
Condensed Matter Physics I

a.y. 2023/24

15 January 2024

(Time available: 2h 30')

Hand in the assignment giving also this sheet of paper (solve here the Exercise n. 3)

Exercise 1: *Electronic specific heat in the Sommerfeld model*

1. Show that the electronic density of states varies as $g(E) \sim E^{-1+d/2}$ in spatial dimension d . $n \propto k^d \propto E^{d/2}$ (since $E \propto k^2$) $\implies g(E) = \frac{dn}{dE} \propto E^{d/2-1}$; alternatively: $\int d\mathbf{k}^{(d)}(\dots) \implies \int k^{d-1} dk d\Omega(\dots) \implies \int k^{d-2} (k dk) \implies \int E^{(d-2)/2} dE$; therefore $g(E) \propto E^{(d-2)/2}$
2. Show that the electronic heat capacity varies as $C_{el} \sim T^p$, where the power p is independent of dimension d . Briefly explain the value of p and why C_{el} does not depend on d .

Answer: By thermal excitation, the electron energy increases from $k_B T$ below the Fermi energy to $k_B T$ above. The volume density of excited electrons is $g(E_F) k_B T$, so that $E_{el} \sim g(E_F) (k_B T)^2$ and hence the specific heat $c_{el} \sim g(E_F) k_B^2 T \implies p = 1$, independent on d (d may enter into the heat capacity in the area/volume, counting the number of electrons).

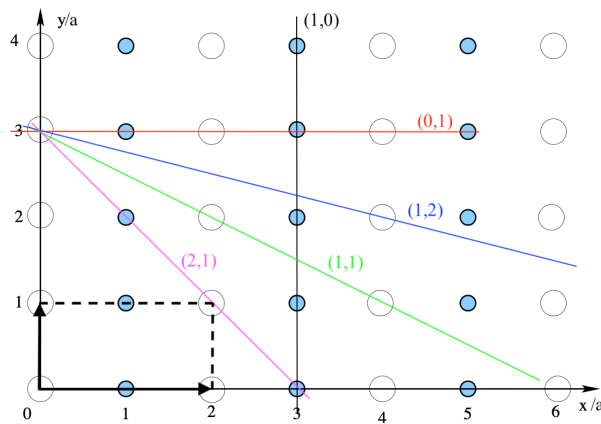
Exercise 2: *Semiclassical model of the electron dynamics and Bloch oscillations*

Consider an electron moving in a one-dimensional periodic potential $V(x+a) = V(x)$. Let the dispersion relation be $E(k) = -A \cos(bk)$ within the band where the electron is confined.

1. What is the value of b ? What is the group velocity of the electron with wavenumber k ?
Answer: The first reciprocal lattice vector is $G = 2\pi/a$. Requiring $E(k+G) = E(k)^{(*)} \implies bG=2\pi$ so that $b=a$ ($^{(*)}$ general for Bloch electrons; but it is also possible to consider that $E(k)$ comes from the simplest possible tight binding nearest neighbor approach). Group velocity is $v_g = (1/\hbar)dE/dk = (aA/\hbar) \sin(ak)$.
2. A constant applied electric field $E = -E\hat{x}$ accelerates the electron. What is dk/dt ? Neglect any scattering by impurities, defects or phonons, i.e. take the mean free time $\tau \rightarrow \infty$.
Answer: According to semiclassical dynamics, $dk/dt = F_{ext}/\hbar = eE/\hbar$. We may restrict k to the first Brillouin zone, but this is not necessary.
3. Assume the electron is initially localized around the origin $x = 0$ in a wave packet with $k = 0$. Qualitatively describe the motion of the electron in response to the applied field, and comment on how the motion differs from that of a free particle.
Answer: $k(t) = eEt/\hbar$ increases linearly in $t \implies v_g$ must oscillate sinusoidally. Thus the particle initially moves to the right, until $k = \pi/a$ at which point its velocity reverses, and the particle returns to the origin $x = 0$ once $k = 2\pi/a$. Thus the motion is oscillatory, while a free electron in contrast will simply accelerate.

Exercise 3: Crystalline structures

- In the hypothetical 2D crystal structure shown below, the large and small circles are atoms of different type, spaced by a . Sketch one unit cell. Write the expression for the primitive translation vectors \mathbf{a}_1 and \mathbf{a}_2 chosen parallel to the axes and draw them on the figure. Write also the vectors of the unit cell internal basis $\{\mathbf{d}_i\}$.



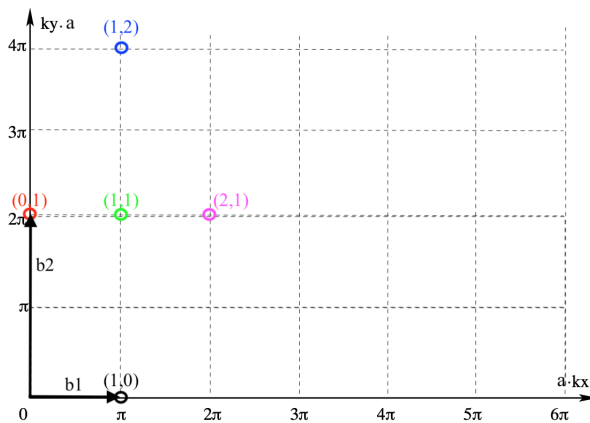
$$\mathbf{a}_1 = a(2, 0)$$

$$\mathbf{a}_2 = a(0, 1)$$

$$\mathbf{d}_1 = (0, 0)$$

$$\mathbf{d}_2 = a(1, 0)$$

- Write the primitive reciprocal lattice vectors \mathbf{b}_1 and \mathbf{b}_2 and draw them in the plane (k_x, k_y) shown below.



$$\mathbf{b}_1 = \frac{2\pi}{a} \left(\frac{1}{2}, 0 \right)$$

$$\mathbf{b}_2 = \frac{2\pi}{a} (0, 1)$$

$$\text{Miller indices } (1,0): \mathbf{K} = \frac{2\pi}{a} \left(\frac{1}{2}, 0 \right)$$

$$\text{Miller indices } (0,1): \mathbf{K} = \frac{2\pi}{a} (0, 1)$$

$$\text{Miller indices } (1,1): \mathbf{K} = \frac{2\pi}{a} \left(\frac{1}{2}, 1 \right)$$

$$\text{Miller indices } (1,2): \mathbf{K} = \frac{2\pi}{a} \left(\frac{1}{2}, 2 \right)$$

$$\text{Miller indices } (2,1): \mathbf{K} = \frac{2\pi}{a} (1, 1)$$

- Remember that the Miller indices (h, k) of a lattice plane are the coefficients of the vector normal to the plane in terms of the primitive reciprocal lattice vectors. Consider the planes corresponding to the Miller indices $(1,0)$, $(0,1)$, $(1,1)$, $(1,2)$, $(2,1)$; for each plane write the corresponding reciprocal lattice vector \mathbf{K} normal to it and place and label the points in the plane (k_x, k_y) . See 2nd figure
- Draw and label in the figure (x,y) the lattice planes corresponding to the Miller indices above (one plane for each Miller index). See 1st figure
- Consider the diffraction corresponding to the family of planes of the Miller indices above. For each diffraction peak calculate the value of the structure factor $S_{(h,k)}$. $S(h, k) = f_1 + (-1)^h f_2$
- If the form factors of the two atoms are equal, for which values of the Miller indices the structure factor vanishes? $h = 2m + 1$ (odd)