4.7 THE CHOICE OF EQUILIBRIUM

i) some fundamental physical principle (nature)

ii) initial condition + .. (nurture)

(i)

PRINCIPLE OF MAXIMUM ENTROPY

$$S \equiv -\int_{\text{phase space}} p \ln p \, d\tau \qquad \boxed{4.106}$$

$f = DF \implies p = f$ density probability

$$S \equiv - \int f \ln f \, d^3\bar{x} \, d^3\bar{v} + \text{cost.}$$

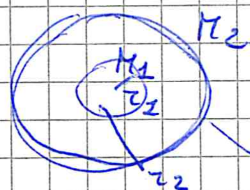
$\times N$ ↓ if N stars in the system

Lynden-Bell 1967 → S has a maximum only if the DF is the isothermal function

(but M mass and E energy are ∞ !)

if M and E are finite $S \to \infty$

EXAMPLE



envelope

$M = M_1 + M_2$

$|E|$

$\boxed{M_2 \ll M_1}$

system can be divided in $M_1 + M_2$

$$|E_1| \sim \frac{GM_1^2}{r_1} \qquad |E_2| \sim \frac{G M_1 M_2}{r_2}$$

binding energy

Shrink the main body

$r_1 \to (1-\varepsilon) r_1$

This will change energy by a small factor $\Delta E \sim \varepsilon \dfrac{GM_1^2}{r_1}$

anche + negativo! $\cancel{E_1}$

$|E_1| \uparrow$ ↓ so energy will be deposited on M_2

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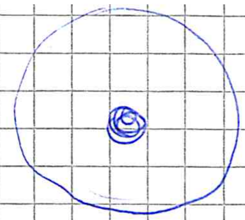
E_2'

⊖ ⊕

$$|E_2'| = |E_2 + \Delta E| \sim \frac{G M_1 M_2}{r_2'}$$



$$|E_2'| < |E_2| \qquad r_2' > r_2$$

The velocity dispersion in the envelope

$$\sigma_2' \sim \sqrt{\frac{G M_1}{r_2'}}$$

↓

VT theorem

The 6D volume of the envelope is

$V \sim \sigma_2'^3 \, r_2'^3$ of the phase space

$$V \sim (G M_1 r_2')^{3/2}$$

For M_1
the ΔS_1 is finite
$S_1 \downarrow$ no limitation.

4-206 → for the envelope

$$S_2 = - \int f \ln f \, d^3x \, d^3v + \text{cost}$$

$$f = \frac{N_2}{V} \qquad \sim -\int \frac{N_2}{V} \ln \frac{N_2}{V} \underbrace{d^3v \, d^3x}_{dV} + \text{cost}$$

$$\sim -\frac{N_2}{\cancel{V}} \ln \frac{N_2}{V} \cdot \cancel{V} + \text{cost} \sim$$

N_2 is a const.

$$\sim -N_2 \ln V^{-1} + \text{cost} \sim -N_2 \ln (r_2')^{-3/2} + \text{cost}$$

$$\sim +\frac{3}{2} N_2 \ln r_2' + \text{cost} \sim -\frac{3}{2} N_2 \ln |E_2 + \Delta E| + \text{cost}$$

$$r_2' \downarrow \frac{G M_1 M_2}{|E_2 + \Delta E|}$$

$\Delta E \to |E_2|$

$\ln \to \ln(0)$

$$\boxed{S_2 \to +\infty}$$

we have to investigate (ii)

the processes of relaxation

(a) Phase mixing $\Longrightarrow$ CHANGE the $\bar{f}$
the coarse grained phase space density

