb{Z}^d}\langle\ell\rangle^s\,\widehat{f}(\ell)\,\langle\ell\rangle^{-s}\,\overline{\widehat{g}(\ell)}.$$

Then we get

$$\left|(f,g)_{L^2(\mathbb{T}^d)}\right|\le\|f\|_{H^s(\mathbb{T}^d)}\|g\|_{H^{-s}(\mathbb{T}^d)}.$$

This shows that $(\cdot,\cdot)_{L^2(\mathbb{T}^d)}: H^s(\mathbb{T}^d)\times H^{-s}(\mathbb{T}^d)\to\mathbb{C}$ is a bounded bilinear map. It is easy to conclude from this that there exists an isomorphism $H^{-s}(\mathbb{T}^d)\ni g\to(\cdot,g)_{L^2(\mathbb{T}^d)}\in\left(H^s(\mathbb{T}^d)\right)'$. This sort of identification, arising concretely from the inner product in $L^2(\mathbb{T}^d)$, is much more common in practice than the somewhat more abstract identification of $H^s(\mathbb{T}^d)$ and $\left(H^s(\mathbb{T}^d)\right)'$.

18 Compact Operators

Definition 18.1. A bounded linear operator $T:E\to F$ between two Banach spaces is said *compact* if it sends bounded sets into relatively compact sets.

Example 18.2. A bounded linear operator $T:X\to Y$ between two Banach spaces is a finite rank operator if $\dim R(T)<\infty$. Finite rank operators between Banach spaces are compact operators.

Exercise 18.3. Let $T\in\mathcal{L}(X,Y)$ and $S\in\mathcal{L}(Y,Z)$ and suppose that one of the two is compact. Then $S\circ T$ is compact.

Exercise 18.4. Consider a compact operator $T : X \to Y$ between two Banach spaces. Show that if $x_n \rightharpoonup x$ in the $\sigma(X, X')$ topology then $Tx_n \xrightarrow{n\to\infty} Tx$ in the strong topology in Y.

Remark 18.5. Consider a compact operator $T : X' \to Y$ between two Banach spaces with X' the dual of a Banach space X. It is not true in general that if $x'_n \rightharpoonup x'$ in the $\sigma(X', X)$ topology then $Tx'_n \xrightarrow{n\to\infty} Tx'$ in the strong topology in Y. Indeed, consider $Y = \mathbb{R}$ and consider $\mathrm{ev}_0 f := f(0)$, which is bounded from $C^0_0(\mathbb{R}^d) \to \mathbb{R}$. Let $T : L^\infty(\mathbb{R}^d) \to \mathbb{R}$ be an extension of ev_0 using the the Hahn–Banach Theorem, Obviously, T is a compact operator. Consider any $\psi \in C^0_c(\mathbb{R}^d)$ with $\psi(0) = 1$ and let $\psi_n(x) = \psi(nx)$. Then, by Dominated Convergence we have $\psi_n \rightharpoonup 0$ in the $\sigma(L^\infty(\mathbb{R}^d), L^1(\mathbb{R}^d))$ weak topology, yet $T\psi_n = \mathrm{ev}_0\psi_n = \psi_n(0) = 1$ for all n. So it is not true that $T\psi_n \xrightarrow{n\to\infty} T0 = 0$, and this gives a desired example.

Example 18.6. Lack of compactness of an operator $T : E \to F$ is often related to the action by a non–compact group. For example, take a convolution

$$Tf = \kappa * f.$$

We know that by Young's inequality (16.23),

$$\|Tf\|_{L^r(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)}\|\kappa\|_{L^q(\mathbb{R}^d)} \text{ for } \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}.$$

The operator $T : L^p(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$ is never compact if $\kappa \neq 0$ when $p > 1$.
Indeed, if we take any sequence $x_n \xrightarrow{n\to+\infty} \infty$ in $\mathbb{R}^d$, then by the commutation property (16.29), we have $T\tau_{x_n} f = \tau_{x_n} Tf$. Now, if $1 < p < \infty$ we have $\tau_{x_n} f \rightharpoonup 0$ in $\sigma\left(L^p, L^{p'}\right)$ if $1 < p < \infty$ for any $f \in L^p(\mathbb{R}^d)$ (See Example 11.17) but, if $Tf \neq 0$, we have $\|\tau_{x_n} Tf\|_{L^r} = \|Tf\|_{L^r}$, and so $T\tau_{x_n} f$ does not converge strongly to 0 in $L^r(\mathbb{R}^d)$.
Case $p = \infty$ is similar. Indeed, $\tau_{x_n} f \rightharpoonup 0$ in $\sigma\left(L^\infty, L^1\right)$ for any $f \in L^\infty(\mathbb{R}^d)$ with compact support but, if $Tf \neq 0$, we have $\|\tau_{x_n} Tf\|_{L^\infty} = \|Tf\|_{L^\infty}$, and so $T\tau_{x_n} f$ does not converge strongly to 0 in $L^\infty(\mathbb{R}^d)$ (recall, $r = \infty$). Notice that here $\kappa \in L^1(\mathbb{R}^d)$ and, if $\kappa \neq 0$, there is certainly an $f \in L^\infty(\mathbb{R}^d)$ of compact support such that $\kappa * f \neq 0$ in $L^\infty(\mathbb{R}^d)$.
When supp κ is compact, we can capture also case $p = 1$.

Example 18.7. A similar effect of translation invariance is obtained considering scale invariance. So here, in a specific example, we will use scaling as an alternative to translation. A very important theorem states that

$$\text{for any } \gamma \in (0, d) \text{ and } 1 < p < q < \infty \text{ with } \frac{1}{p} = \frac{1}{q} + \frac{d-\gamma}{d} \tag{18.1}$$

and for

$$Tf(x) := \int_{\mathbb{R}^d} |x - y|^{-\gamma} f(y) dy \tag{18.2}$$

there exists a constant C s.t.

$$\|Tf\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}. \tag{18.3}$$

This is the Hardy-Littlewood-Sobolev Inequality. It is related to the Sobolev Embedding Theorem, although not discussed in Brezis [3] and not in the 2nd part of this Course. We refer for it to Stein [11]. Notice that

$$T\delta_{p,\lambda}f(x) = \int_{\mathbb{R}^d} |x-y|^{-\gamma}\lambda^{\frac{d}{p}} f(\lambda y)dy = \lambda^{\gamma-d}\int_{\mathbb{R}^d} |\lambda x-\lambda y|^{-\gamma}\lambda^{\frac{d}{p}} f(\lambda y)\lambda^d dy = \lambda^{\gamma-d-\frac{d}{q}}\lambda^{\frac{d}{p}}Tf(x).$$

So we have shown that

$$T\delta_{p,\lambda}f = \lambda^{\gamma-d-\frac{d}{q}+\frac{d}{p}}\delta_{q,\lambda}Tf.$$

It is easy to see (we leave this as an exercise, you'll see something similar when discussing the Gagliardo–Nirenberg Sobolev Inequality in the next semester) that, for (18.3) to be true, we need to have $\gamma - d - \frac{d}{q} + \frac{d}{p} = 0$, which is indeed the condition in (18.1).
So here we have that, under the conditions (18.1), then

$$T\delta_{p,\lambda}f = \delta_{q,\lambda}Tf. \tag{18.4}$$

This can be used to show that the operator T is not compact. In fact, taking $\lambda_n \xrightarrow{n\to+\infty} +\infty$. we recall that for $1 < p < \infty$ we have $\delta_{p,\lambda_n}f \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$ for the $\sigma(L^p, L^{p'})$ topology, see Example 11.23. If T was a compact operator, we would have $\delta_{q,\lambda_n}Tf \xrightarrow{n\to+\infty} 0$ in norm in $L^q(\mathbb{R}^d)$, but this is not true, because $\|\delta_{q,\lambda_n}Tf\|_{L^q(\mathbb{R}^d)} = \|Tf\|_{L^q(\mathbb{R}^d)} \neq 0$ for all nonzero f.

Notice that here, we could have used translation instead of dilation. But there are examples where translation is not available but dilation is, and in fact you will see it, in relation to Remark 10 p. 214 in Brezis [3], in the next semester.

Remark 18.8. It is an important topic to find why certain operators fail to be compact, for example the operator $e^{it\triangle} : L^2(\mathbb{R}^d) \to L^q(\mathbb{R}, L^r(\mathbb{R}^d))$ for an admissible pair (q, r), see Remark 17.54. There are results which state that there is a sort of compactness up to scaling and translation. An important paper is Bahouri and Gerard [2], but there are earlier papers. The most famous paper exploiting these facts in PDE's is probably Kenig and Merle [7]. An expert on the failure of Sobolev Embeddings to be compact is Sergio Solimini, now in Bari but many years ago professor at SISSA.

Exercise 18.9. Prove that $\ell^p(\mathbb{Z}^d) \subset \ell^q(\mathbb{Z}^d)$ for $p < q$ and check if the immersion $\ell^p(\mathbb{Z}^d) \hookrightarrow \ell^q(\mathbb{Z}^d)$ is compact, at least for some $p < q$.

Answer. One can exploit the existence of translation in $\mathbb{Z}^d$ which induces translation in the above spaces, to exclude that these embeddings are compact operators.

Exercise 18.10. Check if the immersion $L^2(0, 1) \hookrightarrow L^1(0, 1)$ is compact.

Answer. It is not compact. Notice that we have

$$\begin{array}{ccc} L^2(0,1) & \longrightarrow & L^1(0,1) \\ \downarrow & & \downarrow \\ \ell^2(\mathbb{Z}) & \longrightarrow & \ell^\infty(\mathbb{Z}) \end{array}$$

and if our map is compact, then also the immersion $\ell^2(\mathbb{Z}) \hookrightarrow \ell^\infty(\mathbb{Z})$ is compact, while we know from Exercise 18.9 that the latter immersion is not compact.

Exercise 18.11. Check if, for some pair $p > q$ other than the previous one, the immersion $L^p(0,1) \hookrightarrow L^q(0,1)$ is compact.

Answer. They are never compact. It is equivalent to consider $L^p(\mathbb{T}) \hookrightarrow L^q(\mathbb{T})$. It is not restrictive to pick $q = 1$ (since we have a continuous embedding $L^q(\mathbb{T}) \hookrightarrow L^1(\mathbb{T})$ for all q). From Exercise 18.10 we suspect that translations in phase space are equivalent to multiplications by $e^{\mathrm{i}xn}$. So, for any $f \in L^p(\mathbb{T})$ consider the sequence $e^{\mathrm{i}xn}f$. If a subsequence $e^{\mathrm{i}xn_k}f$ is convergent in $L^q(\mathbb{T})$, then for any $\epsilon > 0$ there exists $N(\epsilon)$ such that

$$j,k > N(\epsilon) \Longrightarrow \int_{\mathbb{T}} \left|e^{\mathrm{i}xn_k} - e^{\mathrm{i}xn_j}\right| \; |f(x)|dx = \int_{\mathbb{T}} \left|e^{\mathrm{i}x(n_k-n_j)} - 1\right| \; |f(x)|dx < \epsilon.$$

But, using (16.35),

$$\begin{aligned} \epsilon &> \int_{\mathbb{T}} \left|e^{\mathrm{i}x(n_k-n_j)} - 1\right| \; |f(x)|dx = \int_{\mathbb{T}} |(\cos(x(n_k-n_j)) - 1) + \mathrm{i}\sin(x(n_k-n_j))| \; |f(x)|dx \\ &\geq \int_{\mathbb{T}} |\sin(x(n_k-n_j))| \; |f(x)|dx \xrightarrow{(n_k-n_j)\to+\infty} \frac{2}{\pi}\int_{\mathbb{T}} |f(x)|dx, \end{aligned}$$

which yields

$$\frac{2}{\pi}\int_{\mathbb{T}} |f(x)|dx \leq \epsilon.$$

Obviously, by the arbitrariness of $\epsilon > 0$, this implies $f = 0$. □

Lemma 18.12. *The space of compact operators $K(E,F)$ is closed for the uniform norm, in the space of bounded linear operators $\mathcal{L}(E,F)$*

Proof. Consider $\overline{D_E(0,1)}$ the unit closed ball in E and let $T \in \overline{K(E,F)}$. To show that $T\overline{D_E(0,1)}$ is relatively compact it suffices to show that for any $\epsilon > 0$ we can cover $T\overline{D_E(0,1)}$ by a finite number of balls in F of radius ϵ. Let $S \in K(E,F)$ with $|S - T| < \epsilon/2$ and cover $S\overline{D_E(0,1)}$ by balls $D_F(f_j, \epsilon/2)$ for $j = 1...n$. Then $D_F(f_j, \epsilon)$ for $j = 1...n$ cover $T\overline{D_E(0,1)}$. □

Theorem 18.13. *Given two Banach spaces X and Y, $T \in K(X,Y)$ if and only if $T^* \in K(Y', X')$*

Proof. Assume that $T \in K(X,Y)$. We need to show that $\overline{T^*(D_{Y'}(0,1))}$ is compact. Let $K := \overline{T(D_X(0,1))}$. We know that K is compact. Consider a sequence y_n' in $D_{Y'}(0,1)$. Obviously $\{y_n'|_K\}$ are elements of $C^0(K,\mathbb{R})$. It is easy to see that we can apply Ascoli–Arzelá and conclude that there is a subsequence, which is not restrictive to assume equal to the initial sequence, such that $y_n'|_K \xrightarrow{n\to+\infty} \varphi$ in $C^0(K,\mathbb{R})$. So

$$\sup_{x\in D_X(0,1)} \left|\langle y_n', Tx\rangle_{Y'\times Y} - \varphi(Tx)\right| \xrightarrow{n\to+\infty} 0.$$

This implies

$$\sup_{x\in D_X(0,1)} \left|\langle T^*y_n', x\rangle_{X'\times X} - \langle T^*y_m', x\rangle_{X'\times X}\right| \xrightarrow{n\to+\infty, m\to+\infty} 0.$$

This is equivalent to

$$\|T^*y_n' - T^*y_m'\|_{X'} \xrightarrow{n\to+\infty, m\to+\infty} 0.$$

This implies that $\overline{T^*(D_{Y'}(0,1))}$ is compact.

