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Precalculus - An Inquiry-Based Approach

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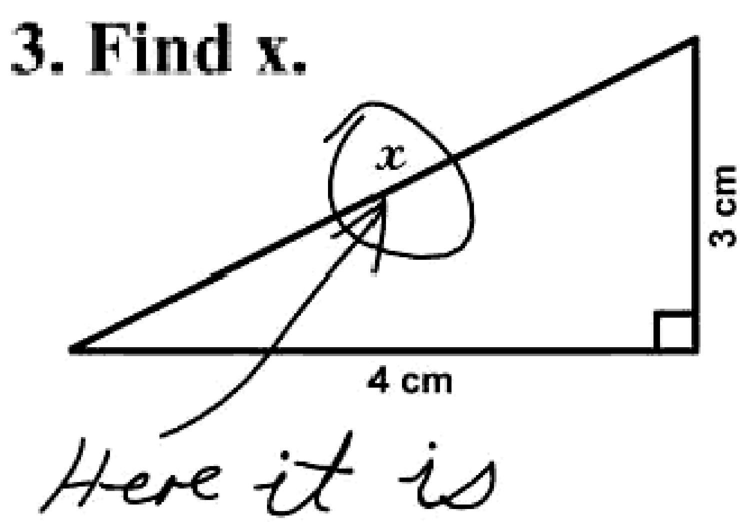
Contents

Dedication

These notes are dedicated to you, our students. The joy of watching you struggle and then succeed has driven us beyond all our personal expectations. If we are honest, neither of us consider ourselves to be particularly bright or talented. We never had aspirations to publish mathematics, create new things, or lead people and projects. In fact, both of us had a bit of a "wasted" youth while we "found" ourselves. However, watching you succeed over the years has made us push ourselves to new successes as well. You see, by asking you to do more than you think you can, and seeing you succeed, we recognize that we too can do more than we think we can! We would not be where we are without the example that you set for us. Thank you.



To the Instructor

Here we address the history and content of the course, each author's perspective on his implementation, and conclude with a few technical points on which we both agree.

History A long, long time ago, in a land far, far away, Mahavier authored and published the IBL course *Trigonometry* at jiblm.org. Howard modified these notes to meet his needs, added algebra to them, and published *Precalculus* at jiblm.org. Mahavier took this version, modified it, and taught out of it for three years. The two then worked together to polish and publish this final version. All together, the notes have been taught in one form or another since 1996.

Content The notes constitute a self-contained problem set designed for a college or high school precalculus course covering polynomials, rationals, exponentials, logarithmics, planar trigonometry, inverse functions, polar coordinates, conic sections, and an introduction to rates of change. As the algebra and trigonometry chapters are mutually exclusive, one could use these notes for either college algebra (the odd numbered chapters) or trigonometry (the even numbered chapters).

Paul K Howard

I teach at a private mid-western university with an undergraduate enrollment of 2200 students. Enrollment in my precalculus courses has ranged from ten to forty. Semesters are typical fifteen weeks long with the class meeting four times per week for fifty minutes for each class period. For me, the optimal class size while using these notes would be fifteen to twenty students.

I have found that, especially with larger class sizes, it is best to divide the work into fifteen to twenty problems per week. Each weekend the students are expected to study the set number of problems from the previous week, expecting a quiz on Mondays covering problems from the previous week, This often requires strategically placed focused class discussions to insure that all problems from the week get discussed in class.

The class structure is pretty straight forward. At the beginning of each week students are given problems for the week. Students are explicitly told not to use outside resources as they work through the problems for the week. Each day students volunteer to present problems to the class. Rarely will I explicitly point out mistakes in class. The class is expected to discuss all presentations. Students are expected to treat each other with dignity and respect. When algebraic deficiencies arise, the students are forced to realize the significance of these shortcomings. The goal is to force students to understand the struggle they will run into in Calculus if they do not address their algebraic weaknesses. This does require instructors to keep a very close eye on what algebraic weaknesses need to be addressed. A few key areas I have noticed consistently: 1). Wheel motion problems reveal proportionality weaknesses as well as the ability to convert between units. 2). Trigonometric identities reveal weaknesses working with complex fractions. In any case, it is very important for instructors to notice when there are algebraic weaknesses and to address them accordingly.

When algebraic weaknesses are revealed, they should be dealt with in the context where students understand that their weaknesses will hinder the development of future mathematical knowledge. Mini-lectures, class discussions and quizzes, give direct help in overcoming algebraic weaknesses. These weaknesses are always distinct for each class. Most importantly, one-on-one discussions lets students know that they need to take serious responsibility to work on their algebraic weaknesses. Otherwise they will struggle in a Calculus series.

Throughout the notes students are asked to prove various statements or theorems. Student's first efforts usually fall short of what would be considered a rigorous proof. After any student attempts to write a proof it is quite important to have class discussions examining areas of their proof that lack rigor and to help students modify their work to illustrate what rigorous proofs should look like. This will help the class learn how to write rigorous proofs. As the class progresses the hope is that student proofs become more rigorous.

Please feel free to send me an email with questions and/or concerns.

W. Ted Mahavier

These brief introductory remarks may be insufficient to assure a successful implementation of an inquiry-based course. For extensive writings on my thoughts, please refer to "The Moore Method: A Pathway to Learner-Centered Instruction," by Coppin, Mahavier, May, and Parker. For mentoring using these notes, contact me directly at ted.mahavier@gmail.com. I enjoy assisting those who implement IBL courses.

This course attempts to hone my students' communication, presentation, and problem-solving skills by studying the properties of the functions necessary to succeed in the STEM fields. My class sizes have ranged from seven to thirty-five students. An optimal class size might be twenty. For class sizes over twenty-five, I have students work in groups where one person represents the group for the presentation of a given problem.

While too long to include here, Dr. Roberson, who taught an early version of my Trigonometry course, wrote the following, which I think captures the essence of the value of inquiry-based learning:

I am having a wonderful time with the trigonometry class in which we are using your problem sequence. The students were very skeptical at the beginning, but not having to pay for a high priced book was enough to keep most of them from dropping out after the first day. No matter what happens the rest of the semester, I have already had enough rewarding experiences to make it worthwhile. One young lady spoke with me after class last Friday and said that she had taken intelligence tests in the past and was told that she was extremely talented in mathematics, but she never made good grades in math classes in school. She said that no one had ever bothered to tell her "why" things worked like they did, but just asked her to memorize lists of formulas. She says now she has more confidence than she ever had and very much enjoys working on these problems. Another student piped up in class and said he had finally figured me out. I asked him what he meant and he said that he realized that every time they ask me a question, I direct it to the class to deal with. He said, "we're going to end up practically writing the book ourselves!" The rest of the *class got a good laugh out of it and told him they were surprised it took him that long to figure it out. I think one of the most fun things to watch as the students work through these problems is that some of the problems are almost trivial and some are quite sophisticated, yet the students never seem to notice the difference. They just dive in and tackle each problem as a new challenge.*

The structure of the class is simple. I provide the notes to the students one chapter at a time. Students work on the problems outside of class and volunteer to present the problems each day, with priority going to the student who has the least number of presentations. I track presentations on a Surface Pro in real time. It is clearly stated that they are to look only to themselves and to me for guidance; no book, web, or outside help of any other kind is allowed. As a problem is presented, the class asks questions and if there are none, I ask if there are questions. While I might lead with questions to the audience, I rarely point out mistakes at the board and early in the semester, I place the burden of determining the correctness of each problem completely on the class. If I am asked if a problem is correct, I merely take a vote from the class or ask what they are worried about. The key is to develop their ability to politely question a speaker. I spend considerable time discussing the benefits of carefully phrased questions versus blunt statements. We discuss the benefits and dangers of statements such as, “I think there is a mistake on line 3,” versus questions such as “Dominique, can you tell me again how you got from line two to line three, I think I missed a step?” To help clarify a potential problem, I may rephrase a student’s question by saying, “Are you asking...” and then accurately paraphrasing the question. Addressing the skill set of gracefully asking questions in ways that set the speaker at ease and gracefully answering questions in ways that puts the audience at ease makes the environment a positive one.

Generally, I offer minimal guidance until the students have nothing to present. If such a day arises, then I will chat informally about what we have covered, using examples to solidify understanding of previous work and discussing the upcoming definitions and problems to provide guidance. Direction of this type can easily double the speed of a class and I definitely use this technique to assure that we cover what I consider “sufficient material,” if needed. These are rarely lectures. They are typically reactive discussions where my questions (“Where are we stuck?”) and students’ questions (“Why do my graphs take so long?”) are addressed. The most important aspect of my successful classes has been constant open discourse between the students so that they feel comfortable presenting material at the board, asking questions of the students who are presenting, asking questions of me, and defending their arguments.

At the beginning of the semester, I always fear that we are making minimal progress, since students may spend an entire class struggling to put up a few problems. As the course progresses, the rate of increase of problems per day accelerates and by the end of the course, they are putting up many correct problems per class period. It is common for me to attend a conference and leave the students in charge of the class. In such cases, I require pictures uploaded to some social media site (Facebook Group, Discord, Slack, ...) of the presentations along with an email from each student as to their level of understanding of each problem presented. The learning curve is exponential and the patience required at the beginning is rewarded as they learn to read carefully and do the mathematics on their own. The approach and demeanor of the instructor in the beginning is the single-most important element for success.

During the semesters when I gave two or three full class length tests during the semester, I created practice sheets several times per semester. Discovering and understanding a concept is a distinct skill from being proficient at working through multiple problems quickly. Without these simple drill sheets, the students became very proficient at discovering mathematical truths during the course, but lacked the ability to quickly knock out simple problems on tests. These sheets were typically a page with ten to twenty problems on it, three to five from each major concept that the students have discovered. These sheets offer the additional benefit that the student who had *not* successfully discovered the mathematics, but was one step behind the rest of the class, could quickly catch up, rather than fall further behind. These sheets are not included here because they were written specifically to address what students requested and what I felt students needed.

During semesters when I gave weekly quizzes in place of longer tests, I printed paper (old-school) with a one page quiz on it over the week's problems. They placed their name on the blank side. They flipped them over, worked the quiz, and then flipped them over again. I collected, shuffled, and passed them back out with the names down. I presented the solution at the board while students graded them as either: very little is right (1 point), half right (2 points), mostly right (3 points), or completely right (4 points). I collected them and later I regraded them myself, entered the grades, and returned them. Having students review another student's work anonymously turns out to be quite powerful. Grading a good paper teaches one what is possible. Grading a poor paper shows that we all make mistakes. Being responsible for estimating a grade makes them try to parse another student's mathematics and focus on the problem while we are going over it. This leads to very detailed "What if they did this?" type discussions, often leading to multiple solutions. I've found it to be a great way to spend thirty minutes of class time, but no more. Aftewards, we present a few problems.

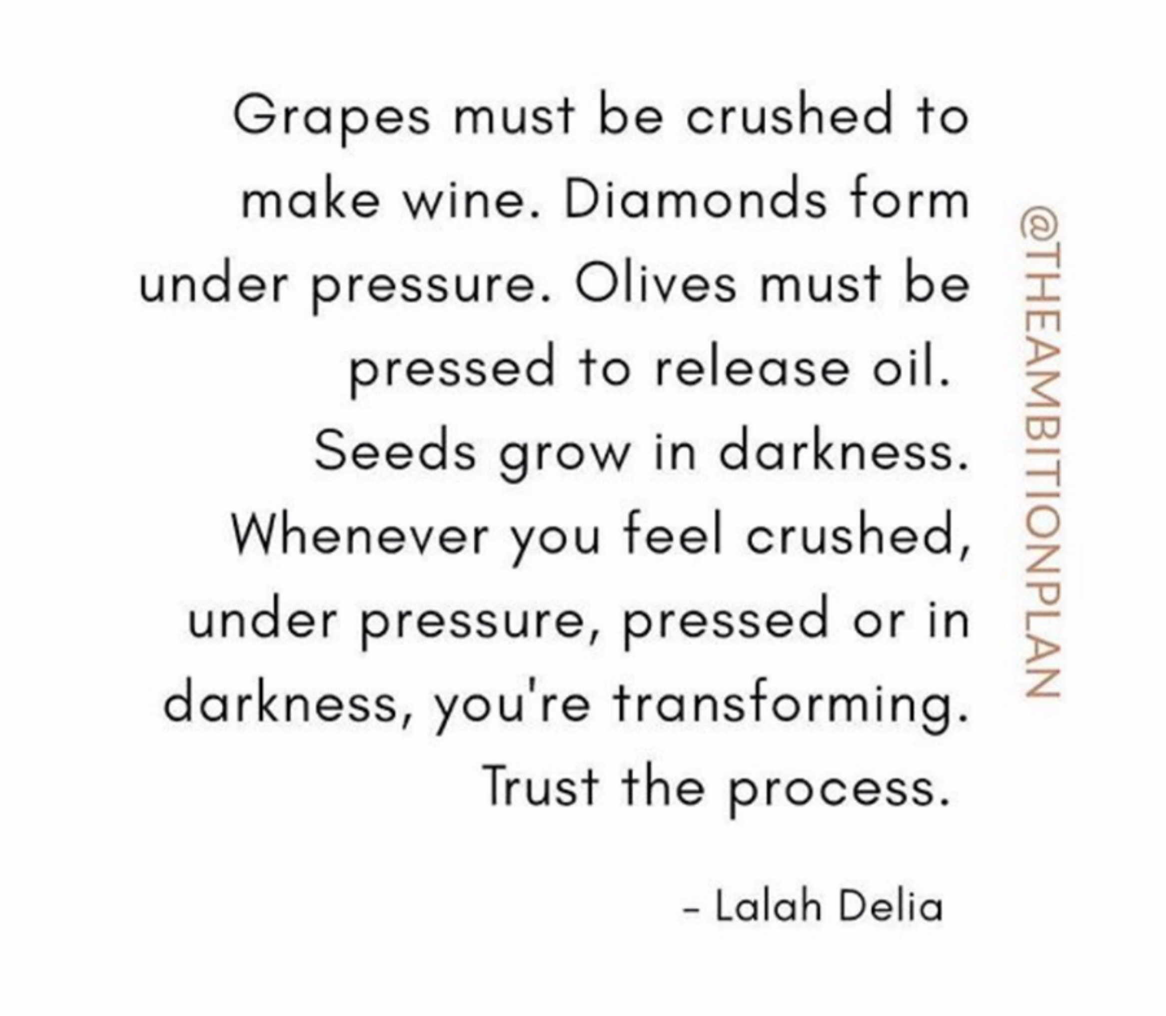
Grading. I have used three grading schemes for this class, all of which worked out just fine. (1) Midterm and Final, both comprehensive, 50%. Presentations 50%. (2) Presentations 20%, three tests 20% each, and a comprehensive final 20%. (3) Presentations 50% and weekly quizzes 50%. The latter was definitely the best from my perspective and, based on the reaction of the students, theirs. For each method, there was a comprehensive final which replaced the test/midterm/quiz average if it improved the grade.