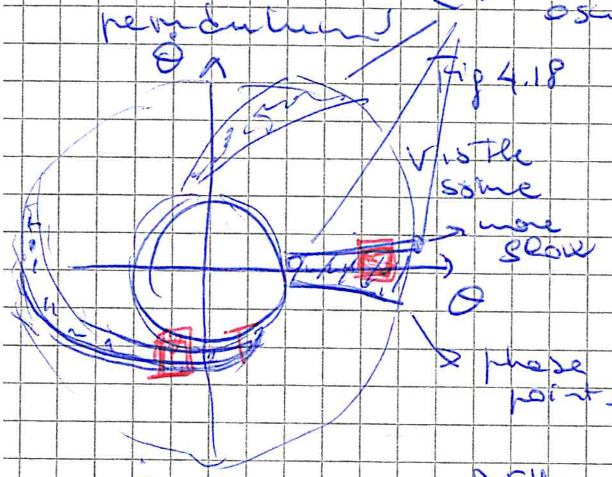
N independent pendulum
with length L

$\Theta \to \theta$ in $\theta_0 \pm \frac{1}{2}\Delta\theta$

$\Delta\theta \ll \theta_0$

+ a small random velocity

representation of an array pendulums

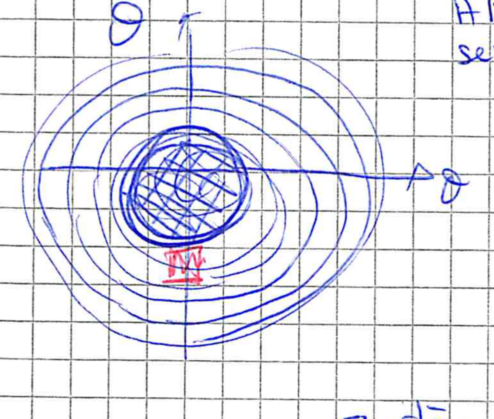


momentum $p = L^2 \dot{\theta}$ per unit mass

in the rang $\pm \Delta p$

start on phase $\longrightarrow$ phase mixing

After several dynamical times



the coarse-grained phase space density $\to$ uniformity in the annulus

system of pendulums $\Updownarrow$ f } evolution described by the B. eq.

macroscopic observer $\bar{f} = f$

after $\bar{f} < f$

It can be shown $\bar{f}\downarrow$ $\bar{S}\uparrow$

eq. of B. $\not\Rightarrow$ $\bar{f}$ is not increasing!

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→ distribution of velocities is maxwellian! (1D is gaussian)

(b) violent relaxation → CHANGES THE ENERGIES

Lynden-Bell 1967

when a star moves in a fixed potential ϕ

its energy (x unit mass) is const $\quad E = \frac{1}{2} v^2 + \phi$

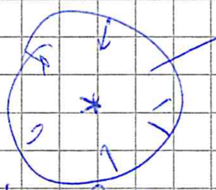
Now $\phi = \phi(\bar{x}, t)$ the Energy is not constant

$$\frac{dE}{dt} = \frac{1}{2}\frac{dv^2}{dt} + \frac{d\phi}{dt} = \cancel{\bar{v} \cdot \frac{d\bar{v}}{dt}} + \frac{\partial \phi}{\partial t} + \cancel{\bar{v}\,\bar{\nabla}\phi} = \frac{\partial \phi}{\partial t}$$

$\frac{d\bar{x}}{dt} \cdot \frac{\partial \phi}{\partial x}$; $\frac{d\bar{v}}{dt} = -\bar{\nabla}\phi$

4.208

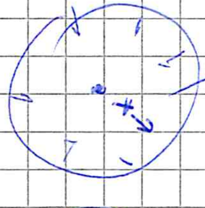
EXAMPLE



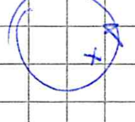
star at the center at the rest

if system collapse $\phi \downarrow$ $\quad$ v is still 0

protogalaxy $\rightarrow E_{star} \downarrow$

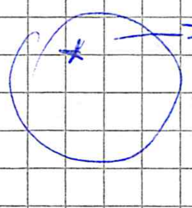


at half-mass radius



when star is reapproaching, the system is more compact

more kinetic energy



same ϕ but with more kinetic energy

mass does not appear in eq. 4.208 (e.g. multiply by m, but the variation will be $m\frac{\partial \phi}{\partial t}$)

⇒ equipartition of velocity (= energy x unit mass)

2-body relaxation

⇒ equipartition of energy $\quad$ S.M., Fig 4.19, 4.22