Let now $T^* \in K(Y',X')$. Then, by the first part of the proof we have $T^{**} \in K(X'',Y'')$. Recall that from Lemma 6.22 we have $T^{**}J_X = J_Y T$. Since $T^{**}|_{J_X X} \in K(J_X X, Y'')$ and, having image in $J_Y Y$, it is $T^{**}|_{J_X X} \in K(J_X X, J_Y Y)$, and since $J_Y : Y \to J_Y Y$ and $J_X : X \to J_X X$ are isometries and isomorphisms, $T = J_Y^{-1}\, T^{**}|_{J_X X}\, J_X \in K(X,Y)$. □

Theorem 18.14. *If F is a Hilbert space, then any $T \in K(E,F)$ is the uniform limit of finite rank operators.*

Proof. Let $T\overline{D_E(0,1)} \subseteq \cup_{j=1}^n D_F(f_j,\epsilon)$. Let G be the space generated by the f_j, $j = 1...n$, and P_G the orthogonal projection on G. Then for any $x \in \overline{D_E(0,1)}$ there is f_j such that $\|Tx - f_j\|_F < \epsilon$. So $\|P_G \circ Tx - f_j\|_F < \epsilon$ and so $\|P_G \circ Tx - Tx\|_F < 2\epsilon$. This implies that $\|P_G \circ T - T\|_{\mathcal{L}(E,F)} \leq 2\epsilon$. □

Exercise 18.15. Show that, for $\kappa \in L^1(\mathbb{T}^d)$, the operator

$$Tf = \kappa * f \tag{18.5}$$

is a compact operator $T : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$.

Remark 18.16. The crucial difference between Example 18.6 and Exercise 18.15 is that $\mathbb{T}^d$ is a bounded manifold.

Exercise 18.17. More generally, show that, for $\kappa \in L^q(\mathbb{T}^d)$ with $q < \infty$, the operator

$$T_\kappa f = \kappa * f \tag{18.6}$$

is a compact operator $T_\kappa : L^p(\mathbb{T}^d) \to L^r(\mathbb{T}^d)$, where $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$.

Answer. Let us consider the case $\kappa \in C^0(\mathbb{T}^d)$. Then κ is uniformly continuous and for any $\epsilon > 0$ there exists $\delta > 0$ s.t., if $\Omega \subseteq \mathbb{T}^d$ is such that $\text{diam}\Omega < \delta$, then $\text{osc}_\Omega \kappa < \epsilon$. So, let us consider $\bigcup_{j=1}^{N_\epsilon} D_{\mathbb{T}^d}(y_j, \delta/2)$ a covering of $\mathbb{T}^d$ and a partition of unity $\sum_{j=1}^{N_\epsilon} \chi_j(x) = 1$ such that $\chi_j \in C_c^\infty(D_{\mathbb{T}^d}(y_j, \delta/2), [0,1])$. Then

$$\begin{aligned}\kappa * f(x) = &\overbrace{\sum_{j=1}^{N_\epsilon} \kappa(x-y_j) \int_{\mathbb{T}^d} \chi_j(y) f(y) dy}^{L_{N_\epsilon} f(x)} \\ &\underbrace{+ \sum_{j=1}^{N_\epsilon} \int_{\mathbb{T}^d} \left(\kappa(x-y) - \kappa(x-y_j)\right) \chi_j(y) f(y) dy}_{S_{N_\epsilon} f(x)} .\end{aligned}$$

Now, L_{N_ϵ} is a finite rank operator, while

$$\|S_{N_\epsilon} f\|_{L^p(\mathbb{T}^d)} \le \epsilon \left\| \sum_{j=1}^{N_\epsilon} \int_{\mathbb{T}^d} \chi_j(y) |f(y)| dy \right\|_{L^p(\mathbb{T}^d)} = \epsilon \|f\|_{L^1(\mathbb{T}^d)} \left(\text{vol}(\mathbb{T}^d)\right)^{\frac{1}{p}} \le \epsilon \text{vol}(\mathbb{T}^d) \|f\|_{L^p(\mathbb{T}^d)}.$$

So $\|L_{N_\epsilon} - T_\kappa\|_{\mathcal{L}\left(L^p(\mathbb{T}^d)\right)} \le \epsilon \text{vol}(\mathbb{T}^d) \xrightarrow{\epsilon \to 0^+} 0$ and we conclude that T_κ is compact. In general, if $q < \infty$, we can take $C^0(\mathbb{T}^d) \ni \kappa_n \xrightarrow{n \to +\infty} \kappa$ in $L^q(\mathbb{T}^d)$, and then

$$\|T_{\kappa_n} - T_\kappa\|_{\mathcal{L}\left(L^p(\mathbb{T}^d)\right)} \le \|\kappa_n - \kappa\|_{L^q(\mathbb{T}^d)} \xrightarrow{n \to +\infty} 0$$

and so, since all the T_{κ_n} are compact, also T_κ is so. $\square$

Lemma 18.18. *Let $W \subsetneqq X$, W closed and X Banach space. Then there exists a sequence v_n such that $\|v_n\|_X = 1$ and*

$$\mathit{dist}(v_n, W) \xrightarrow{n \to +\infty} 1.$$

Proof. Given $v \in X \backslash W$, there exists a sequence $w_n \in W$ such that

$$\|v - w_n\|_X \xrightarrow{n \to +\infty} \text{dist}(v, W) > 0.$$

Let $v_n = \frac{v - w_n}{\|v - w_n\|_X}$. Obviously $\text{dist}(v_n, W) \le \text{dist}(v_n, 0) = \|v_n\|_X = 1$. Suppose now that

$$S := \liminf_{n \to \infty} \text{dist}(v_n, W) < 1. \tag{18.7}$$

Let $S < a < 1$. Then there would be a subsequence of n and $\widetilde{u}_n \in W$ such that

$$\|v - w_n - (\|v - w_n\|_X) \widetilde{u}_n\|_X < a \|v - w_n\|_X \xrightarrow{n \to +\infty} a \, \text{dist}(v, W).$$

So, setting $W \ni u_n := w_n + \|v - w_n\|_X \widetilde{u}_n$, there is a subsequence of n and $u_n \in W$ such that

$$\|v - u_n\|_X < \text{dist}(v, W) - \epsilon$$

for a fixed small ϵ. Absurd. This means that we have $S = 1$.

□

Exercise 18.19. Let X be an infinite dimensional Banach space. Show that for any $r \in (0, 1/2)$ there exists a sequence $\{v_n\}$ in X such that $\|v_n\|_X = 1$ and the closed balls $\overline{D_X(v_n, r)}$ are pairwise disjoint. Show also that $\bigcup_{n=1}^{\infty} \overline{D_X(v_n, r)}$ is a closed set in X.

Corollary 18.20. *V Banach with $\overline{D_V(0,1)}$ compact. Then* $\dim V < \infty$.

Proof. If $\dim V < \infty$ we know $\overline{D_V(0,1)}$ compact. Let us prove the opposite. Suppose $\dim V = \infty$. Then there is a strictly increasing sequence E_n of closed vector spaces such that for any n there is a $u_n \in E_n - E_{n-1}$ with $\|u_n\|_V = 1$ and with $\text{dist}(u_n, E_{n-1}) \geq \frac{1}{2}$. Then $\|u_n - u_m\|_V \geq \frac{1}{2}$ for $n \neq m$, and in particular this sequence does not admit a convergent subsequence. This implies that $\overline{D_V}$ cannot be compact.

□

Remark 18.21. The following theorem is a very important tool. As we know, if $T \in \mathcal{L}(\mathbb{R}^d)$, then $R(T) = \mathbb{R}^d \iff \ker T = 0$. This is not true if $T \in \mathcal{L}(X)$ with $\dim X = \infty$. However Theorem 18.22 implies that if $T = A + K$ with A an isomorphism in $\mathcal{L}(X)$ and K a compact operator, then in fact $R(T) = X \iff \ker T = 0$.

Theorem 18.22 (Fredholm alternative). *Let X be a Banach space. Let $K \in K(X)$ and set $T = I - K$. Then:*

1 $\dim \ker T < \infty$

2 $R(T) = (\ker T^*)^{\perp}$

3 $\ker T = 0 \Leftrightarrow R(T) = X$

4 $\dim \ker T = \dim \ker T^*$.

Proof.

1 For $N := \ker(I - K)$, we have $D_N(0,1) \subseteq K D_X(0,1)$ and so $D_N(0,1)$ is relatively compact. Then, by Corollary 18.20, N is finite dimensional.

2 We have $\overline{R(T)} = \ker^{\perp} T^*$ by (6.39), so here we need to show that $R(T) = \overline{R(T)}$. Consider a sequence $Tx_n \xrightarrow{n \to +\infty} f$ in X, we need to show that $f \in R(T)$. Notice that $x_n = Tx_n + Kx_n$. If $\{x_n\}_{n \in \mathbb{N}}$ is a bounded sequence in X, then up to a subsequence, which is not restrictive to assume equal to the whole series, we have $Kx_n \xrightarrow{n \to +\infty} g$ in X. Then $x_n \xrightarrow{n \to +\infty} f + g$ and hence, by continuity, $Tx_n \xrightarrow{n \to +\infty} f = T(f + g)$, and so $f \in R(T)$.

The whole point in the above argument was the boundedness of the sequence $\{x_n\}$, which in general is not true a priori. However we claim that

$$\exists \text{ a sequence } \{y_n\} \text{ in } \ker T \text{ s.t. } \{x_n - y_n\}_{n\in\mathbb{N}} \text{ is a bounded sequence in } X. \tag{18.8}$$

This in turn yields Claim 2 of the statement, because

$$Tx_n = T(x_n - y_n). \tag{18.9}$$

Notice that $\ker T = N$ and has finite dimension. Let $d_n := \text{dist}(x_n, N)$. It is an elementary, by the Weierstrass Theorem, that since $\|x_n - .\|_X \in C^0(N, \mathbb{R})$ with $\lim_{y\to\infty} \|x_n - y\|_X = +\infty$, there is an absolute minimum $y_n \in N$, with therefore $d_n = \|x_n - y_n\|_X$. It is enough to prove now that $\{d_n\}$ is a bounded sequence. Suppose that this is false, and that there is a subsequence with limit $+\infty$. It is not restrictive to assume $d_n \xrightarrow{n\to+\infty} +\infty$. By (18.9) and $Tx_n \xrightarrow{n\to\infty} f$ in X,

$$\frac{Tx_n}{\|x_n - y_n\|_X} = w_n - Kw_n \xrightarrow{n\to+\infty} 0 \text{ where } w_n := \frac{x_n - y_n}{\|x_n - y_n\|_X}. \tag{18.10}$$

By compactness, up to a subsequence which, again, is not restrictive to take the whole sequence, $Kw_n \xrightarrow{n\to+\infty} g$ in X. By (18.10) we get $w_n \xrightarrow{n\to+\infty} g$ in X and $g \in N$. This obviously implies

$$\text{dist}(w_n, N) \le \|w_n - g\|_X \xrightarrow{n\to+\infty} 0. \tag{18.11}$$

However

$$\text{dist}(w_n, N) = \text{dist}\left(\frac{x_n - y_n}{\|x_n - y_n\|_X}, N\right) = \frac{\text{dist}\,(x_n - y_n, N)}{\|x_n - y_n\|_X} = \frac{\text{dist}\,(x_n, N)}{d_n} = \frac{d_n}{d_n} = 1.$$

So we get a contradiction to (18.11), and this shows that $\{d_n\}$ is a bounded sequence. This completes the proof of the claim in (18.8).

3 Assume T is injective. Suppose T is not surjective. We know $X_1 = R(T) \subsetneqq X$ is closed. Then $T : X \to X_1$ is an isomorphism between Banach spaces. Set by induction $X_{n+1} = TX_n$. By induction, this is a strictly decreasing sequence of closed spaces. Indeed if $X_n \subsetneqq X_{n-1}$, then by injectivity $X_{n+1} = TX_n \subsetneqq TX_{n-1} = X_n$. Next consider $x_n \in X_n$ such that $\|x_n\|_X = 1$ and $\text{dist}(x_n, X_{n+1}) > 1/2$, see Lemma 18.18. Now for $n > m$

$$Kx_n - Kx_m = x_m + [-x_n - (Tx_m - Tx_n)] = x_m + x_{n,m}$$

with $x_{n,m} \in X_{m+1}$. Hence

$$\|Kx_n - Kx_m\|_X \ge \text{dist}(x_m, X_{m+1}) \ge \frac{1}{2}.$$

But then $\{Kx_n\}$ is not a relatively compact sequence, contradicting the compactness of the operator K. We conclude that T injective implies T surjective.

Now we consider the opposite implication assume that T is surjective. Then, by $\ker T^* = R(T)^\perp$, see (6.37), the dual $T^* = I - K^*$ is injective. Since K^* is compact and X' is a Banach space, we conclude $R(T^*) = X'$ and, therefore, from $\ker T = R(T^*)^\perp$, see (6.36), that T is injective.