Howard and Mahavier Both authors agree on the following technical points.

- We do not need to cover all of Chapter 15. We like giving the geometric definitions for parabolas, circles, and even ellipses to show the alternate definitions to the algebraic definitions, yet we do not feel working the full chapters is necessary.
- We do not necessarily proceed linearly through the notes. On occasion, a solution is not available for a problem, but a solution to a subsequent problem is available that uses this first problem. In this case, we assume the former to solve the latter, leaving the former problem for later. One must be careful to avoid circular reasoning. Hence, the first problem must now be solved independently of the latter.
- There are points in the notes, for example the end of the chapter on the Law of Sines and Law of Cosines, where, in order to keep the class steadily moving forward, we allow students to move into the next chapter while still accepting presentations from the end of the previous chapter.
- There are theorems listed among the problems which we do not necessarily prove during class, although we encourage them to work on them and present them if a student finds a proof.
- Sprinkled throughout are optional essays. We have each mandated them some semesters and made them optional others, offering some type of extra credit such as replacing a low

quiz grade or adding points to the presentation grade. If assigned, we recommend giving guidance on length, sources, and plagiarism. We believe they are an opportunity to explain the difference in scientific writing (clear, concise, correct) and writing for the liberal arts where there may be a minimum word length or page length. A powerful theorem proved in five lines trumps a weak theorem published in 200 pages.

To the Student



Welcome to Pre-calculus! The structure of this course will likely be quite different from previous mathematics courses you have taken. There will be no traditional textbook. You and your classmates will work through these notes, which will become your book. Problems will be worked out and presented in class by you and your classmates. You will be actively involved in doing mathematics rather than being told how to do mathematics. The process of doing mathematics involves thinking through ideas and trying to find solutions to problems. This is often a messy process involving lots of mistakes. When an author or instructor presents ideas, they have spent time polishing their thoughts. Hidden to the student is the messy work of seeking to understand the ideas. You will not be protected from the messy process of doing mathematics. Nor will you be protected from the process of presenting clear and precise solutions.

We will all make mistakes and learn from these without judgment. Our presentation goals will always be whole class understanding. Mistakes will be our pathway to learning and points are not taken away for mistakes. In fact, an original approach with a mistake may earn more credit than a perfect presentation! Just as I learned to ride a bike by getting on and falling off, I expect that you will learn mathematics by attempting it and (sometimes) falling off! Please seriously consider the value of becoming an independent thinker who tackles doing mathematics, and everything else in life, on your own rather than waiting for someone else to show you how to do things.

Nervous? Don't be! Every semester I collect written evaluations from our classes and the students who have taken this course have been overwhelmingly positive in their comments. While challenged by the material and the method, they enjoyed the class. If you'd like, I'll read random comments from the last semester I taught it. I'll print them out, let you randomly select a number and I'll read that comment.

Grading

The official class syllabus has lots of tedious details, but this is the most pertinent information.

1. You may discuss problems with me or with members of this class, but you *may not use any other resources* (internet, books, individuals not in the class,...) without my explicit permission in advance. Doing so will be considered cheating and will have consequences listed in the Student Handbook.
2. Your grade will be the average of
   (a) your participation/presentation grade, and
   (b) your weekly homework/quiz grade.

An optional comprehensive final may improve your grade if the course ends and you are hoping for a better grade than you have at that time.

A common pitfall

There are two ways in which students often approach my classes. The first is to say, "I'll wait and see how this works and then see if I like it and put some problems up later in the semester after I catch on." Think of the course as a forty yard dash. Do you really want to wait and see how fast the other runners are before you start running? If you try to do the problems every night, then either you will get a problem (YEA!) and be able to put it on the board, or you will struggle with the problem and learn a lot in your struggle. If you have worked hard on a problem, you may go to the board with your work and show us what you tried. Doing so will almost surely give you further ideas on how to solve it. If someone else puts the problem on the board, then you will be able to ask questions and help everyone understand it. Doing so also counts toward your presentation grade. You will be able to say to yourself, "Ahhhh, now I see where I went wrong and now I can do this one and a few more for next class." If you do not try problems each night, then you will watch the student put the problem on the board and maybe even believe you understand it. More likely, you will not quite catch all the details and when you study for the tests or try the next problems, then you will have only a loose idea of how to tackle related problems. Basically, you have seen it only once in this case. The student who tried, whether successful or not, saw it once when they tackled it on their own, again when they put it on the board or another student presented it, and a third time when they studied for the next test or quiz. Think of watching someone show you how to play guitar. Can you do it after they show you how? I hope that each of you will tackle this course with the attitude that you will learn this material and thus will both enjoy and benefit from the course.

Boardwork

Because the board work constitutes half of your grade, let's put your mind at ease regarding this part of the class. By coming to class everyday you will earn a 60% on board work. Every problem that you present, right or wrong, pushes your grade higher. You may come see me anytime for an indication of what grade your current level of participation will earn you at the end of the semester. I will give you one presentation credit for an office visit within the first two weeks of the semester. Each day I will call for volunteers and will choose from those with the least presentations. You may inform me that you have a problem in advance, which I appreciate, but the problem still goes to the person with the least presentations on the day I call for solutions. In the beginning, ties are broken randomly or by test grades, with lower test grades taking priority. A student who has not gone to the board on a given day will be given precedence over a student who has gone to the board that day. To "present" a problem at the board means to have written the problem statement up prior to coming to class, to have written a correct solution using complete mathematical sentences, and to have answered all students' questions regarding the problem. When you have worked hard on a problem, even if you aren't sure your work is correct, *go to the board* because going to the board can only help your grade. Many of our students have started the semester "scared" of the board and ended at the top of the class. Since you will be communicating with other students in class on a regular basis, here are several guidelines that will help you. Remember that the whole class is on your side because everyone wants to see the problem presented correctly and to understand it.

When you speak, don't use the words "obvious," "stupid," or "trivial." Don't attack anyone personally or try to intimidate anyone. Don't get mad or upset at anyone, even yourself. And if you do, try to get over it quickly. Don't be upset when you make a mistake. Brush it off and learn from it. Don't let anything go on the board that you don't fully understand. Don't say to yourself, "I'll figure this out at home." Don't use concepts we have not defined. Don't try to put up a problem you have not written up.

Do prepare arguments in advance. Do be polite and respectful. Do learn from your mistakes. Do ask questions such as, "Can you tell me how you got the third line?" Do let people answer when they are asked a question. Do refer to earlier results and definitions by name or number when possible. For example, "by Definition 29" or "by the Definition of the inverse function" we know that...

How to Succeed

1. Read over your notes from class *every* day.
2. Make a list of questions to ask at the beginning of the next class and *ask* them!
3. Redo the problems from each class without looking at your notes.
4. Work on *several* new problems.
5. Write up as many solutions as possible so that you can simply copy your solutions onto the board the next day.
6. Come to my office every time you are stuck!

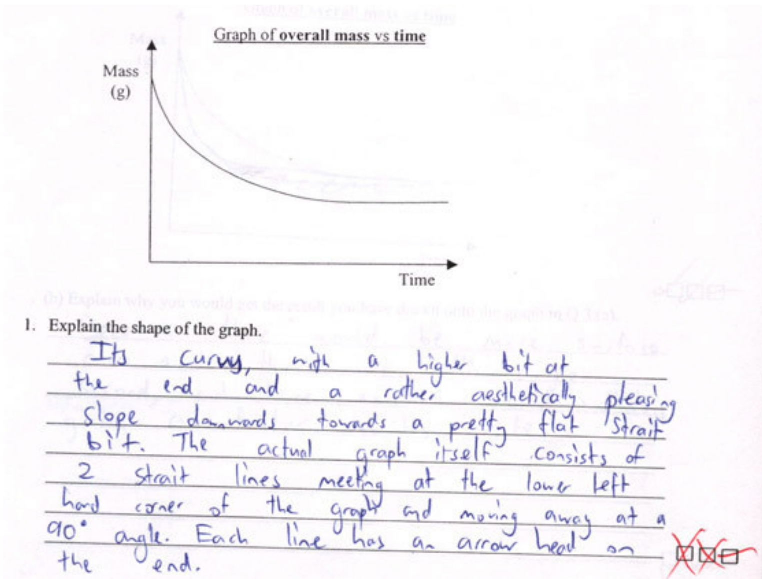
Some problems are harder than others. If you don't get one, don't give up. Save it for later, move on to another problem, and come back to it later. The problems worth solving in life are not solved in five minutes.

Content

The notes alternate between algebra (the study of polynomials, rationals, exponentials and logarithmics) and trigonometry (the study of the six trigonometric functions and their inverses). Together these topics make up the subject loosely referred to as "precalculus." A better name would be "functions," because the one mathematical goal of the course is for you all to walk into calculus with a firm grasp of several classes of functions. In a sense, we only have one thing to learn!

Chapter 1

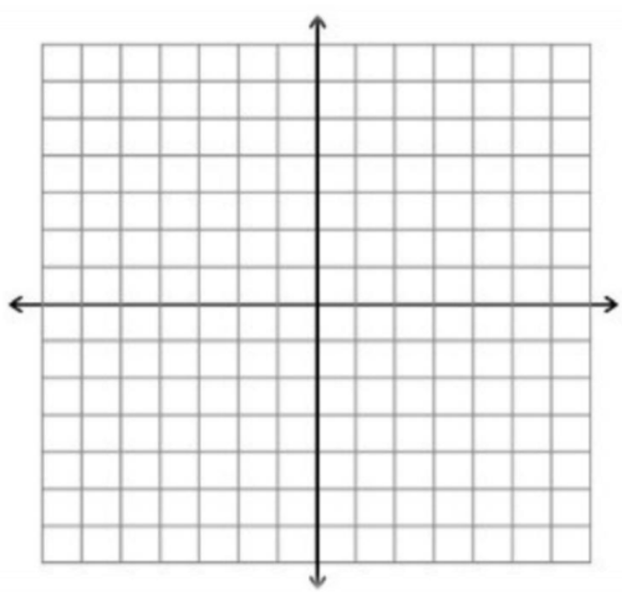
Graphing



Definition 1. [1] *For any real number a, the* ***absolute value of*** *a is the distance on the number line from the number a to the number zero. The absolute value of a is denoted as* $|a|$.

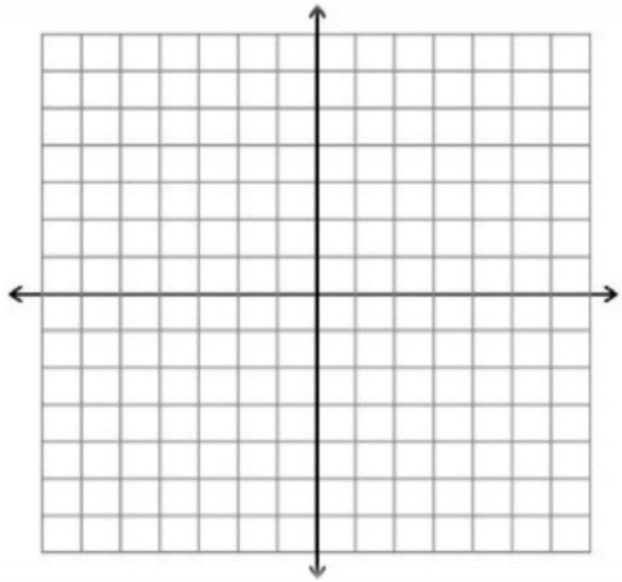
Problem 1. *Create a table of values and a graph for* $y = |x|$. *Give coordinates of x and y intercepts.*

x	y



Problem 2. *Let $y = |x|$. Replace x with $x-2$ creating $y = |x-2|$. Create a table of values and a graph for $y = |x-2|$. Write a short statement describing what happened to the graph of $y = |x|$ when x was replaced by $x-2$.*

x	y



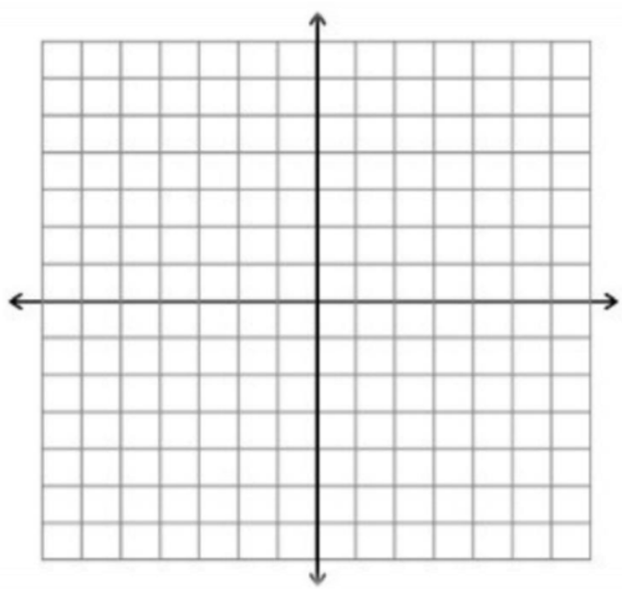
Problem 3. *To "solve an equation" means to find all numbers that satisfy the equation. Use the graph from Problem 1 to solve $|x| \leq 2$.*

Problem 4. *Solve $|x| \leq 3$.*

Problem 5. *Explain why the inequality* $|x-3| \leq .01$ *means that* x *must be within* $.01$ *units of* 3. *The bag of Doritos on my desk contains within .2 grams of 2.3 grams. If x is the weight of the Doritos in the bag, use absolute value to express that x is within .2 grams of 2.3 grams. Can you write the same expression without absolute values?*