4 Let $d = \dim \ker T$ and $d^* = \dim \ker T^*$. We have already proved that both are finite. Let us show first $d^* \le d$. If not, $d < d^*$. Notice that $\mathrm{codim} R(T) = d^*$, since there is a natural identification $(X/R(T))' \sim \ker T^*$. Indeed, there is a natural embedding $(X/R(T))' \hookrightarrow X'$ with image in $(R(T))^\perp$, which by (6.37), equals $\ker T^*$ and, viceversa, given any element of X' in $(R(T))^\perp$, it induces an element in $(X/R(T))'$. The above algebraic isomorphism is continuous, and so it is an isomorphism between Banach spaces.
So we conclude that both $\ker T$ and $R(T)$ are complementary and we have

$$X = \ker T \oplus E = F \oplus R(T) \text{ , where } \dim F = d^*. \tag{18.12}$$

So there is a map $\Lambda \in \mathcal{L}(\ker T, F)$ with $\ker \Lambda = 0$. Let then $S = K + \Lambda P_{\ker T}$, with $P_{\ker T}$ the projection on $\ker T$ associated to the first splitting. Notice that S is a compact operator. Then we claim that $\ker(1-S) = 0$. Indeed, if $(1-S)x = 0$, then

$$0 = (1-K)x - \Lambda P_{\ker T} x = Tx - \Lambda P_{\ker T} x \Rightarrow Tx = 0 = \Lambda P_{\ker T} x,$$

by the 2nd splitting in (18.12). From $\ker(1-S) = 0$ we conclude $R(1-S) = X$. But this is not possible because there exists an element $f \in F$ which is not of the form $\Lambda P_{\ker T} x$ for all $x \in X$. So we proved $d^* \le d$. Similarly

$$\dim \ker T^{**} \le \dim \ker T^* \le \dim \ker T. \tag{18.13}$$

But it is obvious from $T^{**} J_X = J_X T$ and the fact that J_X is an isometry, that we have an embedding $J_X : \ker T \hookrightarrow \ker T^{**}$, and so that $\dim \ker T^{**} \ge \dim \ker T$. Then in (18.13) we have equalities. $\square$

Remark 18.23. A consequence of the Theorem 18.24 below, is that, given $K \in K(X)$, there is a K–invariant decomposition

$$X = X_0 \bigoplus_{\lambda \in \sigma(K)\backslash\{0\}} N_g(K - \lambda) \tag{18.14}$$

where $\sigma\left(K|_{X_0}\right) = \{0\}$ and where inside each $N_g(K-\lambda)$, up to an appropriate choice of basis, K decomposes in a finite direct sum of finite rank Jordan blocks like in Sect. 5.1. So one can get a sense of the meaning of some of the statements in Theorem 18.22 splitting and looking singularly at $1 - K|_{X_0}$ and at each $1 - K|_{N_g(K-\lambda)}$ with $\lambda \in \sigma(K)\backslash\{0\}$, further splitting the latter in the Jordan blocks. The idea is that, up to the 0 spectrum part, K is a (possibly infinite) sum of finite dimensional operators.

So, for example, if we focus on a Jordan block, K leaves a space $\mathrm{Sp}\{e_1, ..., e_n\}$ invariant and for this basis has associated matrix

$$K = \begin{bmatrix} \lambda & 1 & 0 & 0 & \dots & & \\ 0 & \lambda & 1 & 0 & \dots & & \\ 0 & 0 & \lambda & 1 & 0 & \dots & \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & \\ & & & \dots & 0 & \lambda & 1 \\ & & & \dots & 0 & 0 & \lambda \end{bmatrix} \text{ and } 1-K = \begin{bmatrix} 1-\lambda & -1 & 0 & 0 & \dots & & \\ 0 & 1-\lambda & -1 & 0 & \dots & & \\ 0 & 0 & 1-\lambda & -1 & 0 & \dots & \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & \\ & & & \dots & 0 & 1-\lambda & -1 \\ & & & \dots & 0 & 0 & 1-\lambda \end{bmatrix}.$$

The interesting case is when $\lambda = 1$. So, in $\mathrm{Sp}\{e_1, ..., e_n\}$,

$$T := 1-K = \begin{bmatrix} 0 & -1 & 0 & 0 & \dots & & \\ 0 & 0 & -1 & 0 & \dots & & \\ 0 & 0 & 0 & -1 & 0 & \dots & \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & \\ & & & \dots & 0 & 0 & -1 \\ & & & \dots & 0 & 0 & 0 \end{bmatrix}.$$

Now, let $X' = \mathrm{Sp}^{\perp}\{e_1, ..., e_n\} \bigoplus \mathrm{Sp}\{e_1^*, ..., e_n^*\}$ with

$$\langle e_j, e_k^* \rangle_{X \times X'} = \delta_{jk}. \tag{18.15}$$

Then notice that

$$\delta_{jk} = \langle e_j, e_k^* \rangle_{X \times X'} = \langle -Te_{j+1}, e_k^* \rangle_{X \times X'} = -\langle e_{j+1}, T^* e_k^* \rangle_{X \times X'}$$

implies $T^* e_k^* = -e_{k+1}^*$ with $Te_n^* = 0$. In $\mathrm{Sp}\{e_1^*, ..., e_n^*\}$, for this basis, T^* acts as

$$T^* = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & & \\ -1 & 0 & 0 & 0 & \dots & & \\ 0 & -1 & 0 & & 0 & \dots & \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & \\ & & & \dots & -1 & 0 & 0 \\ & & & \dots & 0 & -1 & 0 \end{bmatrix}$$

So $R(T) = \mathrm{Sp}\{e_1, ..., e_{n-1}\}$ and $\ker T^* = \mathrm{Sp}\{e_n^*\}$ and, by (18.15), they are orthogonal to each other, as indicated in Theorem 18.22, but here, in this example, one can see it!

Theorem 18.24. *Let $K \in K(X)$ and let $\dim X = \infty$. Then:*

1 $0 \in \sigma(K)$*;*

2 $\lambda \in \sigma(K)$ *and* $\lambda \neq 0 \Rightarrow \lambda$ *is an eigenvalue;*

3 Either $\sigma(K)$ *is finite or* $\sigma(K)\backslash\{0\}$ *is a sequence convergent to 0;*

4 Each $\lambda \in \sigma(K)\backslash\{0\}$ *has finite algebraic (and so, also geometric) multiplicity.*

1 If $0 \notin \sigma(K)$ then $I = K \circ K^{-1}$ is compact, which is incompatible with $\dim X = \infty$.

2 Let $\lambda \in \sigma(K)$ and $\lambda \neq 0$. If λ is not an eigenvalue then $\ker(K - \lambda) = 0$ and, by Fredholm alternative, $R(K - \lambda) = X$. Then, $(\lambda^{-1}K - 1)^{-1}$ is well defined, with domain X. The

$$\text{graph of } (\lambda^{-1}K - 1)^{-1} = \left\{\left(x, (\lambda^{-1}K - 1)^{-1}x\right) : x \in X\right\} = \left\{\left((\lambda^{-1}K - 1)x, x\right) : x \in X\right\}$$

is closed, because, by the fact that $R\left((\lambda^{-1}K - 1)\right)$ is closed, the graph

$$\text{graph of } (\lambda^{-1}K - 1) = \left\{\left(x, (\lambda^{-1}K - 1)x\right) : x \in X\right\}$$

is closed. But then by the Closed Graph Theorem we have $(\lambda^{-1}K - 1)^{-1} \in \mathcal{L}(X)$. Hence $(K - \lambda)^{-1} \in \mathcal{L}(X)$ and so $\lambda \notin \sigma(K)$.

3 Suppose $\sigma(K)\backslash\{0\}$ is infinite. Then, since K is bounded we have $\sigma(K) \subseteq \{z \in \mathbb{C} : |z| \leq \|K\|_{\mathcal{L}(X)}\}$. So $\sigma(K)$ is compact. Consider a sequence λ_n of distinct elements in $\sigma(K)$ and suppose $\lambda_n \to \lambda \neq 0$. Let $Kx_n = \lambda_n x_n$ and let X_n be the span of $\{x_1, \dots, x_n\}$. There exists $y_n \in X_n$, $\|y_n\|_X = 1$, $\text{dist}(y_n, X_{n-1}) > 1/2$. Now, for $n > m$

$$\frac{Ky_n}{\lambda_n} - \frac{Ky_m}{\lambda_m} = y_n + \left[-y_m + \frac{(K - \lambda_n)y_n}{\lambda_n} - \frac{(K - \lambda_m)y_m}{\lambda_m}\right] = y_n + z_{n,m}$$

with $z_{n,m} \in X_{n-1}$ since $(\lambda_n - K)y_n \in X_{n-1}$. Hence

$$\left\|\frac{Ky_n}{\lambda_n} - \frac{Ky_m}{\lambda_m}\right\|_X = \|y_n + z_{n,m}\|_X \geq \text{dist}(y_n, X_{n-1}) > 1/2$$

and this contradicts the compactness of K.

4 It is easy to see that if $\lambda \in \sigma(K)\backslash\{0\}$ then $\dim\ker(K - \lambda) < \infty$. Otherwise we could consider the usual sequence $X_n = \text{span}\{x_1, ..., x_n\} \subseteq \ker(K - \lambda)$ with $\text{dist}(x_n, X_{n-1}) > 1/2$ and $\|x_n\|_X = 1$. But then $\|Kx_n - Kx_m\| > |\lambda|/2 > 0$ for $n \neq m$ and we contradict compactness of K.

Example 18.25 (Compact operator without eigenvalues). Let f_n be an orthonormal basis in a Hilbert space H, and a decreasing sequence in $\mathbb{R}$ with a strictly decreasing sequence $a_n \xrightarrow{n \to \infty} 0$. Then

$$A := \sum_{n=1}^{\infty} a_n(\cdot, f_n)_H f_{n+1}$$

has no eigenvalues and $\sigma(A) = \{0\}$. It is easy to see that A

$$\left\|A - \sum_{n=1}^{N} a_n(\cdot, f_n)_H f_{n+1}\right\|_{\mathcal{L}(H)} = a_{N+1} \xrightarrow{N\to\infty} 0$$

and so $A \in K(H)$. It is also easy to see that $\ker A = 0$. Next, we claim

$$A^m f = \sum_{n=m}^{\infty} \prod_{j=1}^{m} a_{n-j+1}(f_{n-m+1}, f)_H f_{n+1}. \tag{18.16}$$

Formula (18.16) is trivially true for $m = 1$. Suppose it true for m. Then, for $f_0 = 0$, we have

$$\begin{aligned}
A^{m+1} f &= \sum_{n=m}^{\infty} \prod_{j=1}^{m} a_{n-j+1}(f_{n-m+1}, Af)_H f_{n+1} = \\
&= \sum_{n=m}^{\infty} \prod_{j=1}^{m} a_{n-j+1} \left(f_{n-m+1}, \sum_{l=1}^{\infty} a_l(f, f_l)_H f_{l+1}\right)_H f_{n+1} \\
&= \sum_{n=m}^{\infty} \sum_{l=1}^{\infty} \prod_{j=1}^{m} a_{n-j+1} a_l \underbrace{(f_{n-m+1}, f_{l+1})_H}_{\delta_{m-n,l}} (f, f_l)_H f_{n+1} \\
&= \sum_{n=m}^{\infty} \prod_{j=1}^{m} a_{n-j+1} a_{n-m}(f, f_{n-m})_H f_{n+1} = \sum_{n=m+1}^{\infty} \prod_{j=1}^{m+1} a_{n-j+1}(f, f_{n-m})_H f_{n+1}
\end{aligned}$$

where in the last line we used that $f_0 = 0$. This yields (18.16) is trivially true for $m+1$ proving it for all m.
We have

$$\|A^m\|_{\mathcal{L}(H)}^{\frac{1}{m}} = \left(\prod_{n=1}^{m} |a_n|\right)^{\frac{1}{m}} \leq \frac{1}{m} \sum_{n=1}^{m} |a_n| \xrightarrow{m\to\infty} 0,$$

where the inequality follows from,

$$\log \left(\prod_{n=1}^{m} |a_n|\right)^{\frac{1}{m}} = \sum_{n=1}^{m} \frac{1}{m} \log |a_n| \leq \log \frac{1}{m} \sum_{n=1}^{m} |a_n|,$$

that is, the fact that log (with basis e) is strictly concave and increasing. So $\|A^m\|_{\mathcal{L}(H)}^{\frac{1}{m}} \xrightarrow{m\to\infty} 0$. This implies that there are no nonzero eigenvalues and so $\sigma(A) = \{0\}$.

Remark 18.26. Notice that, if λ is an eigenvalue, from $|\lambda| \leq \|T\|$ we can derive also $|\lambda| \leq \|T^n\|^{\frac{1}{n}}$ for all $n \in \mathbb{N}$. So, in particular, if $\|T^n\|^{\frac{1}{n}} \xrightarrow{n\to+\infty} 0$, we get $\lambda = 0$.

Exercise 18.27. Suppose that $T \in \mathcal{L}(X)$ is an operator with $\sigma(T) = \{0\}$. Is necessarily T compact?

Answer. No. Take in $\ell^2(\mathbb{N})$ the operator

$$T(x_1, x_2, ...) = (x_2, 0, x_4, 0, ..., \underbrace{x_{2n}, 0}_{\text{at } 2n-1 \text{ and } 2n \text{ place}}, ...).$$

It is trivial that $T^2 = 0$. Notice that if $\lambda \neq 0$, then

$$(\lambda - T)\left(\frac{1}{\lambda} + \frac{T}{\lambda^2}\right) = 1 + \frac{1}{\lambda}(T - T) - \frac{1}{\lambda^2}T^2 = 1$$

which means that $\lambda \notin \sigma(T)$. □

Exercise 18.28. Consider the operator $Tf(x) = x^{-1} \int_0^x f(t)dt$, that, as we saw in Example 16.40, defines an bounded linear operator in $L^p(0,1)$ for all $1 < p \leq +\infty$. Show that it is not a compact operator.

Answer. For any $\lambda > 0$ let $f_\lambda(x) := x^{\frac{1}{\lambda}}$ then $\lambda f'_\lambda = x^{-1} f_\lambda$ or also, at east formally,

$$Tf'_\lambda = \lambda f'_\lambda.$$

Notice that for $\lambda \in (0,1)$ we have $f'_\lambda = \frac{1}{\lambda} x^{\frac{1}{\lambda}-1} \in L^\infty(0,1)$. So $\sigma(T) \supseteq [0,1]$,in contrast to Theorem 18.24. □

Example 18.29. For $s > \sigma$, the inclusion $H^s(\mathbb{T}^d) \subset H^\sigma(\mathbb{T}^d)$ is compact.
Indeed, let $\tau = s - \sigma$. Then notice that $\Lambda^\tau : H^s(\mathbb{T}^d) \to H^\sigma(\mathbb{T}^d)$ is a bounded operator. Next, we claim that $\Lambda^{-\tau} : H^\sigma(\mathbb{T}^d) \to H^\sigma(\mathbb{T}^d)$ is compact. Assuming the claim, the lemma follows by the fact that the immersion coincides with $\Lambda^{-\tau} \circ \Lambda^\tau$. So let us prove the claim. Let us split

$$\Lambda^{-\tau} f = \sum_{|\ell| \leq R} \langle \ell \rangle^{-\tau} \widehat{f}(\ell) \frac{e^{i\ell \cdot x}}{(2\pi)^{\frac{d}{2}}} + \sum_{|\ell| > R} \langle \ell \rangle^{-\tau} \widehat{f}(\ell) \frac{e^{i\ell \cdot x}}{(2\pi)^{\frac{d}{2}}} =: T_{1R} f + T_{1R} f.$$

Then, for any R the operator T_{1R} has finite rank while

$$\begin{aligned} \|T_{2R} f\|^2_{H^\sigma(\mathbb{T}^d)} &= \sum_{|\ell| > R} \langle \ell \rangle^{-2\tau} \langle \ell \rangle^{2\sigma} |\widehat{f}(\ell)|^2 \leq \langle R \rangle^{-2\tau} \sum_{\ell \in \mathbb{Z}^d} \langle \ell \rangle^{2\sigma} |\widehat{f}(\ell)|^2 \\ &= \langle R \rangle^{-2\tau} \|f\|^2_{H^\sigma(\mathbb{T}^d)} \end{aligned}$$

so $\|T_{2R}\|_{\mathcal{L}(H^\sigma(\mathbb{T}^d))} \leq \langle R \rangle^{-\tau} \xrightarrow{R \to +\infty} 0$. Hence, since $T_{1R} \xrightarrow{R \to +\infty} \Lambda^{-\tau}$ in $\mathcal{L}(H^\sigma(\mathbb{T}^d))$, it follows that $\Lambda^{-\tau}$ is a compact operator.

Exercise 18.30. Show that the embedding the embedding in Example 17.52 is compact for any $s > d/2$.

Remark 18.31. Notice that the following statement is true:

$$\text{for any } s \in (0, d/2) \text{ we have an embedding } H^s(\mathbb{T}^d) \hookrightarrow L^{p_s}(\mathbb{T}^d) \text{ for } \frac{1}{p_s} = \frac{1}{2} - \frac{s}{d}. \tag{18.17}$$

Notice that there is a natural analogue

$$\text{for any } s \in (0, d/2) \text{ we have an embedding } H^s(\mathbb{R}^d) \hookrightarrow L^{p_s}(\mathbb{R}^d) \tag{18.18}$$

with natural analogue $H^s(\mathbb{R}^d)$ space which can be defined using the Fourier transform. There is also $H^s(\Omega) \hookrightarrow L^{p_s}(\Omega)$ for Ω open in $\mathbb{R}^d$ (where $H^s(\Omega)$ for $s \notin \mathbb{N}$ is more delicate to define). Usually the proof of (18.17), or of the Sobolev embedding theorem in the context of more general Riemannian manifolds than the tori $\mathbb{T}^d := \mathbb{R}^d/2\pi\mathbb{Z}^d$, is obtained using coordinate charts and (18.18).

Example 18.32. Notice that the Sobolev Embeddings in (18.17) are not compact. Since we don't have at disposal the Fourier transform, it is easier for us to check this in the special case $s \in \mathbb{N}$. Take any $u \in C_c^\infty((-\pi,\pi)^d)$ and for $\lambda \geq 1$ consider $\delta_{p_s\lambda} u \in C_c^\infty((-\pi,\pi)^d)$. Then using the equivalence in Exercise 17.49 consider the equivalent norm

$$\begin{aligned}
\|\delta_{p_s\lambda} u\|^2_{H^s(\mathbb{T}^d)} &= \sum_{|\alpha|\leq s} \|\partial_x^\alpha \delta_{p_s\lambda} u\|_{L^2(\mathbb{T}^d)} = \sum_{|\alpha|\leq s} \lambda^{d\left(\frac{1}{p_s}-\frac{1}{2}\right)} \|\partial_x^\alpha \delta_{2\lambda} u\|_{L^2(\mathbb{T}^d)} \\
&= \sum_{|\alpha|\leq s} \lambda^{d\left(\frac{1}{p_s}-\frac{1}{2}\right)} \lambda^{|\alpha|} \|\delta_{2\lambda} \partial_x^\alpha u\|_{L^2(\mathbb{T}^d)} \\
&= \sum_{|\alpha|\leq s} \lambda^{d\left(\frac{1}{p_s}-\frac{1}{2}\right)+|\alpha|} \|\partial_x^\alpha u\|_{L^2(\mathbb{T}^d)} \xrightarrow{\lambda\to+\infty} \sum_{|\alpha|=s} \|\partial_x^\alpha u\|_{L^2(\mathbb{T}^d)},
\end{aligned}$$

where we exploited that $\lambda^{d\left(\frac{1}{p_s}-\frac{1}{2}\right)+|\alpha|} \xrightarrow{\lambda\to+\infty} 0$ for $|\alpha| < s$ and $\lambda^{d\left(\frac{1}{p_s}-\frac{1}{2}\right)+|\alpha|} = 1$ for $|\alpha| = s$. On the other hand we know that

$$\|\delta_{p_s\lambda} u\|_{L^{p_s}((-\pi,\pi)^d)} = \|u\|_{L^{p_s}((-\pi,\pi)^d)} \text{ for all } \lambda \geq 1. \tag{18.19}$$

Consider $\lambda_n \xrightarrow{n\to+\infty} +\infty$. Then the sequence $\delta_{p_s\lambda_n} u$ is bounded in $H^s(\mathbb{T}^d)$. If the above embedding is compact, then $\delta_{p_s\lambda_n} u$ is relatively compact in $L^{p_s}((-\pi,\pi)^d)$. But in fact, we know $2 < p_s < \infty$ and $\delta_{p_s\lambda_n} u \rightharpoonup 0$ in $L^{p_s}((-\pi,\pi)^d)$ so, if a subsequence converges strongly somewhere, it must converge to 0. But by (18.19) we know this is not the case if $u \neq 0$ and we conclude a contradiction, and therefore that the embedding in (18.17) is not compact, at least in the case $s \in \mathbb{N}$. The argument used is similar to that in Example 18.7. In Example 18.7 we exploited the scale equivariance (18.4). Here we used $\nabla^s \delta_{p_s\lambda} = \delta_{2\lambda} \nabla^s$.
The case $s \notin \mathbb{N}$ is similar, but requires the use of the Fourier transform, which will be introduced next semester. This argument is used in Brezis [3] to show that Sobolev Embedding $W^{1,1}(I) \hookrightarrow L^\infty(I)$ is always not compact, for any interval I, and will be discussed in the next semester.

Exercise 18.33. Establish if the operator $R_0(z)$ in Example 5.14 is in $K(L^2(\mathbb{R}))$.

Exercise 18.34. Establish if the operator $R_V(z)$ in Example 5.15 is in $K(L^2(\mathbb{R}))$.

Example 18.35. The operator

$$Tu(x) = \int_0^x u(t)dt \tag{18.20}$$

defines a compact operator $T : L^1(0,1) \to L^1(0,1)$. To get a sense whether or not is compact, it makes sense to consider sequences like $f_n(x) := n\chi_{[-1,0]}(n(x-1)) = n\chi_{[1-1/n,1]}$. Then

$$Tf_n(x) = \begin{cases} 0 \text{ if } 0 \le x \le 1 - 1/n \\ n\,(x - (1-1/n)) \text{ if } 1 - 1/n \le x \le 1. \end{cases}$$

Then

$$\int_0^1 |Tf_n|dx = n\int_{1-1/n}^1 (t - (1-1/n))\,dt = n\left.\frac{(t-(1-1/n))^2}{2}\right]_{1-1/n}^1 = \frac{1}{n}.$$

So $\|Tf_n\|_{L^1(0,1)} \xrightarrow{n\to+\infty} 0$, and this is compatible with T being compact.

In fact, in the Spring Semester you will see that the embedding $W^{1,1}(0,1) \hookrightarrow L^1(0,1)$ is compact and it happens here that $T \in \mathcal{L}\left(L^1(0,1), W^{1,1}(0,1)\right)$. However this is not an adequate answer now, since in this moment we don't even know what $W^{1,1}(0,1)$ is .

Let us see if we can use the compactness criterium by Kolmogorov, Riesz, Frechét. Let us consider

$$Sf(x) = \int_0^x f(t)dt - x\int_0^1 f(t)dt. \tag{18.21}$$

Notice that $S \in \mathcal{L}(L^1(0,1), C^0([0,1]))$ and that $Sf(0) = Sf(1) = 0$ for all $f \in L^1(0,1)$. Extend $Sf(x) = 0$ for $x \in \mathbb{R}\backslash[0,1]$ and let

$$\mathcal{F} = \{Sf \in L^1(\mathbb{R}) : f \in D_{L^1(0,1)}(0,1)\}.$$

Let us check that condition (16.30) is satisfied. For definiteness let $h > 0$. Then

$$\|Sf(\cdot+h) - Sf\|_{L^1(\mathbb{R})} = \int_0^{1-h} |Sf(x+h) - Sf(x)|dx + \int_{1-h}^1 |Sf(x)|dx + \int_0^h |Sf(x)|dx.$$

Since $\|Sf\|_{L^\infty(\mathbb{R})} \le 2\|f\|_{L^1(0,1)}$, the sum of the last two terms is bounded by $2|h|$. Next

$$\int_0^{1-h} |Sf(x+h) - Sf(x)|dx \le h\|f\|_{L^1(0,1)} + \int_0^{1-h} dx \int_x^{x+h} |f(t)|dt$$
$$= h\|f\|_{L^1(0,1)} + \int_0^1 dt|f(t)|\int_{t-h}^t dx = 2h\|f\|_{L^1(0,1)} \le 2|h|.$$

This yields (16.30) taking $h > 0$. With a similar argument we can consider the case $h < 0$, obtaining the desired compactness.

Exercise 18.36. Let $I = [0,1] \subseteq \mathbb{R}$, $X = C^0(I)$ and $Y = L^1(I)$. Set

$$Tu(x) := \int_0^x xyu(y)\,dy.$$

a) Prove that $T \in \mathcal{L}(X)$ and $T \in \mathcal{L}(Y)$.
b) Establish if T is compact in X and in Y, justifying the answer.

18.1 Hilbert–Schmidt operators

The following is a very important class of operators.

Definition 18.37. A linear operator $T : L^2(X, d\mu) \to L^2(X, d\mu)$ is a Hilbert–Schmidt operator if it is of the form

$$Tf(x) = \int_X K(x,y)f(y)d\mu(y) \text{ with } K \in L^2(X \times X, d\mu \times d\mu).$$

We denote $\|T\|_{HS} := \|K\|_{L^2(X\times X)}$.

It is straightforward that $T \in \mathcal{L}(L^2(X, d\mu))$. Indeed

$$\int d\mu(x)|Tf(x)|^2 \le \int d\mu(x)\|K(x,\cdot)\|^2_{L^2(X)}\|f\|^2_{L^2(X)} = \|K\|^2_{L^2(X\times X)}\|f\|^2_{L^2(X)},$$

so that in particular we obtain

$$\|T\|_{\mathcal{L}(L^2(X,d\mu))} \le \|T\|_{HS}.$$

It turns out that T is also compact.

Notice that there exists a sequence of $K_n \in L^2(X, d\mu) \otimes L^2(X, d\mu)$ with $K_n \xrightarrow{n\to+\infty} K$ in $L^2(X \times X)$. But, then, if we set

$$T_n f(x) = \int_X K_n(x,y)f(y)d\mu(y),$$

we have

$$\|T - T_n\|_{\mathcal{L}(L^2(X,d\mu))} \le \|T - T_n\|_{HS} = \|K - K_n\|_{L^2(X\times X)} \xrightarrow{n\to+\infty} 0.$$

Now, for each n, we have $\dim R(T_n) < \infty$, so T_n is compact. Then also T is compact.

Exercise 18.38. Let $Tf = \displaystyle\int_0^x f(t)dt$ in $L^2(0,1)$.
a) Find T^*.
b) Show that T is Hilbert–Schmidt operator.
c) Find $\sigma(T)$.

Remark 18.39. Notice that T is compact in $L^2(0,1)$ but $f \to \frac{1}{x}Tf$, while bounded, is not compact, see Example 16.40 and Exercise 18.28. Notice also that it is possible to compute

$$T^n f(x) = \int_0^x f(t)\frac{(x-t)^{n-1}}{(n-1)!}dt.$$

Notice that this implies $\|T^n\|_{\mathcal{L}(L^p(0,1))} \le \frac{1}{(n-1)!}\|T\|_{\mathcal{L}(L^p(0,1))} \xrightarrow{n\to+\infty} 0$ and in particular, shows in the previous exercise, that $\sigma(T) = \{0\}$.

Exercise 18.40. Let $A, B \in \mathcal{L}(X)$ with X a Banach space..

a) Show that if either A or B is compact, the composition AB is compact.

b) Is the condition that one of A and B be compact, necessary in order for the composition AB to be compact?