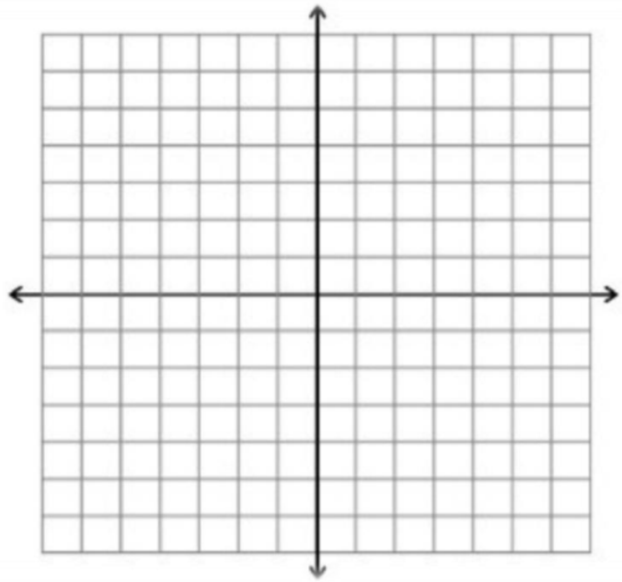
Problem 6. *Let* $y = |x|$. *Replace* y *with* $y-2$ *creating a new equation,* $y-2 = |x|$. *Solve this equation for* y. *Create a table of values and a graph for* $y = |x|+2$. *What do you notice?*

x	y



Problem 7. *Let $y = |x|$. Replace y with $y-1$ and replace x with $x-2$. You should get $y-1 = |x-2|$. Make a prediction for the graph of $y = |x-2|+1$. Create a table and a graph.*

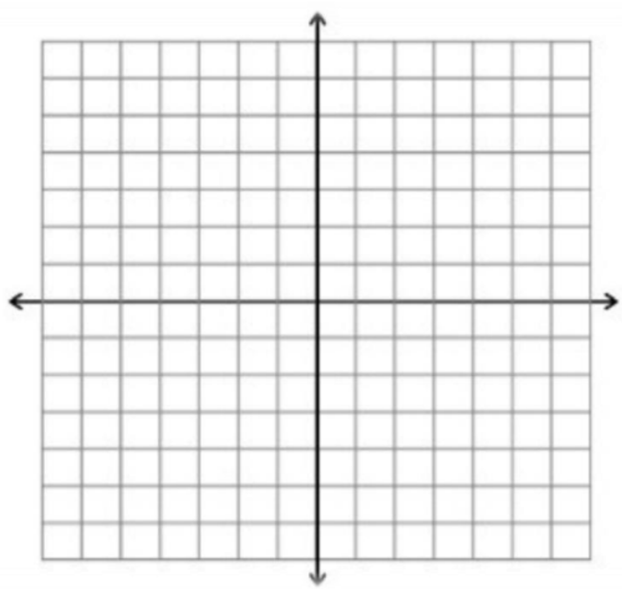
x	y



Definition 2. *Let a be any real number and let n be any positive integer. The expression a^n means $a \times a \times \cdots \times a$ where a is multiplied by itself n times. The* ***square of*** *a means $a \times a$ and is denoted as a^2. The* ***cube of*** *a means $a \times a \times a$ and is denoted as a^3.*

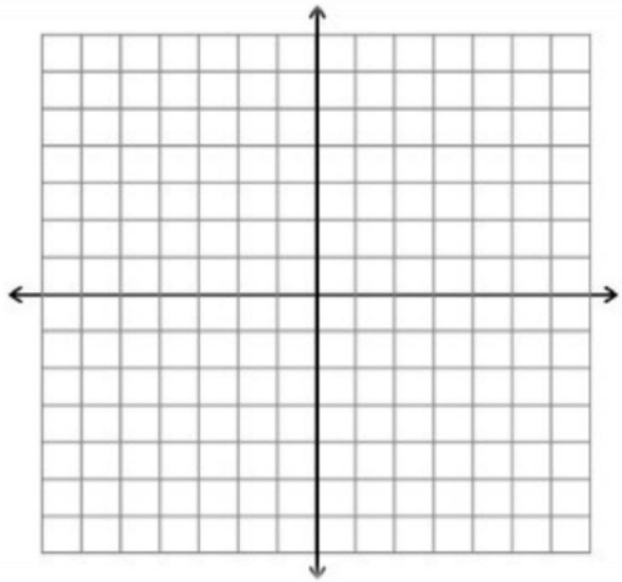
Problem 8. *Create a table and a graph of the equation $y = x^2$.*

x	y



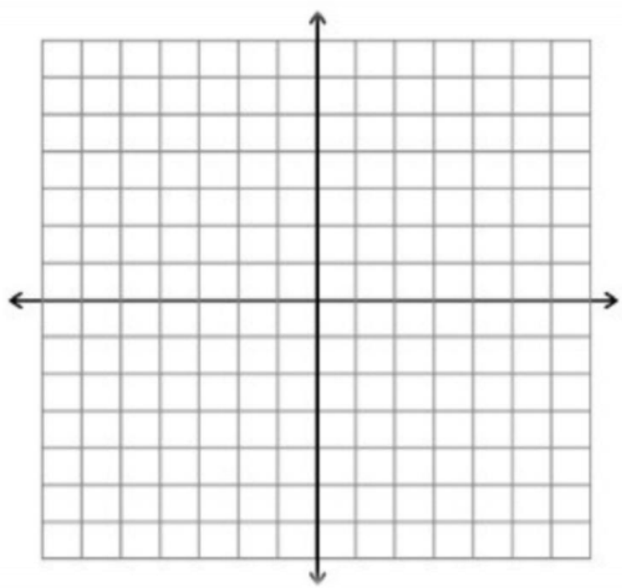
Problem 9. *Let $y = x^2$. Replace x with $x+1$. Make a prediction and graph $y = (x+1)^2$.*

x	y



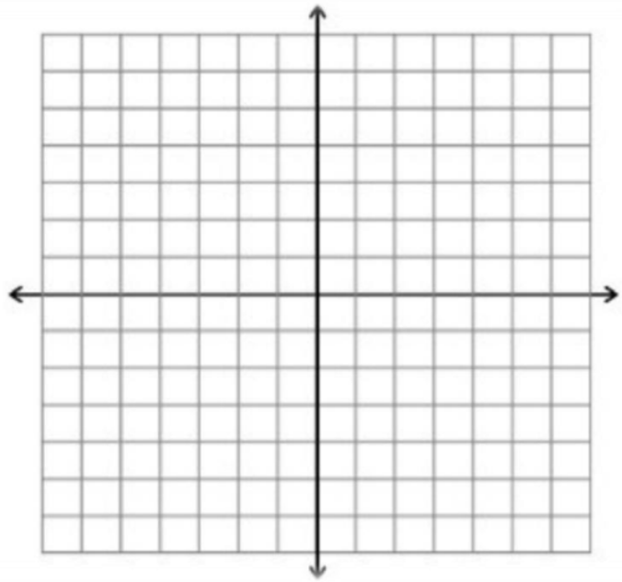
Problem 10. *Let $y = x^2$. Replace y with $y-2$. Make a prediction and graph.*

x	y

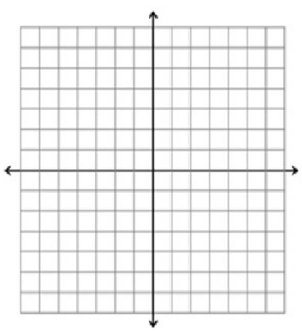


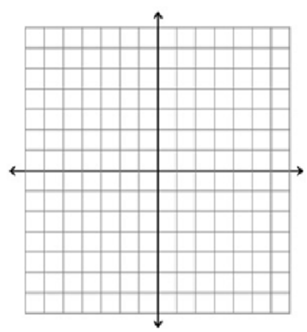
Problem 11. *Let $y = x^2$. Replace y with $y+1$ and replace x with $x+3$. Make a prediction and graph.*

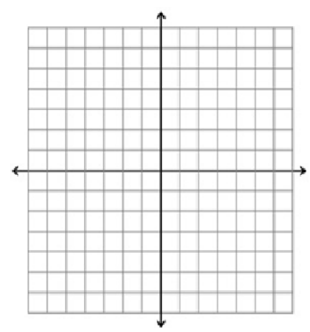
x	y



Problem 12. *Sketch each of these functions:* $y = \dfrac{1}{x}$, $y = \dfrac{1}{x+2}$ *and* $y = \dfrac{1}{x} + 2$.







Chapter 2

Radian Measure

When we begin the study of a subject, certain words are undefined. As mathematicians, we assume an intuitive understanding of these and try to minimize the number of undefined words. Still, *distance*, *number*, and *coordinates* are undefined words we have already used. *Triangle* and *perpendicular* are undefined words we are about to use. Why can't we define every word we use?

Definition 3. *A* ***right triangle*** *is a triangle where two of the sides are perpendicular.*

Definition 4. *The* ***hypotenuse*** *of a right triangle is the side not perpendicular to another side.*

You only need to work the *problems* in these notes. *Theorems* are statements that can be proved, but that we will accept as true.

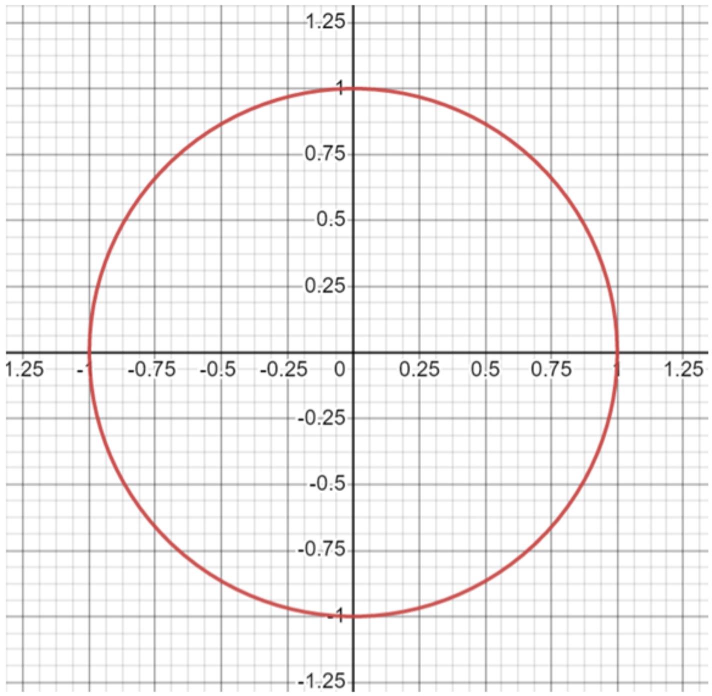
Theorem 1. ***Pythagorean Theorem*** [2] *If we have a right triangle in a plane with sides of lengths a, b and c, where c is the length of the hypotenuse, then* $a^2+b^2=c^2$.

Theorem 2. *If* (x_1,y_1) *and* (x_2,y_2) *are points in the plane, then the distance between these two points is given by*

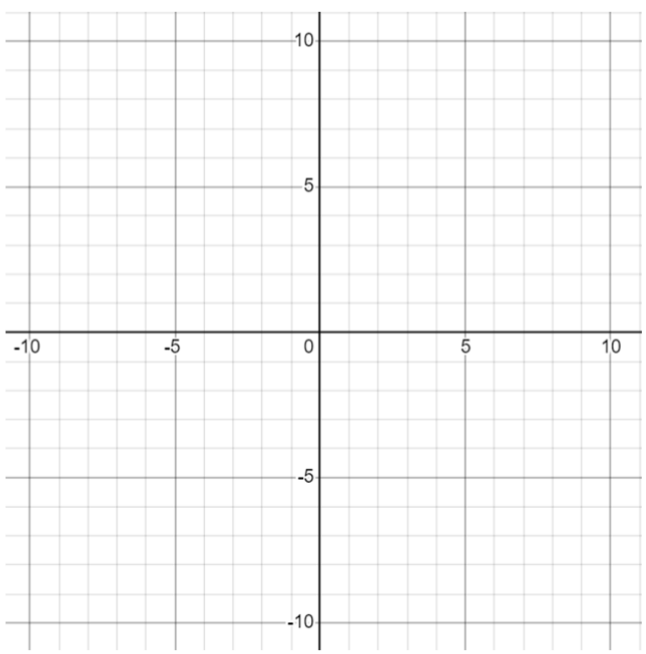
$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

Definition 5. *The* ***unit circle*** *is the circle centered at the origin with radius one unit.*

Problem 13. *Consider the unit circle. Use the Pythagorean Theorem 1 to write an equation relating x and y such that if (x,y) is a point on the circle, then the x,y pair satisfies the equation.*



Problem 14. *Replace y with $y-3$ and x with $x-5$ in the equation from Problem 13. Sketch the graph that you expect to represent this new equation. Find two points on your sketch and verify that the points satisfy the equation.*



Problem 15. *Write down an equation for a circle centered at (h,k) with radius r units.*

Perimeter and *line segment* are undefined words. Ok, we'll stop listing them... Just wanted to point out that we all use lots of words that are undefined and should recognize which words we have working definitions for and which we do not!

Definition 6. *The* circumference $\boldsymbol{C}$ *of a circle is the length of the perimeter of the circle.*

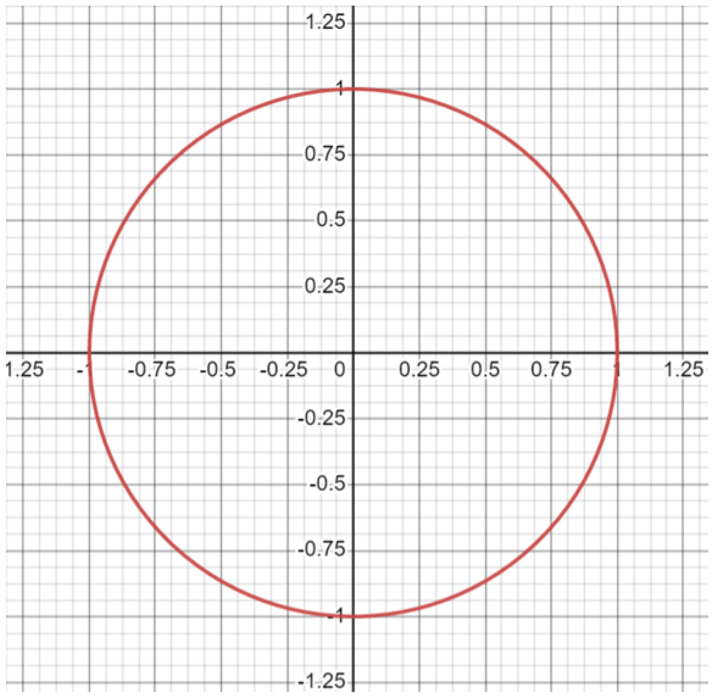
Definition 7. *The* diameter $\boldsymbol{D}$ *of a circle is the length of a line segment starting at one point on a circle, passing through the center, and ending on the circle.*

Definition 8. *The* radius $\boldsymbol{R}$ *of a circle is half the length of the diameter of the circle.*

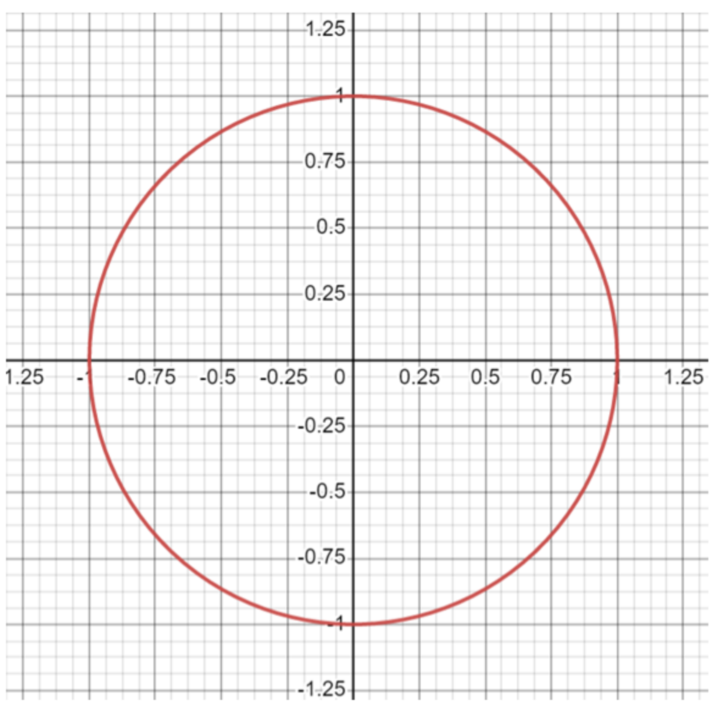
Definition 9. *The number π is the ratio of the circumference of a circle to the diameter of the circle. Restated, $\pi = \frac{\boldsymbol{C}}{\boldsymbol{D}}$.*

One might ask, "how do we know π remains constant for all circles," but we won't.