18.2 The Lax–Milgram Theorem

Definition 18.41. Let H be a Hilbert space on $K = \mathbb{R}$ (resp. $\mathbb{C}$). A bilinear (sesquilinear if $K = \mathbb{C}$) form $B : H \times H \to K$ is said bounded if there is a $\gamma \in \mathbb{R}_+$ such that

$$|B(x,y)| \le \gamma\|x\|_H\,\|y\|_H \text{ for all } x, y \in H. \tag{18.22}$$

and coercive if there is a $\delta \in \mathbb{R}_+$ such that

$$\delta\|x\|_H^2 \le |B(x,x)| \text{ for all } x \in H. \tag{18.23}$$

Example 18.42. Let $V \in C^0(\mathbb{T}^d, [0,+\infty))$ with V positive and $V \not\equiv 0$ and consider the

$$\begin{aligned} &H^1(\mathbb{T}^d,\mathbb{C}) \times H^1(\mathbb{T}^d,\mathbb{C}) \ni (u,v) \to B(u,v) \in \mathbb{C} \\ &B(u,v) := (\nabla u, \nabla v)_{L^2(\mathbb{T}^d,\mathbb{C}^d)} + (Vu, v)_{L^2(\mathbb{T}^d,\mathbb{C})}. \end{aligned} \tag{18.24}$$

By the simple fact that $\nabla \in \mathcal{L}\left(H^1(\mathbb{T}^d,\mathbb{C}), L^2(\mathbb{T}^d,\mathbb{C}^d)\right)$ and that the multiplier operator $u \to Vu$ is bounded from $L^2(\mathbb{T}^d,\mathbb{C})$ into itself, we obtain that the above sesquilinear map is bounded. Now let us check that it is coercive. We have

$$\begin{aligned} &(\nabla u, \nabla u)_{L^2(\mathbb{T}^d,\mathbb{C}^d)} = \|\nabla u\|^2_{L^2(\mathbb{T}^d,\mathbb{C}^d)} = \|\mathbf{n}\widehat{u}(\mathbf{n})\|^2_{\ell^2(\mathbb{Z}^d,\mathbb{C}^d)} = \sum_{\mathbf{n}\in\mathbb{Z}^d} |\mathbf{n}|^2|\widehat{u}(\mathbf{n})|^2 \\ &= (2\pi)^d \sum_{\mathbf{n}\neq 0} |\mathbf{n}|^2|\widehat{u}(\mathbf{n})|^2 \ge 2^{-1}\sum_{\mathbf{n}\neq 0}(1+|\mathbf{n}|^2)|\widehat{u}(\mathbf{n})|^2 = 2^{-1}\|u-(2\pi)^{-\frac{d}{2}}\widehat{u}(0)\|^2_{H^1(\mathbb{T}^d,\mathbb{C})} \ge 0. \end{aligned} \tag{18.25}$$

We have

$$|(Vu,v)_{L^2(\mathbb{T}^d,\mathbb{C})}| \le \|V\|_{L^\infty(\mathbb{T}^d,\mathbb{C})}\|u\|_{L^2(\mathbb{T}^d,\mathbb{C})}\|v\|_{L^2(\mathbb{T}^d,\mathbb{C})} \tag{18.26}$$

Notice that $\|u\|^2_{H^1(\mathbb{T}^d,\mathbb{C})} = |\widehat{u}(0)|^2 + \|u-(2\pi)^{-\frac{d}{2}}\widehat{u}(0)\|^2_{H^1(\mathbb{T}^d,\mathbb{C})}$. Since $V \ge 0$, we have $(Vu,u)_{L^2(\mathbb{T}^d,\mathbb{C})} \ge 0$

If say $|\widehat{u}(0)|^2 \le C^2 \|u - (2\pi)^{-\frac{d}{2}} \widehat{u}(0)\|^2_{H^1(\mathbb{T}^d,\mathbb{C})}$ for a $C > 0$ to be defined momentarily, from (18.25) we conclude

$$\begin{aligned} B(u,u) &\ge \|\nabla u\|^2_{L^2(\mathbb{T}^d,\mathbb{C}^d)} \ge 2^{-1} \|u - (2\pi)^{-\frac{d}{2}} \widehat{u}(0)\|^2_{H^1(\mathbb{T}^d,\mathbb{C})} \\ &= \frac{2^{-1}}{1+C^2} \left(C^2 \|u - (2\pi)^{-\frac{d}{2}} \widehat{u}(0)\|^2_{H^1(\mathbb{T}^d,\mathbb{C})} + \|u - (2\pi)^{-\frac{d}{2}} \widehat{u}(0)\|^2_{H^1(\mathbb{T}^d,\mathbb{C})} \right) \\ &\ge \frac{2^{-1}}{1+C^2} \left(|\widehat{u}(0)|^2 + \|u - (2\pi)^{-\frac{d}{2}} \widehat{u}(0)\|^2_{H^1(\mathbb{T}^d,\mathbb{C})} \right) = \frac{2^{-1}}{1+C^2} \|u\|^2_{H^1(\mathbb{T}^d,\mathbb{C})}. \end{aligned}$$

If instead $|\widehat{u}(0)|^2 > C^2 \|u - (2\pi)^{-\frac{d}{2}} \widehat{u}(0)\|^2_{H^1(\mathbb{T}^d,\mathbb{C})}$, we can consider

$$\begin{aligned} &|(Vu,u)_{L^2(\mathbb{T}^d,\mathbb{C})}| \ge (2\pi)^{-d} (V\widehat{u}(0), \widehat{u}(0))_{L^2(\mathbb{T}^d,\mathbb{C})} \\ &- 2\|V\|_{L^\infty(\mathbb{T}^d,\mathbb{C})} \|u - (2\pi)^{-\frac{d}{2}} \widehat{u}(0)\|_{L^2(\mathbb{T}^d,\mathbb{C})} \|(2\pi)^{-\frac{d}{2}} \widehat{u}(0)\|_{L^2(\mathbb{T}^d,\mathbb{C})} - \|V\|_{L^\infty(\mathbb{T}^d,\mathbb{C})} \|u - (2\pi)^{-\frac{d}{2}} \widehat{u}(0))\|^2_{L^2(\mathbb{T}^d,\mathbb{C})} \\ &\ge \left((2\pi)^{-d} \|V\|_{L^1(\mathbb{T}^d,\mathbb{C})} - 2(2\pi)^{-\frac{d}{2}} \|V\|_{L^\infty(\mathbb{T}^d,\mathbb{C})} C^{-1} - \|V\|_{L^\infty(\mathbb{T}^d,\mathbb{C})} C^{-2} \right) |\widehat{u}(0)|^2. \end{aligned}$$

So, choosing $C \gg 1$, for $|\widehat{u}(0)|^2 > C^2 \|u - (2\pi)^{-\frac{d}{2}} \widehat{u}(0)\|^2_{H^1(\mathbb{T}^d,\mathbb{C})}$ we get

$$\begin{aligned} &|(Vu,u)_{L^2(\mathbb{T}^d,\mathbb{C})}| \ge 2^{-1} (2\pi)^{-d} \|V\|_{L^1(\mathbb{T}^d,\mathbb{C})} |\widehat{u}(0)|^2 \\ &= \frac{2^{-1}(2\pi)^{-d} \|V\|_{L^1(\mathbb{T}^d,\mathbb{C})}}{1+C^2} \left(|\widehat{u}(0)|^2 + C^2 |\widehat{u}(0)|^2 \right) \ge \frac{2^{-1}(2\pi)^{-d} \|V\|_{L^1(\mathbb{T}^d,\mathbb{C})}}{1+C^2} \|u\|^2_{H^1(\mathbb{T}^d,\mathbb{C})}. \end{aligned}$$

So we get the lower bound in (18.23) for

$$0 < \delta \le \min \frac{2^{-1}}{1+C^2} \left\{ 1, (2\pi)^{-d} \|V\|_{L^1(\mathbb{T}^d,\mathbb{C})} \right\}.$$

Notice that

$$B(u,v) = (\nabla u, \nabla v)_{L^2(\mathbb{T}^d,\mathbb{C}^d)} + (Vu,v)_{L^2(\mathbb{T}^d,\mathbb{C})} = (Au,v)_{L^2(\mathbb{T}^d,\mathbb{C}^d)} \text{ where } A := -\triangle + V, \tag{18.27}$$

where $A \in \mathcal{L}(H^1(\mathbb{T}^d,\mathbb{C}), H^{-1}(\mathbb{T}^d,\mathbb{C}))$.

Example 18.43. Recall from the Riesz Frechét Theorem 17.10 that, given a Hilbert space H, there is a natural isomorphism $H \to H'$ given by $u \to (u, \cdot)$. However, often it is natural not to identify H and H'. A case point are the spaces $H^s(\mathbb{T}^d)$ when $s \neq 0$, which are Sobolev spaces, that is, some of the spaces used in applications of Functional Analysis. If we consider two trigonometric polynomials, then we have

$$(f,g)_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} f(x)\overline{g(x)}dx = \sum_{\ell\in\mathbb{Z}^d} \widehat{f}(\ell)\overline{\widehat{g}(\ell)} = \sum_{\ell\in\mathbb{Z}^d} \langle \ell \rangle^s \widehat{f}(\ell) \langle \ell \rangle^{-s} \overline{\widehat{g}(\ell)}.$$

Then we get

$$\left|(f,g)_{L^2(\mathbb{T}^d)}\right| \le \|f\|_{H^s(\mathbb{T}^d)}\|g\|_{H^{-s}(\mathbb{T}^d)}.$$

This shows that $(\cdot,\cdot)_{L^2(\mathbb{T}^d)} : H^s(\mathbb{T}^d)\times H^{-s}(\mathbb{T}^d)\to\mathbb{C}$ is a bounded bilinear map. It is easy to conclude from this that there exists an isomorphism $H^{-s}(\mathbb{T}^d)g \ni\to (\cdot,g)_{L^2(\mathbb{T}^d)} \in \big(H^s(\mathbb{T}^d)\big)'$. This sort of identification, arising concretely from the inner product in $L^2(\mathbb{T}^d)$, is much more common in practice than the somewhat more abstract identification of $H^s(\mathbb{T}^d)$ and $\big(H^s(\mathbb{T}^d)\big)'$.

Theorem 18.44. *Let B be like in Definition 18.41 . Then there exists $S\in\mathcal{L}(H)$ which is invertible, $S^{-1}\in\mathcal{L}(H)$, such that*

$$(x,y)_H = B(x,Sy). \tag{18.28}$$

We have $\|S\|_{\mathcal{L}(H)}\le\delta^{-1}$ and $\|S^{-1}\|_{\mathcal{L}(H)}\le\gamma$. If B is a symmetric (or Hermitian) bilinear form, then S is a symmetric operator.

Proof. Let

$$D := \{y\in H :\ \text{there is a } y^*\in H \text{ such that } (x,y)_H = B(x,y^*)_H \text{ for all } x\in H\ \}.$$

Obviously $0\in D$, with $0^*=0$. y^* if it exists is unique, since $0=B(x,y_1^*-y_2^*)$ for all x, and so in particular for $x=y_1^*-y_2^*$, implies $0=|B(y_1^*-y_2^*,y_1^*-y_2^*)|\ge\delta\|y_1^*-y_2^*\|_H^2$ and so $\|y_1^*-y_2^*\|_H=0$.
So we have a well defined function $S:D\to H$ defined by $Sy=y^*$. It is easy to see that S is linear and that

$$\delta\|Sy\|_H^2\le|B(Sy,Sy)|=|(Sy,y)_H|\le\|Sy\|_H\|y\|_H.$$

So $\|S\|_{\mathcal{L(D,H)}}\le\delta^{-1}$.
Next, we claim that D is closed. First of all, we have an extension $S:\overline{D}\to H$. For $D\ni y_n\xrightarrow{n\to+\infty} z$, then by continuity $Sy_n\xrightarrow{n\to+\infty} w$ for some $w\in H$ with, by the continuity of B,

$$(x,z)_H=\lim_{n\to+\infty}(x,y_n)_H=\lim_{n\to+\infty}B(x,Sy_n)=B(x,w).$$

So $z\in D$ with $w=Sz$.
It remains to be shown that $D=H$. Suppose $D\subsetneqq H$ and consider $w_0\in D^\perp$ nonzero. Then, by $|B(x,w_0)|\le\gamma\|x\|_H\|w_0\|_H$, the Riesz Frechét Theorem 17.10 guarantees the existence of a $w\in H$ such that $B(x,w_0)=(x,w)_H$ for all $x\in H$. This implies $w_0=Sw$. Then

$$\delta\|w_0\|_H^2\le|B(w_0,w_0)|=|(w_0,w)_H|=0\Rightarrow w_0=0.$$

The above argument shows that $D = H$ but also that $S(D) = H$. Since $S : H \to H$ is both surjective and injective, and since it is bounded, it follows that $S^{-1} : H \to H$ is a bounded operator. We have $|(z, S^{-1}y)_H| = |B(z, y)| \le \gamma \|z\|_H \|y\|_H$ and so $\|S^{-1}\|_{\mathcal{L}(H)} \le \gamma$.
Finally, from (18.28) and if B is Hermitian, cf. Definition 17.2, we have

$$B(x, y) = (x, S^{-1}y)_H = ((S^{-1})^* x, y)_H = \overline{B(y, x)} = \overline{(y, S^{-1}x)}_H = (S^{-1}x, y)_H,$$

from which we read that $(S^{-1})^* = S^{-1}$. From this we conclude also $S^* = S$.

□

Corollary 18.45 (Lax–Milgram)**.** *Under the previous hypotheses, let $f' \in H'$ and consider the problem of finding u such that*

$$B(v, u) = \langle v, f' \rangle_{H \times H'} \text{ for any } v \in H. \tag{18.29}$$

Then there exists exactly one solution and is given by $u = Sf$, where $f \in H$ and $f' \in H'$ are related by $\langle \cdot, f' \rangle_{H \times H'} = (\cdot, f)_H$.