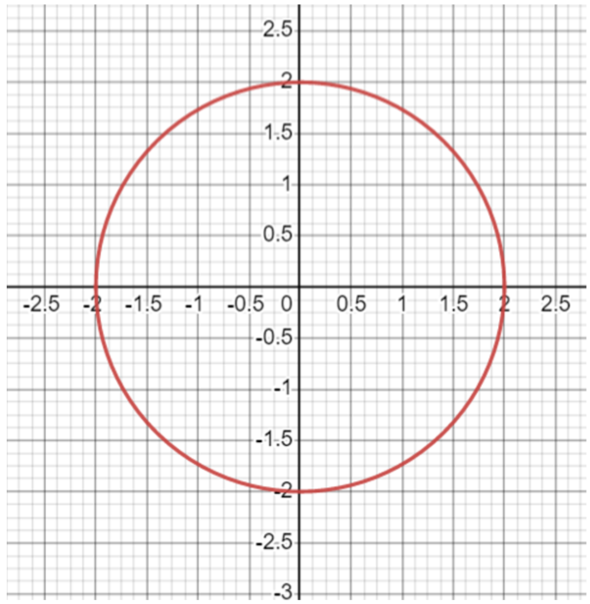
Problem 16. *Consider the unit circle placed onto the x,y coordinate plane with the center of the circle placed at the origin. Mark the point $(1,0)$. Starting at the point $(1,0)$ and working counter-clockwise, mark seven more points on the unit circle that sub-divide the circle into eight arcs of equal length. Label each point with its arc length distance from the point $(1,0)$.*

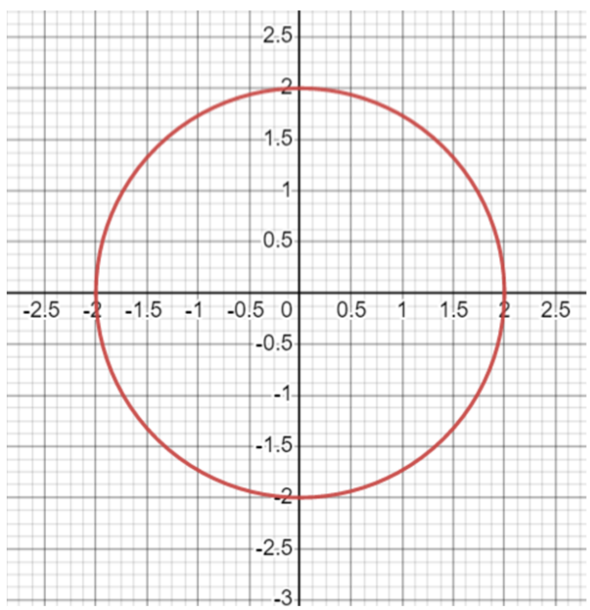


Problem 17. *Repeat Problem 16 using twelve points.*

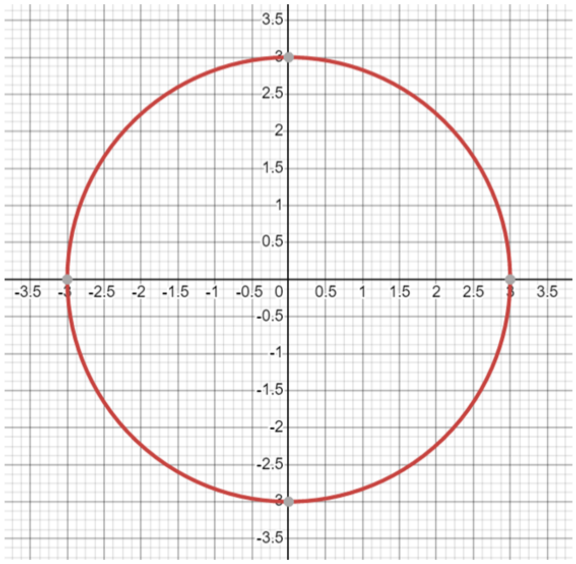


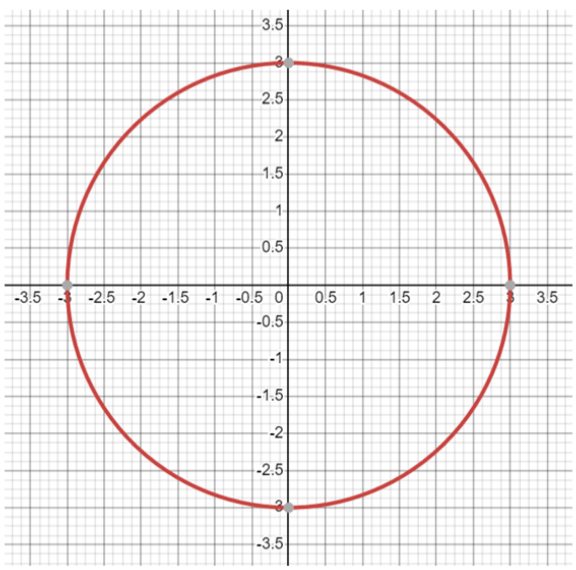
Problem 18. *Repeat Problems 16 and 17 using circles centered at the origin with radius* 2 *units.*



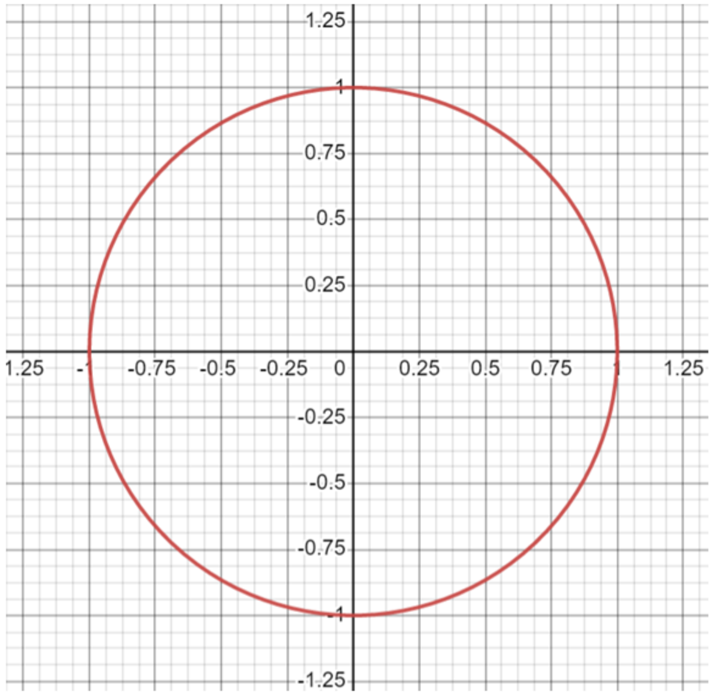


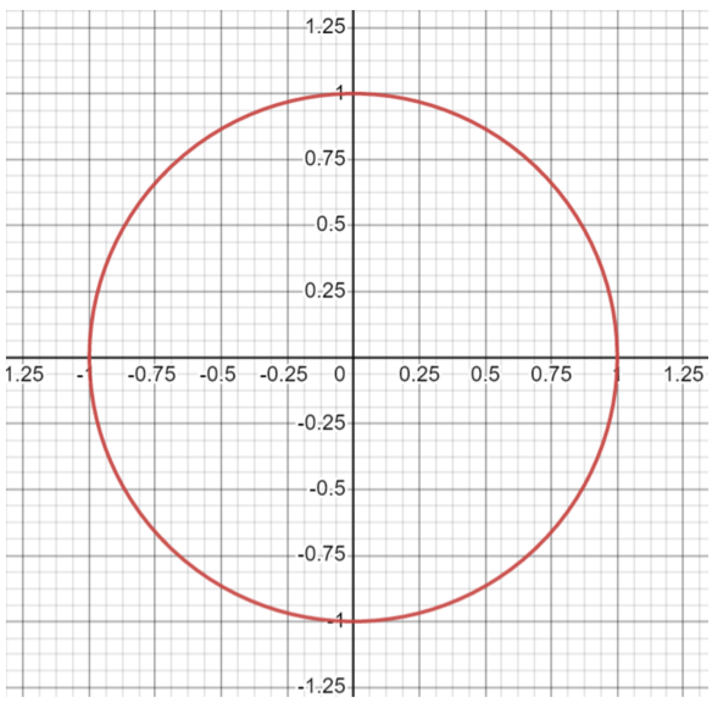
Problem 19. *Repeat Problems 16 and 17 using circles centered at the origin with radius* 3 *units.*



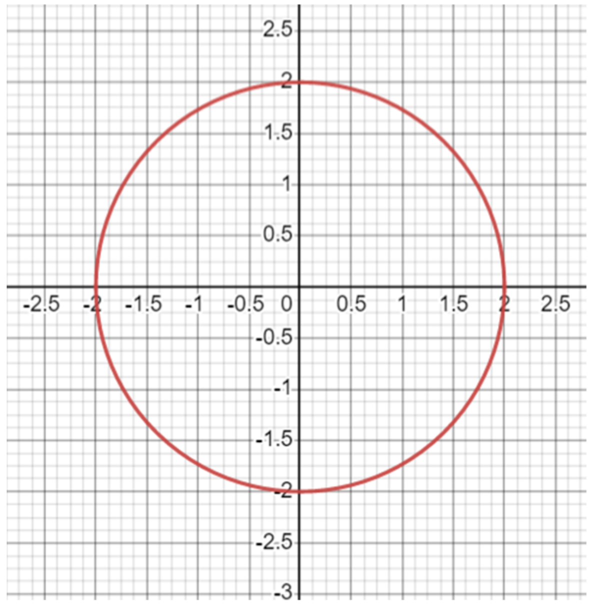


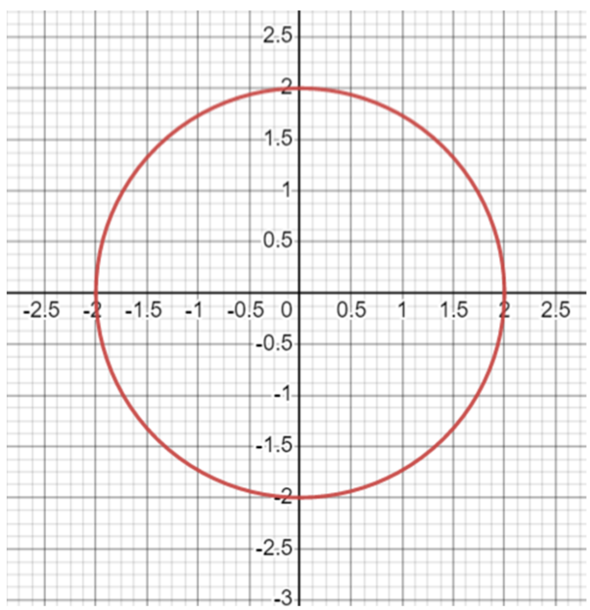
Problem 20. *Consider the unit circles from Problems 16 and 17. Now label each of the points you made with the ratio,* $\dfrac{arclength}{radius}$.