Proof. We know there is an isomorphism $f' \to f$ such that $\langle \cdot, f' \rangle_{H \times H'} = (\cdot, f)_H$. We also have

$$\langle \cdot, f' \rangle_{H \times H'} = (v, f)_H = B(v, Sf) \text{ for all } v \in H.$$

So $u = Sf$.

□

Lemma 18.46. *Let $T \in \mathcal{L}(H)$ be selfadjoint. Then $\sigma(T) \subset \mathbb{R}$.*

Proof. Let $\lambda \in \mathbb{C} \backslash \mathbb{R}$. Then, for

$$B(u, v) := ((T - \lambda)u, v)_H, \tag{18.30}$$

obviously B satisfies (18.22) for $\gamma = \|T\|_{\mathcal{L}(H)} + |\lambda|$.
From

$$B(u, u) = ((T - \lambda)u, u)_H = \underbrace{((T - \lambda_R)u, u)_H}_{\in \mathbb{R}} - \mathrm{i} \underbrace{\lambda_I (u, u)_H}_{\in \mathbb{R}}$$

we obtain

$$|B(u, u)| \ge |\lambda_I| \|u\|_H^2$$

and so we get the lower bound (18.23) with $\delta = |\lambda_I| > 0$. So there exist the $S, S^{-1} \in \mathcal{L}(H)$ with

$$B(u, v) = ((T - \lambda)u, v)_H = (u, S^{-1}v)_H \text{ for all } u, v \in H.$$

Then $(T - \lambda) = (S^{-1})^*$ and so $(T - \lambda)^{-1} = S^*$. Then $\lambda \notin \sigma(T)$ if $\lambda \notin \mathbb{R}$, and this completes the proof $\sigma(T) \subset \mathbb{R}$. □

Exercise 18.47. Let $U \in \mathcal{L}(U)$ be self–adjoint and unitary. Show that $\sigma(U) \subseteq \{-1, 1\}$. Furthermore we have

$$H = \bigoplus_{\lambda \in \sigma(U)} \ker(U - \lambda). \tag{18.31}$$

In particular show that for $U \neq \pm 1$ then we have a nontrivial orthogonal decomposition

$$H = \ker(U + 1) \bigoplus \ker(U - 1). \tag{18.32}$$

Proposition 18.48. *Let $T \in \mathcal{L}(H)$ be selfadjoint. Then*

$$\inf \sigma(T) = m \text{ where } m := \inf\{(Tu, u)_H : u \in H \text{ with } \|u\|_H = 1\} \tag{18.33}$$

$$\sup \sigma(T) = M \text{ where } M := \sup\{(Tu, u)_H : u \in H \text{ with } \|u\|_H = 1\}. \tag{18.34}$$

Furthermore $\|T\|_{\mathcal{L}(H)} = \max\{|m|, M\}$.

Proof. We know already that $\sigma(T) \subset \mathbb{R}$ by in Lemma 18.46. By proceeding like in Lemma 18.46 for

$$\mathbb{R} \ni \lambda \notin [m, M]$$

we get $\lambda \notin \sigma(T)$. Indeed, for B like in (18.30), B satisfies (18.22) for $\gamma = \|T\|_{\mathcal{L}(H)} + |\lambda|$. If say, $\lambda < m$, we have

$$B(u, u) = (Tu, u)_H - \lambda \|u\|_H^2 \geq (m - \lambda) \|u\|_H^2$$

and so we get (18.23) with $\delta = m - \lambda > 0$. We conclude that if $\lambda < m$ then $\lambda \notin \sigma(T)$.
The proof for $\lambda > M$ can be done similarly or, alternatively, one can observe that case $\lambda < m$ implies case $\lambda > M$ by replacing $T \rightsquigarrow -T$.
We now need to show that $m, M \in \sigma(T)$ and it is not restrictive to reduce to the proof of $m \in \sigma(T)$. Let us consider $B(u, v) := ((T - m)u, v)_H$. Since $T - m$ is symmetric and positive, by $B(u, u) \geq 0$ for all $u \in H$, by Theorem 17.39 it has a positive and symmetric square root. So we have

$$\begin{aligned} &|B(u, v)| = |((T - m)u, v)_H| = |((T - m)^{\frac{1}{2}} u, (T - m)^{\frac{1}{2}} v)_H| \\ &\leq \|(T - m)^{\frac{1}{2}} u\|_H \|(T - m)^{\frac{1}{2}} v\|_H = \sqrt{B(u, u)}\ \sqrt{B(v, v)} \leq \sqrt{B(u, u)} \sqrt{\gamma} \|v\|_H. \end{aligned} \tag{18.35}$$

Then,

$$\|(T - m)u\|_H \leq \sqrt{\gamma} \sqrt{B(u, u)} = \sqrt{\gamma} \sqrt{((T - m)u, u)_H}. \tag{18.36}$$

Then, there exists a sequence $\{u_n\}_{n \in \mathbb{N}}$ with $\|u_n\|_H = 1$ such that $\|(T - m)u_n\|_H \xrightarrow{n \to +\infty} 0$. This implies that $m \in \sigma(T)$. Indeed, if $m \notin \sigma(T)$ then $(T - m)^{-1} \in \mathcal{L}(H)$ and

$$1 = \|u_n\|_H = \|(T - m)^{-1}(T - m)u_n\|_H \leq \|(T - m)^{-1}\|_{\mathcal{L}(H)} \|(T - m)u_n\|_H \xrightarrow{n \to +\infty} 0,$$

yielding a contradiction.
Finally, we need to show that $\|T\|_{\mathcal{L}(H)} = \max\{|m|, M\}$. It is not restrictive to assume $M \geq |m|$ (otherwise, by the replacement $T \rightsquigarrow -T$ we can get to this case). By (5.19) we already know that $M \leq \|T\|_{\mathcal{L}(H)}$. If $T \geq 0$, then $\|T\|_{\mathcal{L}(H)} = \|\sqrt{T}\|^2_{\mathcal{L}(H)}$ and

$$\|T\|_{\mathcal{L}(H)} = \sup\{\|Tu\|_H : \|u\|_H = 1\} = \sup\{\|\sqrt{T}u\|^2_H : \|u\|_H = 1\} = \sup\{(Tu, u)_H : \|u\|_H = 1\} = M.$$

If T is not positive, then by Polar Decomposition Theorem 17.41 we have $T = U|T|$, with U unitary. Since T is self–adjoint, by Exercise 17.45 the U introduced in Theorem 17.41 is self–adjoint. If $U = 1$, then T is positive, so here $U \neq 1$. If $U = -1$ then it is not true that $M \geq |m|$. So in our case $U \neq \pm 1$ and, by the solution of Exercise 18.47 we have the orthogonal decomposition

$$H = \ker(U + 1) \bigoplus \ker(U - 1). \tag{18.32}$$

Since $[T, U] = 0$, T leaves the above decomposition invariant. Then,

$$\begin{aligned} T|_{\ker(U-1)} &= U|T||_{\ker(U-1)} = |T||_{\ker(U-1)} \\ T|_{\ker(U+1)} &= U|T||_{\ker(U+1)} = -\,|T||_{\ker(U+1)} \end{aligned}$$

Then

$$\|T\|_{\mathcal{L}(H)} = \max\{\||T|\|_{\mathcal{L}(\ker(U-1))}, \||T|\|_{\mathcal{L}(\ker(U-1))}\} \leq M.$$

□

Theorem 18.49 (Spectral decomposition of a selfadjoint compact operator)**.** *Let $T \in \mathcal{L}(H)$ be selfadjoint and compact operator and let H be separable. Then there exists an orthonormal basis of H formed by eigenvectors of T.*

Proof. Using $T = U|T|$ and the decomposition (18.32), it is easy to show that it is not restrictive to assume $T \geq 0$. So, assuming $T \neq 0$, we have $M > 0$ in (18.33). Notice that $m \geq 0$ in (18.34), and since $0 \in \sigma(T)$ by Theorem 18.24, we have $m = 0$. Since $\sigma(T) \ni M > 0$ it follows that M is an eigenvalue, which has finite multiplicity. Let now $M_1 := M$. Then

$$H = \ker(T - M_1) \bigoplus \ker^{\perp}(T - M_1)$$

Then the restriction of T in $\ker^{\perp}(T - M_1)$ is again a compact positive self–adjoint operator. Let $M_2 = \sup \sigma\left(T|_{\ker^{\perp}(T-M_1)}\right)$. Then $M_2 < M_1$. One gets a sequence, finite or infinite $M_1 > M_2 > ...$ of strictly positive numbers. Finally we consider

$$H = \oplus_{n\geq 1} \ker(T - M_n) \bigoplus (\oplus_{n\geq 1} \ker(T - M_n))^{\perp}$$

Then for the operator T in $(\oplus_{n\geq 1} \ker(T - M_n))^{\perp}$ we must have $m = M = 0$, that is $T = 0$. So

$$H = \oplus_{n\geq 1} \ker(T - M_n) \bigoplus \ker T.$$

□

Theorem 18.50. *Consider Example 18.42. Then there exists a sequence of strictly positive numbers* $\lambda_n \xrightarrow{n\to+\infty} \infty$ *and functions* $e_n \in H^1(\mathbb{T}^d, \mathbb{C})$ *which form an orthonormal basis of* $L^2(\mathbb{T}^d, \mathbb{C})$ *s.t.* $Ae_n = \lambda_n e_n$*, where A is the Schrödinger operator in* (18.27).

Proof. Let us consider the operator $f \in L^2(\mathbb{T}^d, \mathbb{C}) \to Sf \in H^1(\mathbb{T}^d, \mathbb{C}) \subset L^2(\mathbb{T}^d, \mathbb{C})$ which associates to each $f \in L^2(\mathbb{T}^d, \mathbb{C})$ the solution u of (18.29). This means that

$$(\nabla Sf, \nabla v)_{L^2(\mathbb{T}^d,\mathbb{C}^d)} + (VSf, v)_{L^2(\mathbb{T}^d,\mathbb{C})} = (f, v)_{L^2(\mathbb{T}^d,\mathbb{C})} \text{ for all } v \in H^1(\mathbb{T}^d, \mathbb{C}). \tag{18.37}$$

Notice that here v is the so called *test function.*
Let $g \in L^2(\mathbb{T}^d, \mathbb{C})$. Then, for $v = Sg$, from (18.37) we obtain

$$\begin{aligned}(f, Sg)_{L^2(\mathbb{T}^d,\mathbb{C})} &= (\nabla Sf, \nabla Sg)_{L^2(\mathbb{T}^d,\mathbb{C}^d)} + (VSf, Sg)_{L^2(\mathbb{T}^d,\mathbb{C})} \\ &= (\nabla Sf, \nabla Sg)_{L^2(\mathbb{T}^d,\mathbb{C}^d)} + (Sf, VSg)_{L^2(\mathbb{T}^d,\mathbb{C})} = (Sf, g)_{L^2(\mathbb{T}^d,\mathbb{C})},\end{aligned}$$

where the last equality follows reversing the roles of Sg and Sf and thinking of the latter as the test function.
We also know that $\ker S = 0$. Then, there exists a sequence of non–zero numbers $\mu_n \to 0$ and a corresponding orthonormal basis $\{e_n\}$ of $L^2(\mathbb{T}^d, \mathbb{C})$ with $Se_n = \mu_n e_n$. So we have, for $\lambda_n = 1/\mu_n$,

$$(ASe_n - e_n, v)_{L^2(\mathbb{T}^d,\mathbb{C}^d)} = (1/\mu_n Ae_n - e_n, v)_{L^2(\mathbb{T}^d,\mathbb{C}^d)} = 0 \text{ for all } v \in H^1(\mathbb{T}^d, \mathbb{C}).$$

That is

$$(Ae_n - \lambda_n e_n, v)_{L^2(\mathbb{T}^d,\mathbb{C}^d)} = 0 \text{ for all } v \in H^1(\mathbb{T}^d, \mathbb{C}) \Longrightarrow Ae_n - \lambda_n e_n = 0 \text{ in } H^{-1}(\mathbb{T}^d, \mathbb{C}).$$

□

19 Exams

Exercise 19.1. Consider the operator $Tf(x) = \frac{1}{x} f\left(\frac{1}{x}\right)$.

a Show that it is a bounded operator of $L^2(\mathbb{R}_+)$ into itself.

b Find the spectrum of T. In particular, check if there are eigenvalues and if there are eigenvalues of finite multiplicity.

c Establish if T is a compact operator.

Exercise 19.2. Consider a Banach space X and its dual space X'.

a Prove that the $\sigma(X', X)$ topology is the weakest topology in X' which makes the maps $X' \ni x' \to \langle x, x' \rangle_{X \times X'}$ continuous for all $x \in X$.

b Show that for $\dim X = +\infty$ also $\dim X' = +\infty$

c Show that for $\dim X = +\infty$ the closure of $S := \{x' \in X' : \|x'\|_{X'} = 1\}$ for the $\sigma(X', X)$ topology coincides with $\{x' \in X' : \|x'\|_{X'} \leq 1\}$.

d Find a sequence (f_n) in $L^\infty([0,1])$ with $\|f_n\|_{L^\infty([0,1])} = 1$ converging weakly to 0 for the $\sigma(L^\infty([0,1]), L^1([0,1]))$ topology.

f Show that if X is a Hilbert space and (x_n) is an orthonormal sequence in X, then $x_n \rightharpoonup 0$ in X.

e Find a sequence (f_n) in $L^\infty([0,1])$ with $\|f_n\|_{L^\infty([0,1])} = 1$ and $\operatorname{dist}(f_n, V_{n-1}) = 1$ for V_n the space spanned by $f_1, ...f_n$ such that it is not true that f_n converges weakly to 0 for the $\sigma(L^\infty([0,1]), L^1([0,1]))$ topology.

Exercise 19.3. Consider a Banach space X and let $T \in \mathcal{L}(X)$.

a Show that if $\lambda \in \mathbb{C}$ is such that $|\lambda| > \|T^n\|_{\mathcal{L}(X)}^{\frac{1}{n}}$ for a $n \in \mathbb{N}$, then $\lambda \in \rho(T)$.

b Consider the space $X = L^p((0,1),\mathbb{C})$ for some $p \geq 1$, a function $m \in C^0([0,1],\mathbb{C})$ and the operator $T_m f := mf$. Show that it is a bounded operator and that its spectrum $\sigma(T_m)$ satisfies $\sigma(T_m) = m([0,1])$.

c When is the operator T_m of (**b**) compact?

d Recall the exponential of T

$$e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!}.$$

Show that if $A, B \in \mathcal{L}(X)$ commute, that is $[A,B] := AB - BA = 0$, then $e^{A+B} = e^A e^B = e^B e^A$.

Exercise 19.4. Recall that $H^1(\mathbb{T}^d,\mathbb{C})$ is the completion of the set of trigonometric polynomials using the norm

$$\|u\|_{H^1(\mathbb{T}^d,\mathbb{C})}^2 := \sum_{n\in\mathbb{Z}^d} \langle n\rangle^2 \, |\widehat{u}(n)|^2.$$

Consider

$$\|u\|_{\dot{H}^1(\mathbb{T}^d,\mathbb{C})}^2 := \sum_{n\in\mathbb{Z}^d} |n|^2 |\widehat{u}(n)|^2.$$

a Show that $\|\cdot\|_{\dot{H}^1(\mathbb{T}^d,\mathbb{C})}$ is a continuous seminorm in $H^1(\mathbb{T}^d,\mathbb{C})$.

b Prove the following Poincaré inequality:

$$\exists\, C > 0 \text{ s.t. } \left\| u - \frac{1}{\mathrm{vol}(\mathbb{T}^d)} \int_{T^d} u dx \right\|_{L^2(\mathbb{T}^d,\mathbb{C})} \leq C\|u\|_{\dot{H}^1(\mathbb{T}^d,\mathbb{C})} \forall\, u \in H^1(\mathbb{T}^d,\mathbb{C}).$$

c Let X be a topological vector space which is a Banach space for two distinct norms $\|x\|_1$ and $\|x\|_2$. Show that the norms are equivalent, that is that there exists a $C > 1$ such that

$$\frac{1}{C}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \text{ for all } x \in X.$$

d Can we drop the hypothesis of completeness implicit in question (**c**)?

Exercise 19.5. Consider the operator $Tf = \displaystyle\int_0^x f(t)dt$ for $f \in L^1(0,1)$.

a Prove by induction the formula

$$T^n f(x) = \int_0^x f(t)\frac{(x-t)^{n-1}}{(n-1)!}dt.$$

b Show that the above implies that $\sigma(T) = \{0\}$.

c Show that for any $g \in L^1(-1,1)$ then the map

$$f \to g * f = \int_0^1 g(x-t)f(t)dt$$

is a well defined bounded operator of $L^1(0,1)$ into itself.

d Show furthermore that the operator in (**c**) is a compact operator of $L^1(0,1)$ into itself.

e Use statement (**d**) to conclude that T is a compact operator of $L^1(0,1)$ into itself.

Exercise 19.6. Let for $f \in L^2(\mathbb{T}^d,\mathbb{R})$, $\ell^2\left(\mathbb{Z}^d\right) \ni \widehat{f}(n) := (2\pi)^{-d/2}\displaystyle\int_{\mathbb{T}^d} e^{-\mathrm{i}x\cdot n} f(x)dx$. Then consider the Leray projection $\mathbb{P} : L^2(\mathbb{T}^d,\mathbb{R}^d) \to L^2(\mathbb{T}^d,\mathbb{R}^d)$ defined by

$$\left(\widehat{\mathbb{P}u}\right)^j(\mathbf{n}) = \begin{cases} \widehat{u}^j(\mathbf{0}) \text{ if } \mathbf{n} = \mathbf{0} \\ \widehat{u}^j(\mathbf{n}) - \frac{1}{\|\mathbf{n}\|^2_{\mathbb{R}^d}}\sum_{k=1}^d n_j n_k \widehat{u}^k(\mathbf{n}) \text{ if } \mathbf{n} = (n_1,...,n_d) \neq \mathbf{0} \end{cases}$$

where $\|\mathbf{n}\|^2_{\mathbb{R}^d} = n_1^2 + ... + n_d^2$.

a Show $\mathbb{P}$ is a projection.

b Discuss in what sense $\ker\mathbb{P}$ is formed by the conservative fields in $L^2(\mathbb{T}^d,\mathbb{R}^d)$.

c Show that $R(\mathbb{P})$ is formed exactly by the divergence free fields in $L^2(\mathbb{T}^d,\mathbb{R}^d)$, that is the fields such that

$$\sum_{j=1}^d n_j\widehat{u}_j(\mathbf{n}) = 0, \text{ for all } \mathbf{n} \in \mathbb{Z}^d.$$

d Let X be a Banach space on $\mathbb{C}$ and $P \in \mathcal{L}(X)$ a projection. Show that $\sigma(P) \subseteq \{0,1\}$.

Exercise 19.7. Consider in $\ell^1(\mathbb{N}) = \left\{ f : \mathbb{N} \to \mathbb{C} \text{ s.t. } \sum_{n\in\mathbb{N}} |f(n)| < \infty \right\}$ the operator $\tau_1 \in \mathcal{L}\left(\ell^1(\mathbb{N})\right)$ defined by

$$\tau_1 f(n) = \begin{cases} 0 \text{ if } n = 1 \\ f(n-1) \text{ if } n \geq 2. \end{cases}$$

a Show that, for the spectrum, we have $\sigma(\tau_1) \subseteq \overline{D_{\mathbb{C}}(0,1)}$.

b Show that $0 \in \sigma(\tau_1)$, because τ_1 is not algebraically invertible.

c Prove that for $z \neq 0$, there exists an algebraic inverse linear operator of $\tau_1 - z$. In fact, for $(\tau_1 - z)f = g$, prove that for $z \neq 0$ we have the formula

$$f(n) = -\sum_{\ell=1}^{n} \frac{1}{z^\ell}\, g(n+1-\ell). \tag{19.1}$$

d Show that the operator in (19.1) is unbounded for $0 < |z| \leq 1$.

e Prove directly on formula (19.1), that for $|z| > 1$ it yields a bounded operator.

f What changes about $\sigma(\tau_1)$ if we consider instead the operator in $\ell^1(\mathbb{Z})$ defined by $\tau_1 f(n) := f(n-1)$ for any $n \in \mathbb{Z}$?

Exercise 19.8. Consider $\mathbb{R}_+ \ni \lambda \longrightarrow \delta_{d,p,\lambda} \in \mathcal{L}(L^p(\mathbb{R}^d))$ defined by $\delta_{d,p,\lambda} f(x) := \lambda^{\frac{d}{p}} f(\lambda x)$.

a Show that $\mathbb{R}_+ \ni \lambda \longrightarrow \delta_{d,p,\lambda} \in \mathcal{L}(L^p(\mathbb{R}^d))$ is, for $p < \infty$, strongly continuous, that is $\delta_{d,p,\lambda} f \xrightarrow{\lambda \to \lambda_0} \delta_{d,p,\lambda_0} f$ for any $\lambda_0 > 0$ and any $f \in \mathcal{L}(L^p(\mathbb{R}^d))$.

b Do we have $\delta_{d,p,\lambda} \xrightarrow{\lambda \to \lambda_0} \delta_{d,p,\lambda_0}$ in norm inside $\mathcal{L}(L^p(\mathbb{R}^d))$ for $p < \infty$? Justify the answer.

c Consider the induced map $\delta_{1,2,\lambda} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$. Show that $U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ defined by $Uf(x) = e^{x/2} f(e^x)$ is an isomorphism . Show that $U\delta_{1,2,\lambda}U^{-1} = \tau_{-\log\lambda}$, where $\tau_h g(x) := g(x-h)$ for $g \in L^2(\mathbb{R})$.

Exercise 19.9. Consider a $\rho \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ s.t. $\int \rho(x)dx = 1$.

a Establish if the map $L^p\left(\mathbb{R}^d\right) \ni f \to \rho f \in L^p(\mathbb{R}^d)$, where $(\rho f)(x) := \rho(x)f(x)$, is compact for any $1 \leq p \leq \infty$.

b Establish if the map $L^p\left(\mathbb{R}^d\right) \ni f \to \rho * (\rho f) \in L^p(\mathbb{R}^d)$ is compact for any $1 \leq p \leq \infty$.

c Consider $\rho_\epsilon(x) := \epsilon^{-d}\rho(x/\epsilon)$. Establish if in the space

$$C_0^0(\mathbb{R}^d) := \{f \in C^0(\mathbb{R}^d, \mathbb{R}) : \lim_{x\to\infty} f(x) = 0\} \subseteq L^\infty(\mathbb{R}^d)$$

we have $\rho_\epsilon * f \xrightarrow{\epsilon\to 0^+} f$.

d Establish if we have $\rho_\epsilon * \xrightarrow{\epsilon\to 0^+}$Identity, in $\mathcal{L}\left(C_0^0(\mathbb{R}^d)\right)$.

e Establish if we have $\rho_\epsilon * f \xrightarrow{\epsilon\to 0^+} f$ in the space $BC^0(\mathbb{R}^d) := C^0(\mathbb{R}^d, \mathbb{R}) \cap L^\infty(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$.

Exercise 19.10. Consider the space $\ell^\infty(\mathbb{N}) = \{f : \mathbb{N} \to \mathbb{R} : \sup_{n\in\mathbb{N}} |f(n)| < \infty\}$

a Show that $\ell^\infty(\mathbb{N})$ is not separable.

b Show that there exists an isometric embedding $\ell^\infty(\mathbb{N}) \hookrightarrow BC^0(\mathbb{R}) := L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$.

c Show that $BC^0(\mathbb{R})$ is not separable.

Exercise 19.11. Let X be a Banach space, X' its dual space, $\langle \cdot, \cdot \rangle_{X' \times X}$ the duality product, and $D_{X'}(0,1)$ the unit ball in X^*. Consider a bounded sequence $\{x_n, n \in \mathbb{N}\} \subset X$ such that

$$\forall x' \in \partial D_{X'}(0,1) \text{ the sequence } \langle x', x_n \rangle_{X' \times X} \quad \text{converges.}$$

a Show that if X is reflexive, then x_n is weakly convergent in X.

b Is the above conclusion necessarily true if X is not reflexive? Prove if it is true, or find a counterexample if it is false.

Exercise 19.12. Let $I := [0,1]$ and let $I_k^i := [\frac{i-1}{k}, \frac{i}{k}]$ for $k \in \mathbb{N}$ and $i = 1, \dots, k$. For every $k \in \mathbb{N}$ let $T_k \colon L^1(I) \to L^1(I)$ be the linear operator defined by

$$(T_k(f))(x) := k \sum_{i=1}^{k} \chi_{I_k^i}(x) \int_{I_k^i} f(y) dy \text{ for every } f \in L^1(I).$$

a Prove that

$$\|T_k(f)\|_{L^1(I)} \leq \|f\|_{L^1(I)}$$

for every $f \in L^1(I)$.

b Prove that

$$T_k(f) \to f \quad \text{in } L^1(I)$$

for every $f \in C^0(I)$.

c Prove that

$$T_k(f) \to f \quad \text{in } L^1(I)$$

for every $f \in L^1(I)$.

d Is it true that

$$\lim_{k \to +\infty} \sup_{\substack{f \in L^1(I) \\ \|f\|_{L^1(I)} \leq 1}} \|T_k(f) - f\|_{L^1(I)} = 0\,?$$

Exercise 19.13. Consider the operator $T : C^0([0,1]) \to C^0([0,1])$ defined by

$$Tf(x) = \int_0^x e^{t^2} f(t)dt.$$

1. Compute the norm $\|T\|$.
2. Prove that T is compact.
3. Compute the spectrum of T.

Exercise 19.14.

1. Consider the space $C^0([0,1], L^2(\mathbb{T}, \mathbb{C}))$ with norm

   $$\|G\|_{C^0([0,1],L^2(\mathbb{T},\mathbb{C}))} = \sup\{\|G(t)\|_{L^2(\mathbb{T},\mathbb{C})} : t \in [0,1]\}.$$

   Show that it is a Banach space.

2. Denote by $S_n : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ the operator that associates to any $f \in L^2(\mathbb{T})$ its Fourier polynomial of order $n \in \mathbb{N}$ and consider an $F \in C^0([0,1], L^2(\mathbb{T}))$. Show that $S_n F \xrightarrow{n\to+\infty} F$ in $C^0([0,1], L^2(\mathbb{T}))$.

3. Consider for any $n \in \mathbb{N}$ the ordinary differential equation (ODE) in $L^2(\mathbb{T})$,

   $$\begin{cases} \dot{u}_n = S_n \partial_x^2 u_n + S_n F \\ u_n(0) = 0. \end{cases}$$

   Show that $S_n \partial_x^2$ is a bounded operator from $L^2(\mathbb{T})$ into itself and that the solution of the ODE can be written as

   $$u_n(t) = \int_0^t e^{(t-s)S_n\partial_x^2} S_n F(s)ds. \tag{19.2}$$

   Show that $u_n \in C^0([0,1], L^2(\mathbb{T}))$.

4. Show that there exists $u \in C^0([0,1], L^2(\mathbb{T}, \mathbb{C}))$ so that $u_n \xrightarrow{n\to+\infty} u$ in $C^0([0,1], L^2(\mathbb{T}, \mathbb{C}))$.