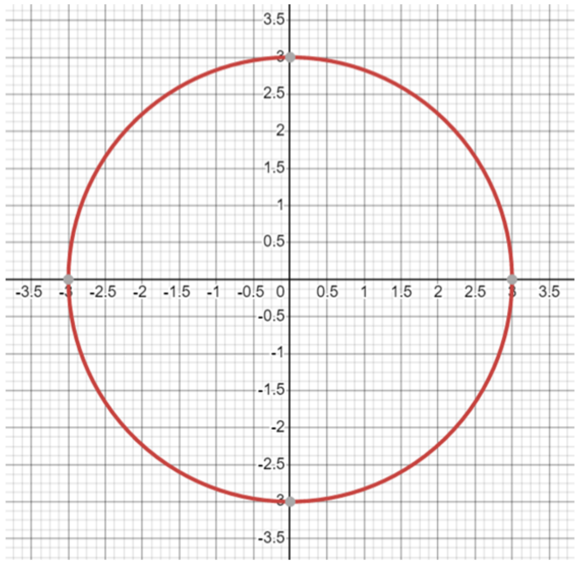


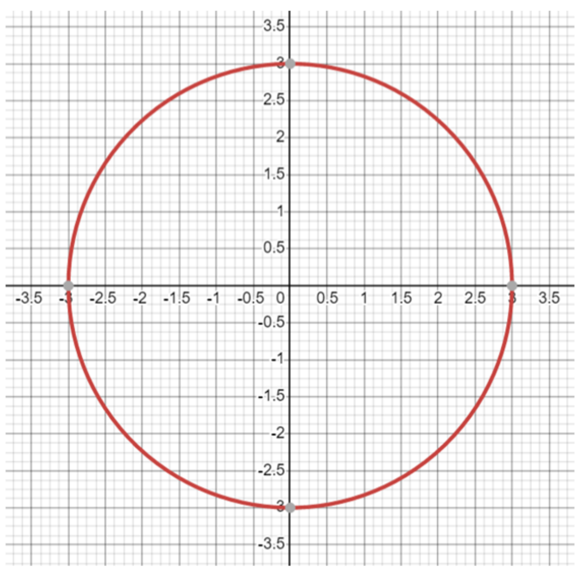
Problem 21. *Repeat Problem 20 using circles with radius* 2 *units.*





Problem 22. *Repeat Problem 20 using circles with radius* 3 *units.*





Definition 10. *An **angle** is a subset of a plane consisting of two distinct rays with a common endpoint called the vertex of the angle.*

Definition 11. *An **angle in standard position** is an angle for which the vertex is at the origin and one of the rays is the positive x-axis. The ray on the positive x-axis is referred to as the **initial side** of the angle. The other ray is referred to as the **terminal side** of the angle.*

Definition 12. *Given any circle centered at the origin and an angle in standard position, let P_1 be the intersection of the circle with the initial side of the angle and let P_2 be the intersection of the circle and the terminal side of the angle. The **arc associated with the angle** is the portion of the circle traced by a point traversing the circle in a counter-clockwise direction from the point P_1 to the point P_2.*

Definition 13. *Given any circle centered at the origin and any angle in standard position, the **radian measure of the angle** is the ratio of the length of the arc associated with the angle and the radius of the circle.*

$$radian\ measure = \frac{arc\ length}{radius}$$

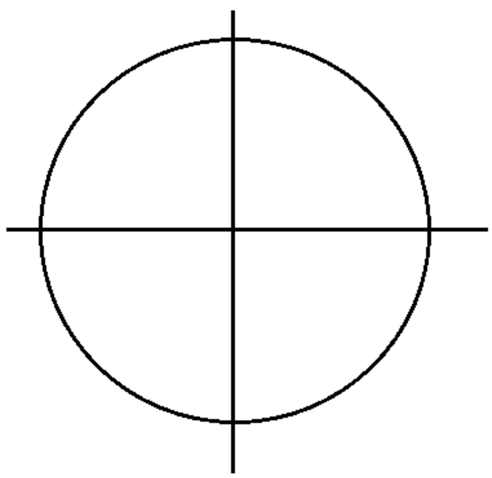
If the radian measure of an angle is θ, then we say the angle has radian measure θ or has measure of θ radians. This is often written as $\theta = \frac{S}{r}$.

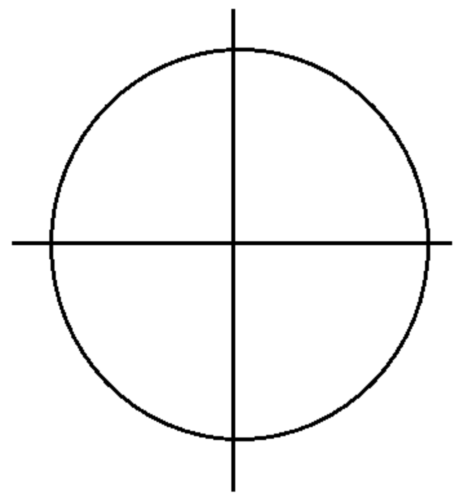
Problem 23. *If θ is radian measure, S is the corresponding arc length, and r is the radius of a circle, explain why $\frac{S}{2\pi r} = \frac{\theta}{2\pi}$.*

Problem 24. *Draw an angle in standard position where the ray forming the initial side begins at* $(0,0)$ *and extends through the point* $(1,0)$ *and the ray forming the terminal side of the angle begins at the point* $(0,0)$ *and passes through the point* $(1,1)$.

Draw a second angle in standard position where the ray forming the initial side begins at $(0,0)$ *and extends through the point* $(1,0)$ *and the ray forming the terminal side of the angle begins at the point* $(0,0)$ *and passes through the point* $(2,2)$.

Find the radian measure for each of these angles.





Definition 14. *The* ***degree measure of an angle*** *is* $\frac{180}{\pi}$ *times the radian measure of the angle.*

Theorem 3. *The sum of the degree measures of the interior angles of a triangle is* $180°$. [3]

Problem 25. *Determine the degree measure of the two angles in Problem 24.*

Problem 26. *What is the degree measure of an angle whose radian measure is* $\frac{\pi}{12}$*?*

Problem 27. *What is the radian measure of* $75°$*?*

Problem 28. *In 2009 Usain Bolt set the world record for the 100 meter sprint with a time of* 9.58 *seconds which is an average speed of* 10.44 $meters/second$. *What was his average speed in* $miles/hour$?

Problem 29. *Car A travels* 60 $miles/hour$ *and car B travels* 80 $miles/hour$.

1. *Convert the speeds of car A and car B into* $feet/second$.

2. *It takes the average person* 2.2 *seconds to read a text message while driving. What is the difference of the distance traveled in feet between the two cars over* 2.2 *seconds?*

Problem 30. *A race horse trains on a circular track that has a radius of* 500 *meters.*

1. *The horse completes* $1/8$ *of a complete lap in* 22 *seconds. What is the average speed of the horse over* $1/8$ *of one lap in* $meters/second$?

2. *A camera is mounted at the center of the racetrack and rotates to track the horse during the training runs. What is the angular velocity in radians/second for the rotating camera as it tracks the horse for* $1/8$ *of a complete lap over* 22 *seconds?*

Chapter 3

Numbers and Lines

Let's start by talking about the different sets of numbers we will be using.

Definition 15. *The **naturals** are the numbers* $1,2,3,4,\ldots.$

Definition 16. *The **integers** are the numbers* $\ldots,-3,-2,-1,0,1,2,3,4,\ldots.$

Definition 17. *The **rationals** are the numbers* $\frac{p}{q}$ *where* p *and* q *are integers and* $q \neq 0$. *Two examples would be* $\frac{2}{3}$ *and* $\frac{-5}{125}$.

Definition 18. *The **irrationals** are the numbers that are not rational. Examples:* π, e *and* $\sqrt{2}$.

It actually requires proof to show that the irrationals even exist, but let's not go there now.

Definition 19. *The **reals** are the integers, rational numbers and irrational numbers.*

In summary, the *reals* are all the numbers you are familiar with except maybe the complex (imaginary) numbers, which we will study later.

Definition 20. *If* n *is a non-negative integer and* $a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n$ *are real numbers, then the **polynomial** defined by these numbers is the function*

$$y = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0.$$

The number n *is called the **degree** of the polynomial and the numbers* $a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n$ *are called the **coefficients** of the polynomial.*

Problem 31. *List the degree and* $a_0, a_1, a_2, a_3, a_4, a_5$ *and* a_6 *for* $y = 6 + 2x - 7x^5 - \pi x^4$.

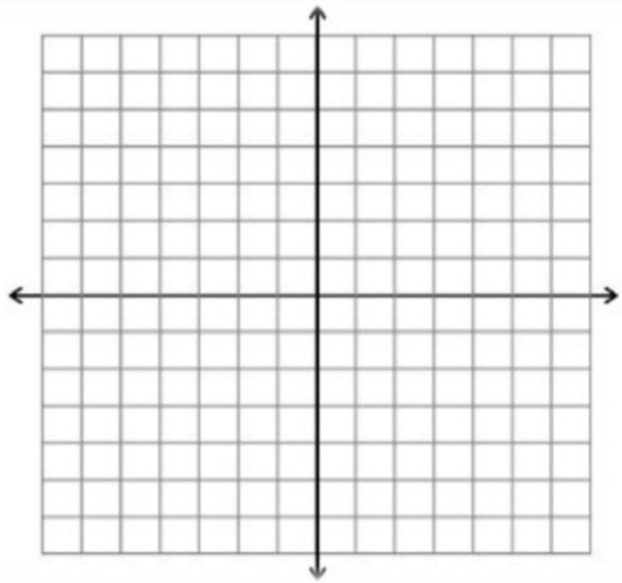
We'll study lines, the simplest example of a polynomial here, and higher degree polynomials in Chapter 5.

Definition 21. ***Lines** are polynomials of degree* 0 *and* 1. *We traditionally use* b *and* m *and write* $y = mx + b$ *instead of* y=$a_1 x + a_0$.

The number m is called the slope and can be computed by taking $\frac{y_2 - y_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are any two points on the line. The number b is called the $y-$intercept and is the y value associated with the value $x = 0$. The point $(0, b)$ is the point where the line intersects the $y-$axis.

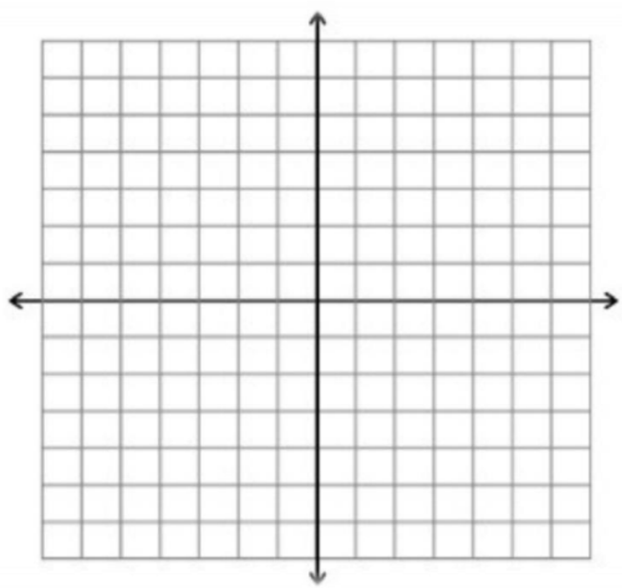
Problem 32. *Consider $y = 3x + 2$. Plot a few points and graph this line. Where does it cross the $x-$axis? Where does it cross the $y-$axis? List the values for m, b, a_0 and a_1.*

x	y



Problem 33. *Sketch and write the equation of the line through the two points $(-1,7)$ and $(3,-1)$. Is the point $(4,-2)$ on this line?*

x	y



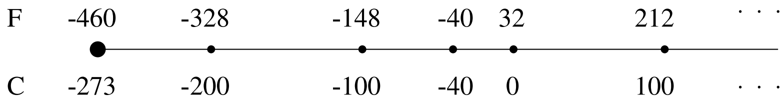


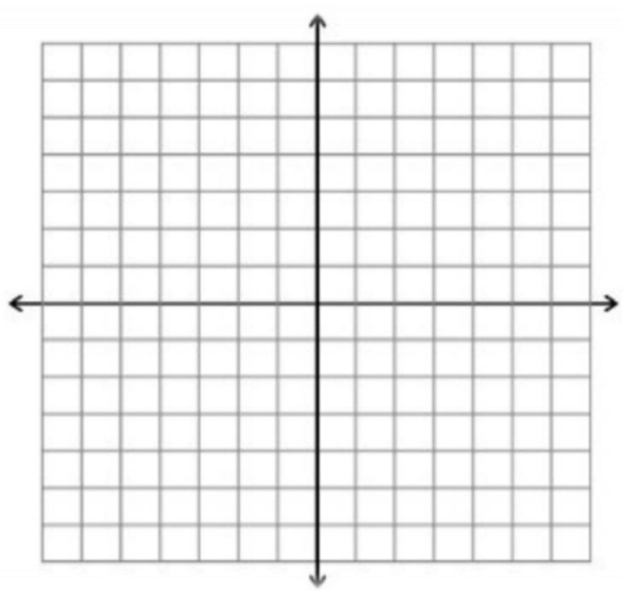
Figure 3.1: Thermometer showing both Fahrenheit and Celsius temperature scales

Problem 34. *Figure 3.1 shows the relationship between the Fahrenheit and Celsius temperature scales. Determine the equation for the line that converts Celsius to Fahrenheit. Determine another line that converts Fahrenheit back to Celsius. Bonus! A student once found a minor flaw in this problem. Can you find it?*

Problem 35. *Reflections.*

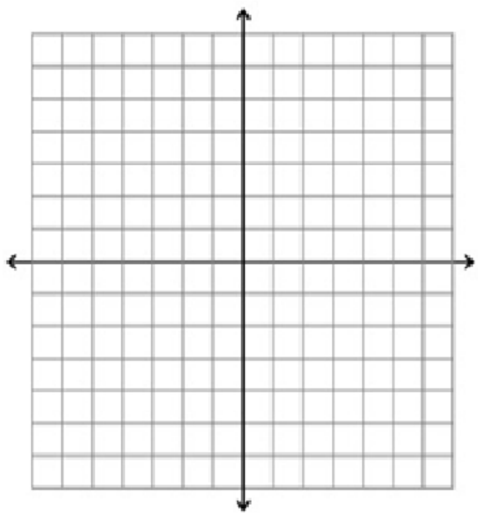
1. *Make a table of points on* $y = 2x + 1$ *and sketch the graph of this line.*
2. *On the same graph, sketch the graph of* $y = x$.
3. *Pick one point you plotted on* $y = 2x + 1$ *and draw the line segment from that point to the closest point on the line* $y = x$.
4. *Extend this line segment across the line* $y = x$ *to double it's length.*
5. *Repeat this process for a second point on* $y = 2x + 1$.
6. *Determine the coordinates of the ends of the line segments you just drew.*
7. *Connect these new dots to form the reflection of* $y = 2x + 1$ *across the line* $y = x$.

x	y



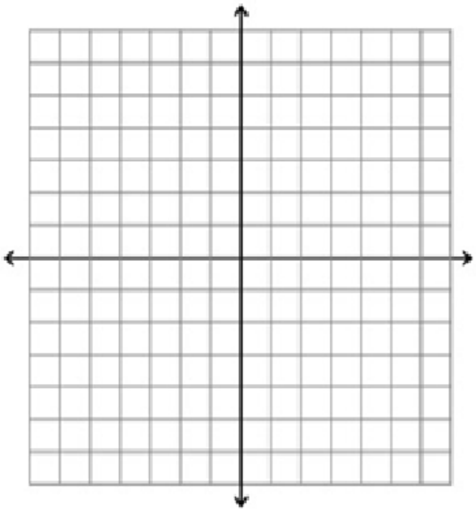
Problem 36. *Do the following.*

1. *Draw any line through the origin. Call the slope m so that the equation of your line is $y = mx$.*
2. *Plot any point which is not on this line. Call this point (h,k).*
3. *Use vertical and horizontal translations to write an equation for the line with slope m through the point (h,k).* [4]

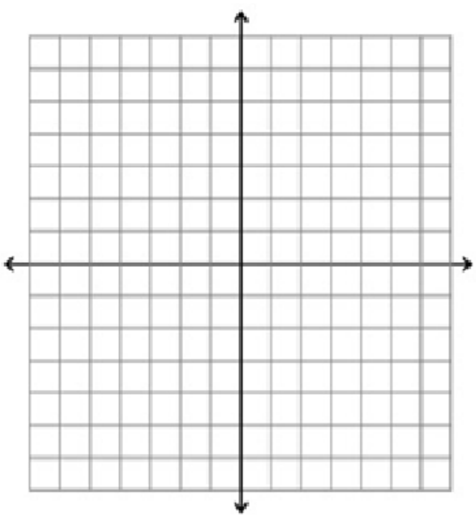


Problem 37. *Find equation of line through the points $(2,-1)$ and $(5,2)$. Write the equation of a line through $(5,2)$ perpendicular to the line you just found.*

Problem 38. *Consider the two lines,* $x+y=3$ *and* $3x-2y=4$. *Sketch a graph of each line below and locate the point on the plane where they intersect.*



Problem 39. *Consdider the two lines,* $y-2=-2x$ *and* $1-3x=2y$. *Compute the intersection of these two lines with and without graphing.*

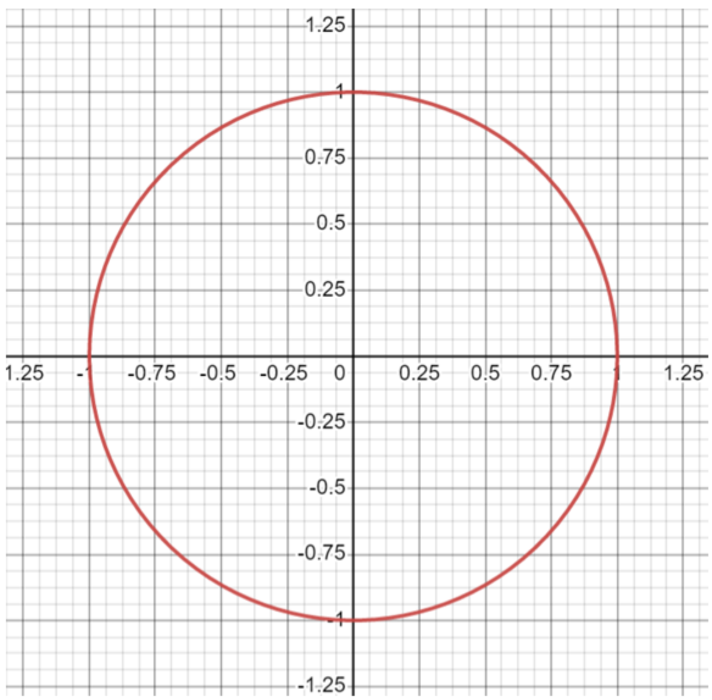


Chapter 4

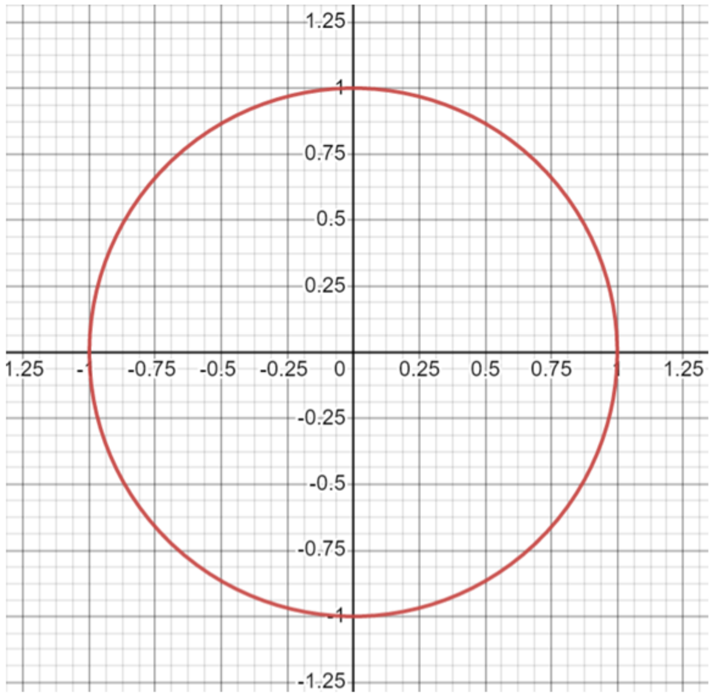
Trigonometrics and Functions

Review Definitions 10 - 14 at the end of Chapter 2.

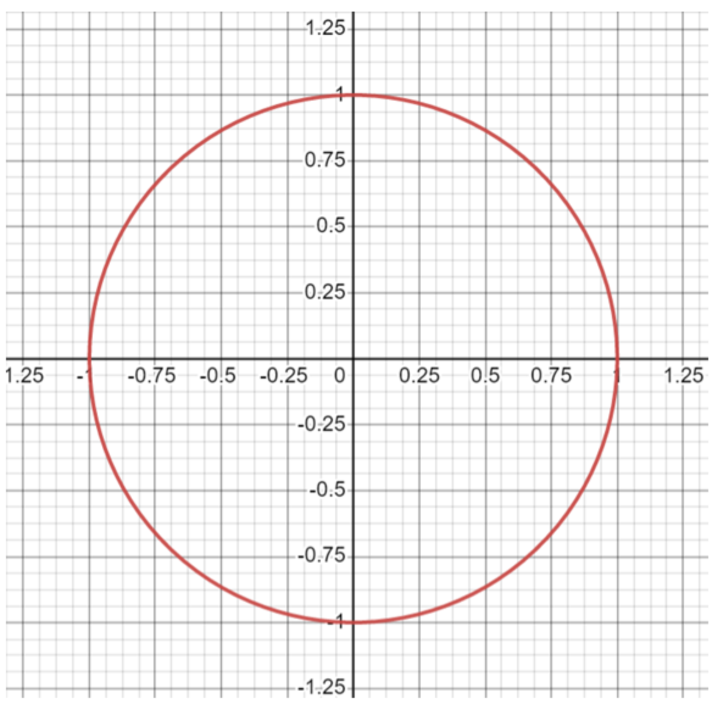
Problem 40. *Consider the unit circle centered at the origin. Draw an angle in standard position that has angle measure* 150 *degrees. What is the radian measure of this angle?*



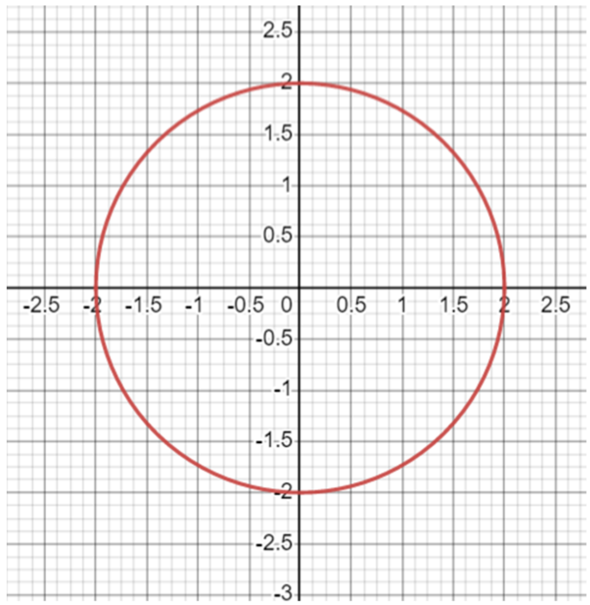
Problem 41. *Consider the unit circle centered at the origin. Draw an angle in standard position that has angle measure* $\frac{\pi}{4}$ *radians. What are the* (x,y) *coordinates of the intersection of the unit circle and the terminal side of the angle?*



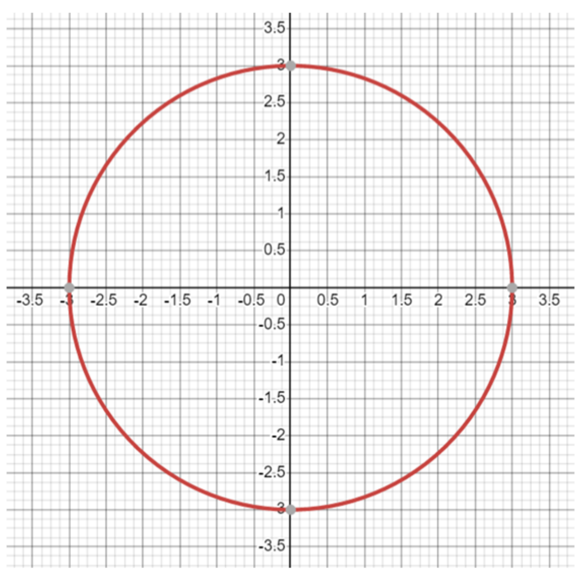
Problem 42. *Consider the unit circle centered at the origin. Draw an angle in standard position that has angle measure* $\frac{\pi}{6}$ *radians. What are the* (x,y) *coordinates of the intersection of the circle with the terminal side of the angle? What is the length of the arc associated with this angle?*



Problem 43. *Consider the circle of radius* 2 *units centered at the origin. Draw an angle in standard position so that the length of the arc associated with the angle is* $\frac{3\pi}{2}$ *units. What is the radian measure of this angle? What is the degree measure?*



Problem 44. *Consider the circle of radius* 3 *units centered at the origin. Draw an angle in standard position that has angle measure* $\frac{7\pi}{4}$ *radians. Determine the length of the arc associated with this angle.*



Problem 45. [5] *Suppose that a unicycle with a wheel of radius 9 inches is rolled 4 feet. Through what radian measure has one spoke on this wheel traveled? How many revolutions has the wheel made?*

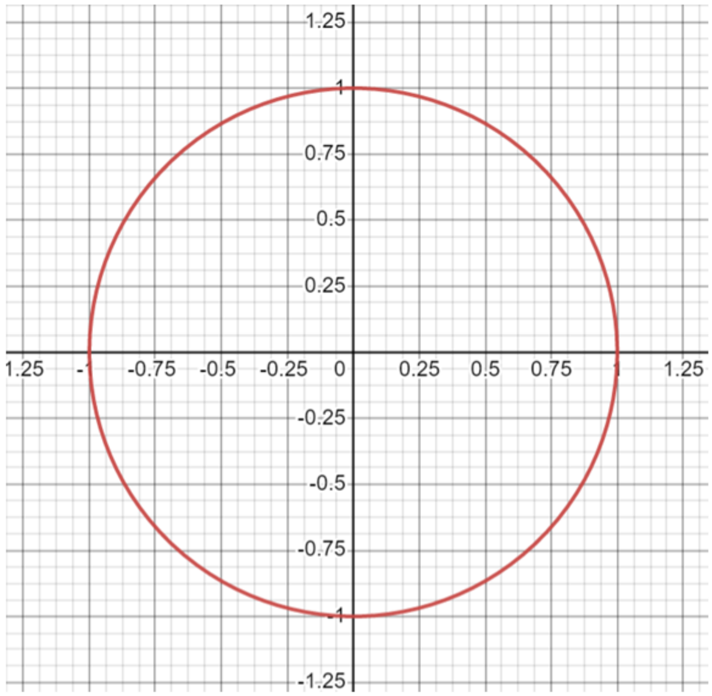
Problem 46. *Consider the unicycle wheel of Problem 45. Suppose it takes* 2 *seconds to roll the unicycle* 2 *feet. What is the speed of the unicycle measured in feet/second? What is the speed in mph? What is the angular speed of the unicycle measured in radians/second? How fast is the wheel spinning measured in revolutions/second?*

Problem 47. *Given an angle with measure* 270°*, what is the measure of the angle in radians?*

Problem 48. *Given an angle with measure* $\frac{2\pi}{3}$ *radians, what is the measure of the angle in degrees?*

Our angles have all been measured in a *counter clockwise* direction from the $x-$axis. Such angles have positive measure and are referred to as positive angles. When measured with a *clockwise* direction from the positive $x-$axis, angles have negative measure and are referred to as negative angles. [6]

Problem 49. *Locate the point on the unit circle so that the angle formed by the radius of the circle containing this point and the positive x−axis has radian measure* $-\frac{3\pi}{4}$. *What is the measure of this angle in degrees?*

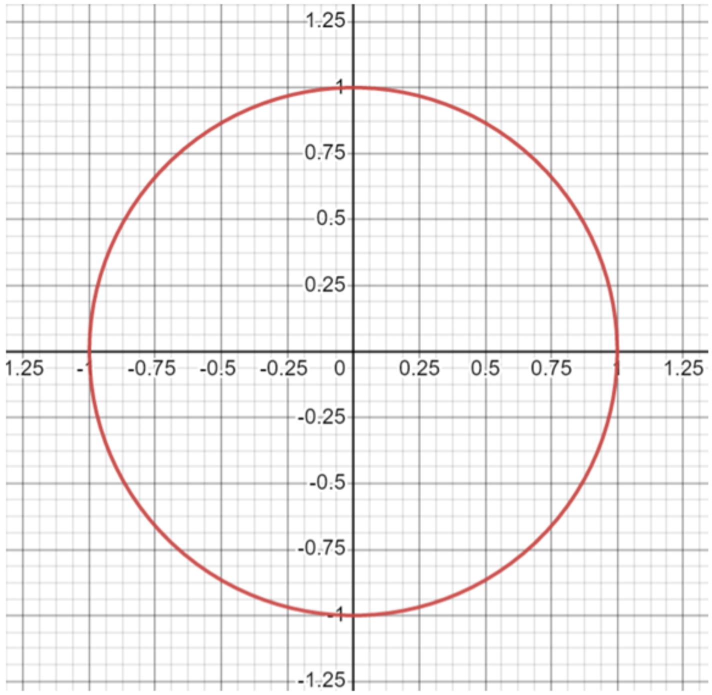


Problem 50. *Given an angle with measure* $-405°$, *determine the radian measure.*

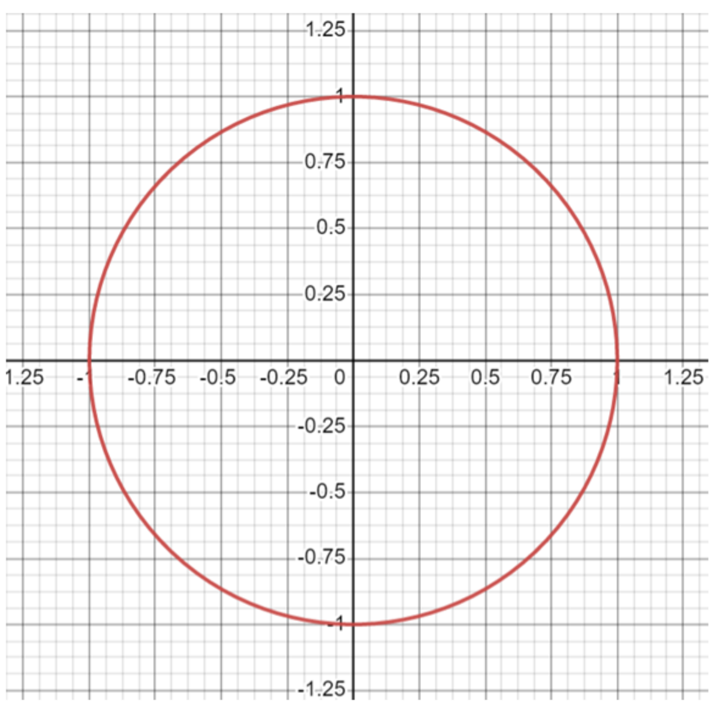
Problem 51. *Given an angle with radian measure* $-\frac{24\pi}{7}$, *determine the degree measure.*

Essay 1. Write an essay addressing the value of mathematics to society. Why should anyone take a mathematics course? How is it used in society? What value, if any, does it add? Cite your sources.