5. Check for the equation

   $$\begin{cases} \mathrm{i}\dot{u}_n = S_n \partial_x^2 u_n + S_n F \\ u_n(0) = 0 \end{cases}$$

   what the analogue of (19.2) is and if statement 4 continues to be true.

Exercise 19.15. Consider the operator $T : C^0([0,1]) \to C^0([0,1])$ defined by

$$Tf\,(x) := f\left(\frac{1}{1+x}\right)$$

1. Compute the norm $\|T\|$.
2. Show that 1 is an eigenvalue and determine explicitly $\ker(T-1)$
3. Show that 0 is an eigenvalue of T and determine explicitly all elements of $\ker T$.
4. Check if T is a compact operator (without using any of the statements below).
5. Check that if $\lambda \in \mathbb{C}$ is an eigenvalue and if $\lambda \neq 1$, then $|\lambda| < 1$.
6. Show that any $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ is an eigenvalue.
7. Find the spectrum $\sigma(T)$.

Answer. It is obvious that $\|T\| \le 1$ and by $T1 = 1$ it follows $\|T\| = 1$. Notice that we have shown that 1 is an eigenvalue. Notice that if $Tf = f$, then for any $x \in [0,1]$ we have $T^n f(x) = f\,(x_n(x)) = f(x)$ for the continuous fraction

$$\begin{aligned} x_0(x) &= x \\ x_{n+1}(x) &= \frac{1}{1+x_n(x)}. \end{aligned}$$

This sequence for any x converges to the value $\widehat{x} := \dfrac{\sqrt{5}-1}{2}$. So by continuity $f\,(\widehat{x}) = f(x)$ for any $x \in [0,1]$. So $\ker(T-1)$ is formed exactly by the constant functions.

Notice that $x \to \varphi(x) := \dfrac{1}{1+x}$ is a homeomorphism from $[0,1]$ into $[1/2,1]$. So $Tf\,(x) = f\left(\frac{1}{1+x}\right) = 0$ if and only if $f|_{[1/2,1]} = 0$. So $\ker T$ can be identified with the space of all continuous functions in $[0,1/2]$ equal to 0 at the extreme point $1/2$. In fact, each of these functions can be extended into a function in $C^0([0,1])$ identically equal to 0 in $[1/2,1]$.

By a general result we know that $\sigma(T) \subseteq \overline{D_{\mathbb{C}}(0,\|T\|)} = \overline{D_{\mathbb{C}}(0,1)}$. So all eigenvalues satisfy $|\lambda| \le 1$. Next, suppose that $\lambda \neq 1$ and $|\lambda| = 1$. Then, since the sequence $x_n(\widehat{x})$ is constant, from $Tf(\widehat{x}) = f\,(\widehat{x}) = \lambda f(\widehat{x})$, we have necessarily $f(\widehat{x}) = 0$. But now, since by continuity, for a nontrivial eigenfunction we have

$$0 = f(\widehat{x}) = \lim_{n\to+\infty} f\,(x_n(x)) = \lim_{n\to+\infty} T^n f(x) = f(x) \lim_{n\to+\infty} \lambda^n.$$

which in turn requires that $f(x) = 0$ for all $x \in [0,1]$, that is a contradiction. So we need to have $|\lambda| < 1$.

So, now let us pick a $0 < |\lambda| < 1$ and let us set $I_0 := (0,1/2)$ and $I_n := \varphi^n\,(I_0)$. Notice that $I_n \cap I_m = \emptyset$ for $n < m$. For $n = 0 < m$ follows immediately from the fact that

$I_0 \cap \varphi([0,1]) = \emptyset$ and that $\varphi^m(I_0) \subseteq \varphi([0,1])$ for $m \geq 1$. On the other hand, if $n > 0$, for $y \in I_n \cap I_m$ there exists a unique x s.t. $y = \varphi(x)$. We need to have $x \in I_{n-1} \cap I_{m-1}$ and so, going backwards, we reduce to the case $n = 0$.

Having established that $I_n \cap I_m = \emptyset$ for $n < m$, let $0 \neq f_0 \in C_c^0(I_0, \mathbb{R})$. Then let

$$f(x) := \begin{cases} f_0(x) \text{ for } x \in I_0 \\ \lambda^n f_0(y) \text{ for } x \in I_n \text{ with } x = \varphi^n(y) \text{ with } y \in I_0 \\ 0 \text{ for } x \in [0,1] \setminus \bigcup_{n=0}^{\infty} I_n. \end{cases}$$

Then we have $f \in C_c^0([0,1], \mathbb{R})$. Indeed, either a point x is in the interior of $\bigcup_{n=0}^{\infty} I_n$ or of its complement, and then f is continuous at that point, or in the frontier, where f has value 0. In this last case, if $x_\alpha \xrightarrow{\alpha \to +\infty} x$, it is enough to focus on the case when $f(x_\alpha) \neq 0$. Then we must have $x_\alpha \in I_{n(\alpha)}$ with $n(\alpha) \xrightarrow{\alpha \to +\infty} +\infty$. But then

$$|f(x_\alpha)| \leq \lambda^{n(\alpha)} \|f_0\|_{L^\infty(I_0)} \xrightarrow{\alpha \to +\infty} 0 = f(x).$$

It is possible to show that $Tf = \lambda f$. Indeed, $Tf(x) = f(\varphi(x))$ and $\varphi(x) \in I_{n+1} \iff x \in I_n$ and, by the definition

$$Tf(x) = \lambda^{n+1} f_0(y) = \lambda f(x) \text{ for } x \in I_n \text{ with } x = \varphi^n(y) \text{ with } y \in I_0.$$

At other points, $Tf(x) = f(x) = 0$.

Finally, since the spectrum $\sigma(T)$ is closed, we have $\sigma(T) = \overline{D_{\mathbb{C}}(0,1)}$. This implies that T cannot be compact.

Exercise 19.16. Let $I := [0,1]$ and let $I_k^i := [\frac{i-1}{k}, \frac{i}{k}]$ for $k \in \mathbb{N}$ and $i = 1, \ldots, k$. For every $k \in \mathbb{N}$ let $T_k \colon L^1(I) \to L^1(I)$ be the linear operator defined by

$$(T_k(f))(x) := k \sum_{i=1}^{k} \chi_{I_k^i}(x) \int_{I_k^i} f(y)dy \text{ for every } f \in L^1(I).$$

a Prove that

$$\|T_k(f)\|_{L^1(I)} \leq \|f\|_{L^1(I)}$$

for every $f \in L^1(I)$.

b Prove that

$$T_k(f) \to f \quad \text{in } L^1(I)$$

for every $f \in C^0(I)$.

c Consider the space $C^0(I, L^1(I))$ with the norm $\sup\limits_{t \in [0,1]} \|F(t,x)\|_{L^1(I)}$. Show that if we define for any function $F(t,x)$ in $C^0(I, L^1(I))$

$$T_k F(t,x) := k \sum_{i=1}^{k} \chi_{I_k^i}(x) \int_{I_k^i} F(t,y)dy$$

this defines a bounded operator of $C^0(I, L^1(I))$ into itself.

d Show that

$$T_k(F) \to F \quad \text{in } C^0(I, L^1(I)).$$

Exercise 19.17. Show the following

1. Both $\ell^2(\mathbb{N})$ and $L^2(\mathbb{R}^d)$ are separable for any $d \geq 1$.
2. Show that for any $d \geq 1$, the spaces $\ell^2(\mathbb{N})$ and $L^2(\mathbb{R}^d)$ are isomorphic.
3. Establish, justifying the answer, which of these pair of spaces are formed by isomorphic spaces.

   a $c_0(\mathbb{N})$ and $\ell^2(\mathbb{N})$

   b $c_0(\mathbb{N})$ and $\ell^1(\mathbb{N})$

   c $\ell^2(\mathbb{N})$ and $\ell^1(\mathbb{N})$
4. Show that for $1 < p < \infty$ a bounded linear operator $T : \ell^p(\mathbb{N}) \to \ell^1(\mathbb{N})$ is compact (Hint: exploit that a sequence in $\ell^1(\mathbb{N})$ converging $\sigma(\ell^1(\mathbb{N}), \ell^\infty(\mathbb{N}))$ weakly to 0, does so also strongly[3]).

Answer to the third question. Let $Y := \overline{TD_{\ell^p(\mathbb{N})}(0,1)}$ be the closure in $\ell^1(\mathbb{N})$ of $TD_{\ell^p(\mathbb{N})}(0,1)$. Since $TD_{\ell^p(\mathbb{N})}(0,1)$ is bounded in $\ell^1(\mathbb{N})$, also Y is bounded. Let $\{y_n\}$ be a sequence in Y. Then there exists a sequence x_n in $D_{\ell^p(\mathbb{N})}(0,1)$ s.t. $\|y_n - Tx_n\|_{\ell^1(\mathbb{N})} \xrightarrow{n\to+\infty} 0$. On the other hand, since $\ell^p(\mathbb{N})$ is reflexive and separable, and so $D_{\ell^p(\mathbb{N})}(0,1)$ is a relatively compact metrizable space for the $\sigma(\ell^p(\mathbb{N}), \ell^{p'}(\mathbb{N}))$ topology, there exists a subsequence of x_n, that it is not restrictive to assume the whole sequence, such that $x_n \rightharpoonup \overline{x} \in \overline{D_{\ell^p(\mathbb{N})}(0,1)}$ weakly $\sigma(\ell^p(\mathbb{N}), \ell^{p'}(\mathbb{N}))$. Then, by the continuity of $T : \left(\ell^p(\mathbb{N}), \sigma(\ell^p(\mathbb{N}), \ell^{p'}(\mathbb{N}))\right) \to \left(\ell^1(\mathbb{N}), \sigma(\ell^1(\mathbb{N}), \ell^\infty(\mathbb{N}))\right)$, it follows that $Tx_n \rightharpoonup T\overline{x}$ weakly $\sigma(\ell^1(\mathbb{N}), \ell^\infty(\mathbb{N}))$. But this implies that $\|Tx' - Tx_n\|_{\ell^1(\mathbb{N})} \xrightarrow{n\to+\infty} 0$. In turn, this means that $y_n \xrightarrow{n\to+\infty} Tx'$ in $\ell^1(\mathbb{N})$. So we have shown that Y is sequentially compact. This implies that Y is compact and the operator T is compact.

Exercise 19.18. It is a known fact, called Pitt's Theorem (partially contained in the previous exercise and proved in the general case in [1]), that if $T : \ell^a(\mathbb{Z}) \to \ell^b(\mathbb{Z})$ with $\infty > a > b \geq 1$ is a bounded linear operator, then T is compact.

1. For $1 < p < 2$ give at least one example of non compact bounded linear operator defined in $L^p(\mathbb{T})$ with values in $L^2(\mathbb{T})$.
2. Use Pitt's Theorem and the conclusion of the previous answer to show that the map $L^p(\mathbb{T}) \ni f \to \{\widehat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^{p'}(\mathbb{Z})$ is not an isomorphism for $1 < p < 2$.

Answer to Exercise 19.18. Consider the map (18.2) for $d = 1$, $q = 2$ and T multiplied by the characteristic function $\chi_{[0,2\pi]}$ if we identify $\mathbb{T}$ with $[0, 2\pi]$. The operator obtained in this way, which we denote by S, is not compact. If now the map $L^p(\mathbb{T}) \ni f \to \{\widehat{f}(n)\}_{n\in\mathbb{Z}} \in$

[3] See Exercise 11.28

$\ell^{p'}(\mathbb{Z})$ is an isomorphism, then we would have a commutative diagram

$$\begin{tikzcd} L^p(\mathbb{T}) \arrow[r, "S"] \arrow[d, "\cong"] & L^2(\mathbb{T}) \arrow[d, "\cong"] \\ \ell^{p'}(\mathbb{Z}) \arrow[r] & \ell^2(\mathbb{Z}) \end{tikzcd}$$

with the horizontal arrow in the bottom necessarily a compact operator, by Pitt's Theorem. Since the vertical arrows are isomorphisms, then S would be compact. This is a contradiction and, since we know that $L^2(\mathbb{T}) \ni f \to \{\widehat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ is an isomorphism, we conclude that $L^p(\mathbb{T}) \ni f \to \{\widehat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^{p'}(\mathbb{Z})$ is not an isomorphism for $1 < p < 2$.

References

[1] F. Albiac, N. J.Kalton, *Topics in Banach Space Theory*, Universitext, Graduate Texts in Mathematics, n. 233 Springer, 2006.

[2] H.Bahouri, P.Gérard, *High frequency approximation of solutions to critical nonlinear wave equations*, Amer. J. Math. 121 (1999), 131–175.

[3] H.Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, 2011.

[4] G. Folland, *Real Analysis: Modern Techniques and Their Applications*, John Wiley & Sons, 1999.

[5] A. Jensen, T. Kato, *Spectral properties of Schrödinger operators and time-decay of the wave functions*, Duke Math. J. 46 (1979), no. 3, 583–611.

[6] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), 955–980.

[7] C. Kenig and F. Merle, *Global well-posedness, scattering and blow–up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math. 166 (2006), 645–675.

[8] Munkres, *Topology* (2001).

[9] W.Rudin, *Analisi Reale e Complessa*, Boringhieri (1982).

[10] W.Rudin, *Functional Analysis*, McGraw–Hill (1973).

[11] E.M.Stein, *Harmonic Analysis*, Princeton Un. Press (1993).

[12] R.S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. 44 (1977), no. 3, 705–714

[13] F. Treves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press (1967).

[14] S.R.S.Varadhan, *Probability Theory*, Courant Lecture Series, American Mathematical Society (2001).

[15] K. Yoshida, *Functional Analysis*, Springer, Berlin Heidelberg (1980).