Problem 52. *Determine the coordinates in the (x,y) plane for each point on the unit circle whose distance from the point $(1,0)$ along the unit circle in a counter-clockwise direction is a positive integer multiple of $\frac{\pi}{4}$.*



Problem 53. *Determine the coordinates in the (x,y) plane for each point on the unit circle such that the angle that is in standard position with the terminal side containing the point has radian measure of a positive integer multiple of $\frac{\pi}{6}$.*



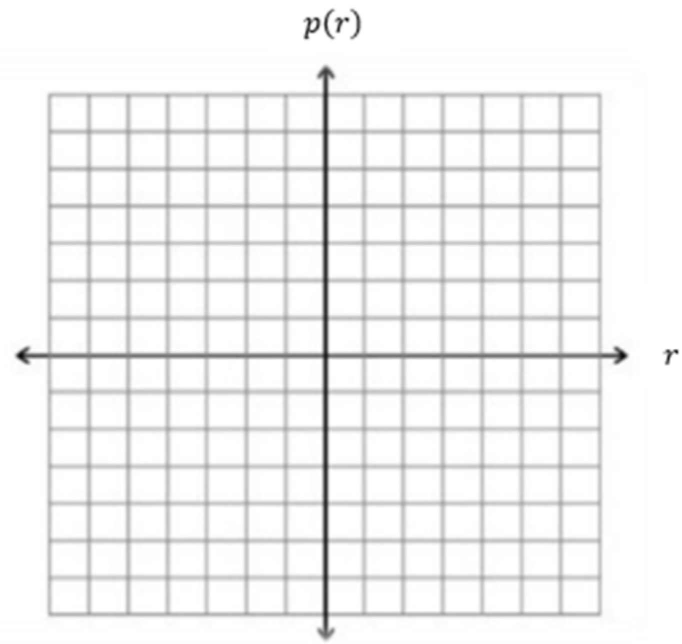
Definition 22. [7] *A **function** is a collection of points in the plane with the property that no two of these points are on the same vertical line.*

Definition 23. *The set of all first coordinates (x- coordinates) of all ordered pairs of a function is referred to as the **domain of the function**. The set of all second coordinates (y- coordinates) of all ordered pairs of a function is referred to as the **range of the function**.*

In Chapter 1 we graphed functions like $y = x^2 + 2$. Mathematicians use a bit more sophisticated notation for such functions. We would more likely write this as $f(x) = x^2 - 2$. We would then write $f(3) = 3^2 - 2 = 7$ and $f(-4) = (-4)^2 - 2 = 14$. Both f and x are just random names we chose. If you graph $f(x) = |x - 3|$ and I graph $g(t) = |t - 3|$, then we will have graphed the same set of coordinates, so will have graphed the same function. We'll use this *functional* notation most of the time from this point forward. Even when we use t and g for our variable names, we still talk about graphing the (x, y) coordinates.

Problem 54. *Create a table and graph of $p(r) = 2r^2 - 2r$. List the domain and the range of this function.*

r	$p(r)$



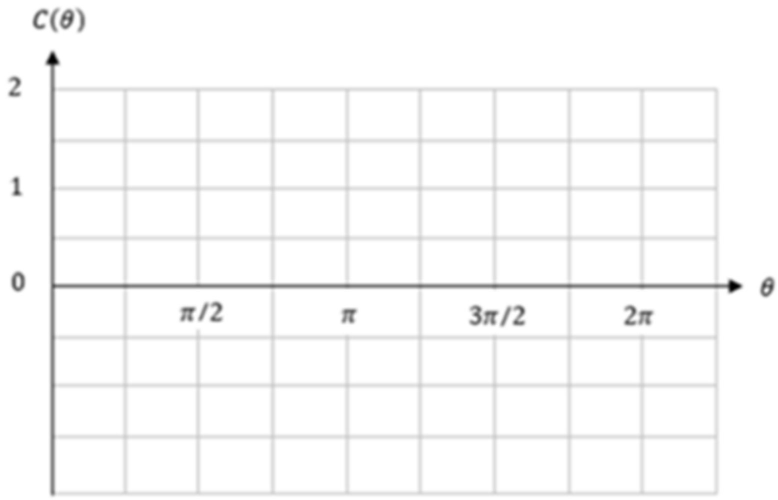
Definition 24. *We define a new function C such that if P is a point on the unit circle and the radius of the circle that contains P forms an angle of radian measure θ with the positive $x-$axis, then $C(\theta)$ is the $x-$coordinate of P.*

Definition 25. *We define a new function S such that if P is a point on the unit circle and the radius of the circle that contains P forms an angle of radian measure θ with the positive $x-$axis, then $S(\theta)$ is the $y-$coordinate of P.*

Definition 26. *We define the function T such that if P is a point on the unit circle and the radius of the circle that contains P forms an angle of radian measure θ with the positive $x-$axis, then* $T(\theta) = \dfrac{S(\theta)}{C(\theta)}$.

Problem 55. *Create a table and a graph for C. Approximate values to three decimal places.*

θ	$C(\theta)$
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
$4\pi/3$	
$3\pi/2$	
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	
2π	



Problem 56. [8] *Determine one value for θ where $C(\theta) = 0$. Are there other values?*

Problem 57. *List all values for θ where $C(\theta) = \frac{1}{2}$.*

Problem 58. *List all values for θ where $C(\theta) = \frac{\sqrt{2}}{2}$.*

Problem 59. *Create a table and a graph for S. Approximate values to three decimal places.*

θ	$S(\theta)$
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
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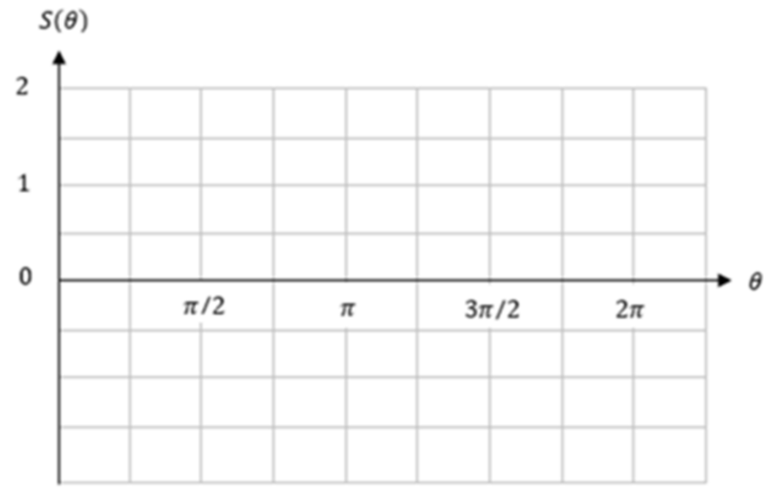
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	
2π	



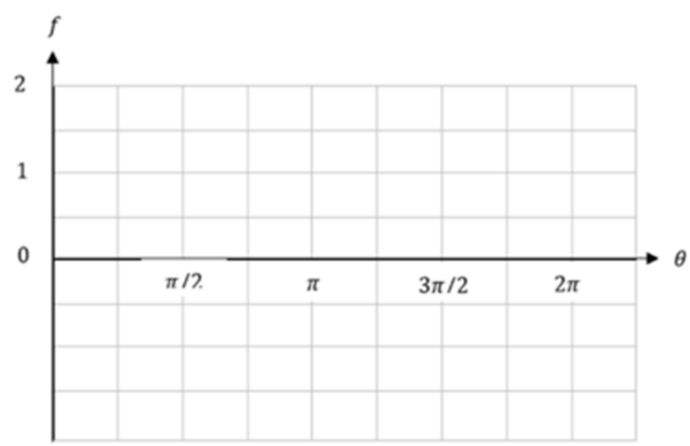
Problem 60. *Solve* $S(\theta) = 1$ *for all possible values of* θ.

Problem 61. *Solve* $S(\theta) = \frac{\sqrt{3}}{2}$ *for all possible values of* θ.

Problem 62. *Solve* $S(\theta) = -\frac{1}{2}$ *for all possible values of* θ.

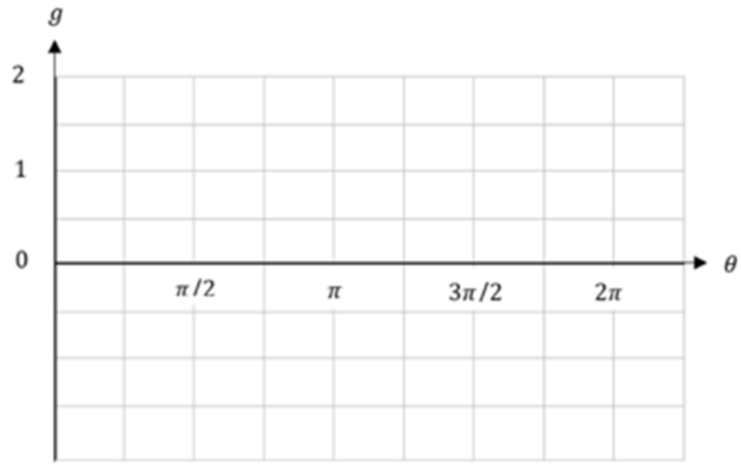
Problem 63. *Let* $f(\theta) = -S(\theta)$ *for every real number* θ. *Graph* f.

θ	f
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
$4\pi/3$	
$3\pi/2$	
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	
2π	



Problem 64. *Let $g(\theta) = C(\theta - \frac{\pi}{2})$ for every real number θ. Graph g.*

θ	g
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
$4\pi/3$	
$3\pi/2$	
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	
2π	

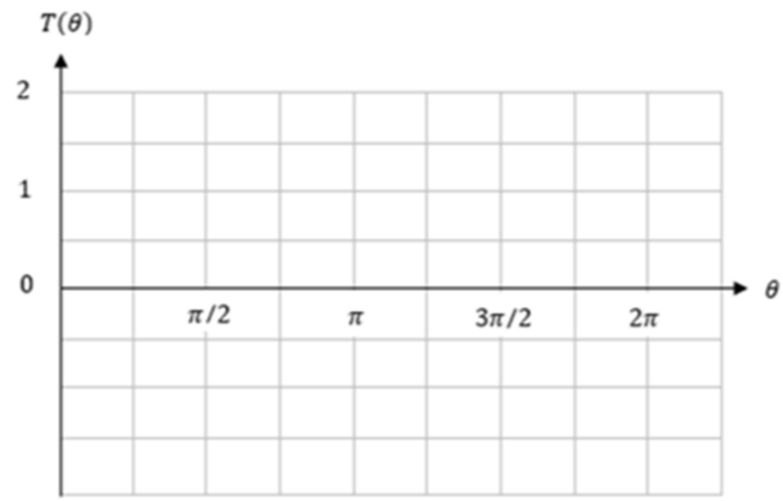


Problem 65. *For what values of θ will $T(\theta)$ be undefined? Why?*

Problem 66. *What is the domain of $T(\theta)$?*

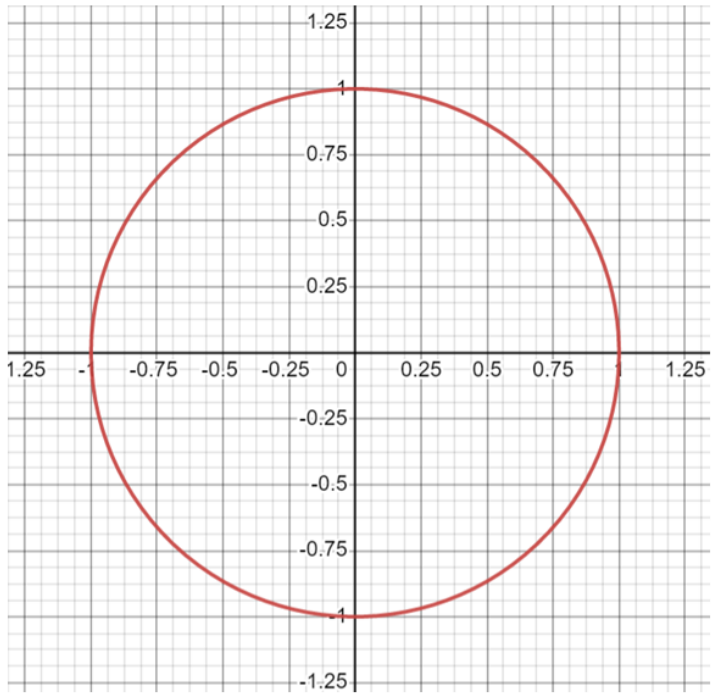
Problem 67. *Create a table and graph for T. Approximate values to three decimal places.*

θ	$T(\theta)$
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
$4\pi/3$	
$3\pi/2$	
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	
2π	



Problem 68. *Solve* $T(\theta) = -\frac{\sqrt{3}}{3}$ *for all possible values of* θ.

Problem 69. *Consider the unit circle. Draw the vertical line $x = 1$. Let θ be any number with $0 < \theta < \frac{\pi}{2}$. Draw the line that forms an angle with radian measure θ with the positive $x-$axis. Find the distance from the point $(1,0)$ to the point of intersection of the two lines you drew.*

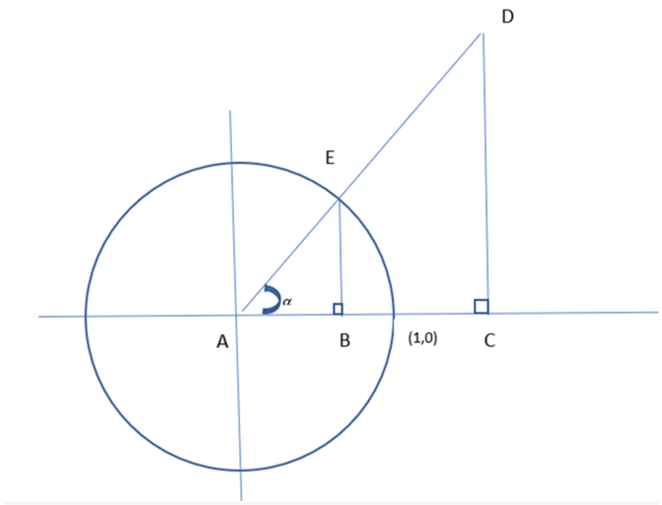


Definition 27. *Two triangles are said to be* ***similar*** *if corresponding angles have equal measure.*

Theorem 4. [9] *Given two similar triangles, the ratios of corresponding sides are equal.*

Problem 70. *Carefully sketch a triangle with sides lengths of 2, 7 and $\sqrt{39}$. Now sketch the triangle similar to this with side lengths 5 and 17.5. Determine the length of the missing side.*

Refer to the following unit circle, Theorem 1 and Theorem 4 for the next three problems.



Problem 71. *Show that* $S(\alpha) = \dfrac{l(CD)}{l(AD)}$.

Problem 72. *Show that* $C(\alpha) = \dfrac{l(AC)}{l(AD)}$.

Problem 73. *Show that* $T(\alpha) = \dfrac{l(CD)}{l(AC)}$.

We have now defined three functions, C, S and T. The last few problems have shown that if we have a right triangle in a plane with side adjacent to the angle having length a, side opposite the angle having length b and hypotenuse having length c, then these functions satisfy:

$$S(\theta) = \frac{b}{c}$$
$$C(\theta) = \frac{a}{c}$$
$$T(\theta) = \frac{b}{a}$$

These last three problems showed that these are the trigonometric functions which some of you were already familiar with, sine, cosine and tangent, usually abbreviated as $\sin, \cos, \tan$.

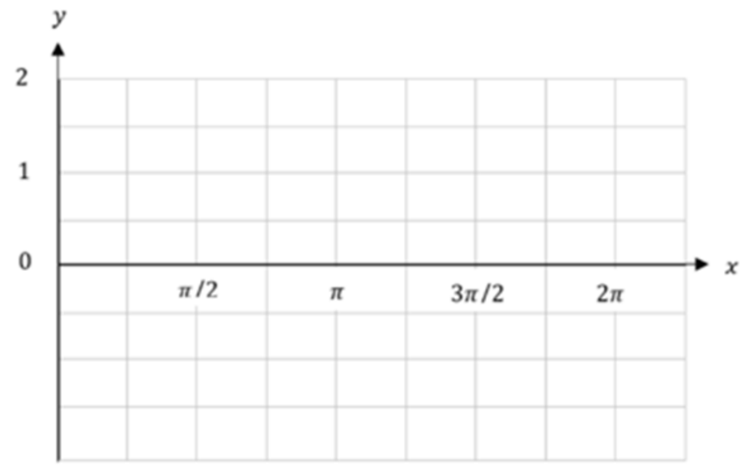
Definition 28. *We now define three more functions.*

1. *The function **secant**, abbreviated as **sec**, is the reciprocal of cosine. I.e.* $\sec(\theta) = \frac{1}{\cos(\theta)}$.
2. *The function **cosecant**, abbreviated as **csc**, is the reciprocal of sine. I.e* $\csc(\theta) = \frac{1}{\sin(\theta)}$.
3. *The function **cotangent**, abbreviated as **cot**, is defined by* $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$.

Problem 74. *Some authors define* $\cot(\theta) = \frac{1}{\tan(\theta)}$ *and others as we did, with cot*$(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$. *Why aren't these the same?*

Problem 75. *Create a table and a graph for the secant function,* $y = \sec(x)$*. Give domain and range with an explanation.*

x	y
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
$4\pi/3$	
$3\pi/2$	
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	
2π	

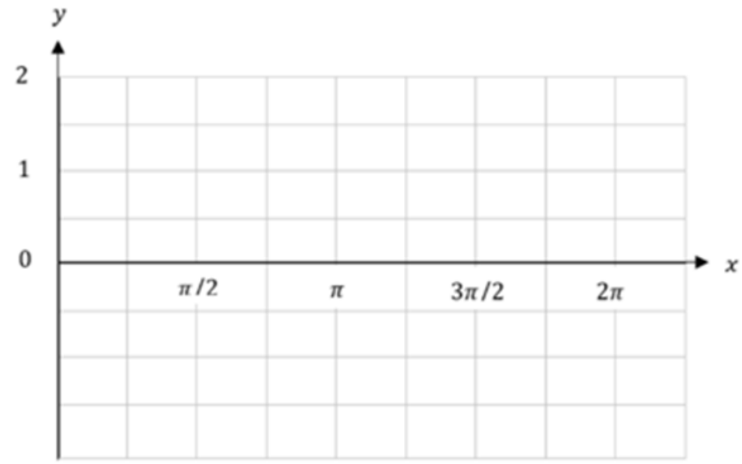


Problem 76. *A road has a grade of z% if the road rises z vertical feet for every 100 horizontal feet. You head for California to pursue a job as a movie director. While driving, you realize that you have driven one mile along a straight road with a 3% upward grade. Reminiscing of your wonderful precalculus class and recalling that one mile is approximately 5,280 feet, how far above ground level are you?*

Problem 77. *You steal Ted's sailboat and he observes you from his 40 meter high radio tower on the shore, which he keeps just so he can evesdrop on presidential campaigns. The angle between a horizontal line at the height of the tower and the line from the top of the tower is called the angle of depression. The angle of depression from the tower to the sailboat is* 6.5°. *How far out at sea are you?*

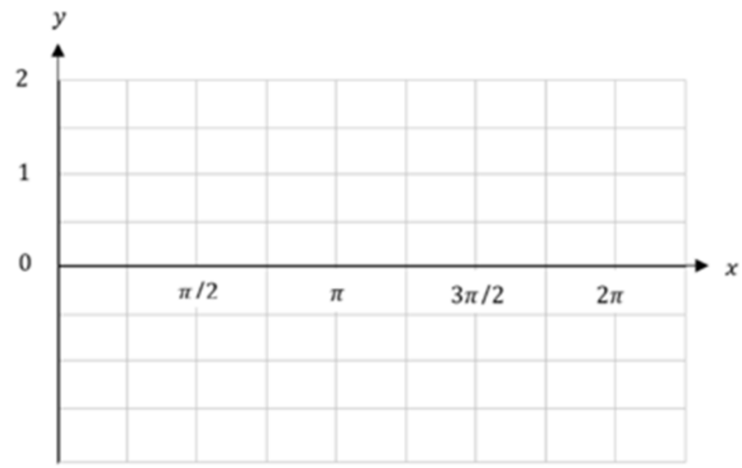
Problem 78. *Create a table and graph for the cosecant function, $y = \csc(x)$. Give the domain and range with an explanation.*

x	y
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
$4\pi/3$	
$3\pi/2$	
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	
2π	



Problem 79. *Create a table and graph for the cotangent function,* $y = \cot(x)$. *Give the domain and range with an explanation.*

x	y
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
$4\pi/3$	
$3\pi/2$	
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	
2π	



Definition 29. [10] *Let f be a function and let A be a positive number. We say the function f has period A if A is the smallest number satisfying $f(x+A) = f(x)$ for all x in the domain of f.*

Problem 80. *Determine the period for each of the six trigonometric functions.*

Problem 81. *Create a table and a graph for $y = 2\sin(x - \pi)$.*

x	y
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
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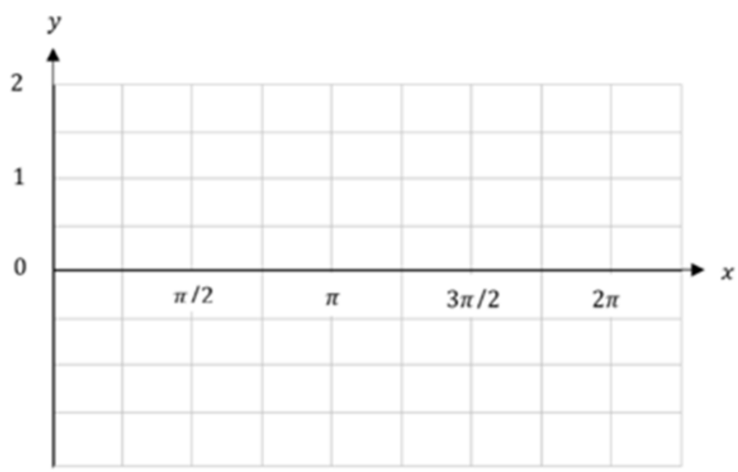
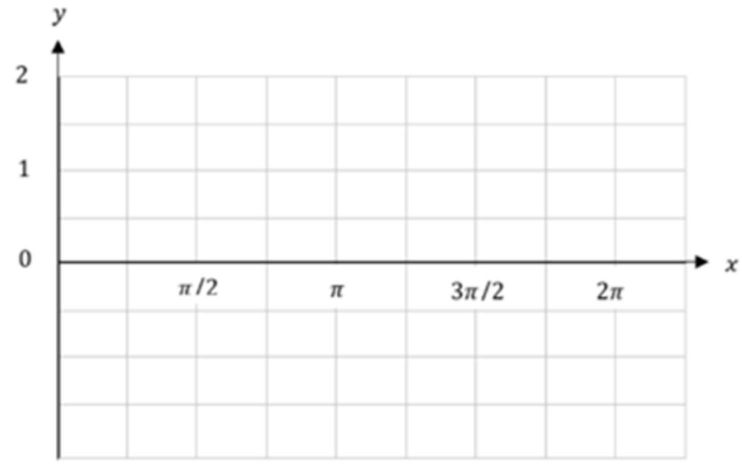
$3\pi/2$	
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	
2π	



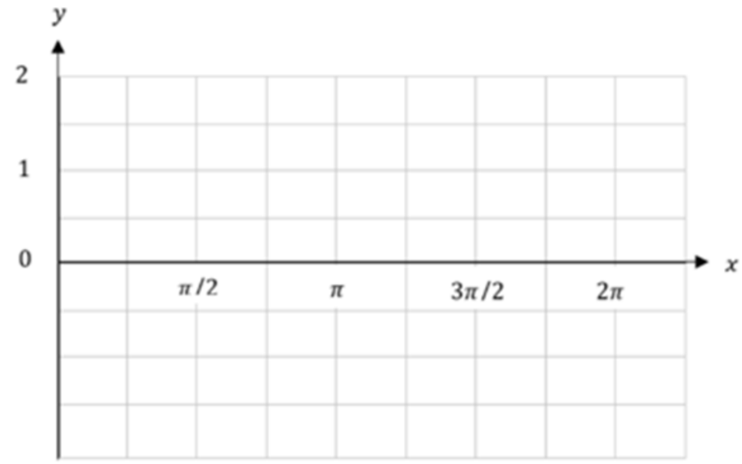
Problem 82. *Create a table and a graph for* $y = \sin(x) + \cos(x)$.

x	y
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
$4\pi/3$	
$3\pi/2$	
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	
2π	



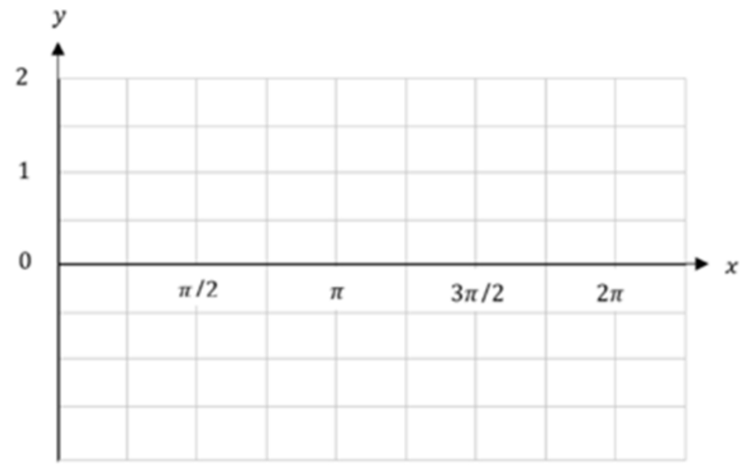
Problem 83. *Create a table and graph for* $y = x\sin(x)$.

x	y
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
$4\pi/3$	
$3\pi/2$	
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	
2π	



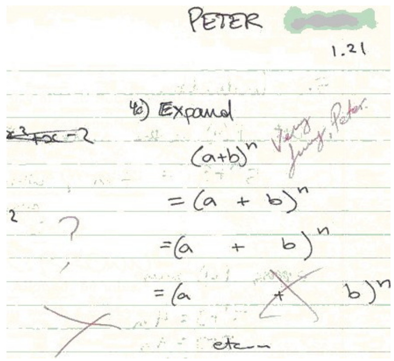
Problem 84. *Consider* $y = 2\sin(2x - \frac{\pi}{6})$. *First, write a sentence describing in detail how this is a transformation of* $y = \sin(x)$. *Create a table and graph. Include two cycles in your graph.*

x	y
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
$4\pi/3$	
$3\pi/2$	
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	
2π	



Chapter 5

Polynomials



Definition 30. *If n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n$ are real numbers, then the* ***polynomial*** *defined by these numbers is the function*

$$y = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0.$$

The number n is called the ***degree*** *of the polynomial and the numbers $a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n$ are called the* ***coefficients*** *of the polynomial.*

Definition 31. *If $p(x) = k$ for some real number k, then we say $p(x)$ is a polynomial of degree 0, or that $p(x)$ is a* ***constant polynomial****. Polynomials of degree 1 are called* ***linear polynomials****. Degree 2 polynomials are called* ***quadratics****, degree 3 polynomials are called* ***cubics****, degree 4 polynomials are called* ***quartics*** *and degree 5 polynomials are called* ***quintics****. No teacher can say what a polynomial of degree 6 is without students snickering, so we don't say it.*

Problem 85. *Show that $f(x) = x^2 + 2x - 7$ and $g(x) = (x+1)^2 - 8$ are the same polynomial.*

Problem 86. *Fill in the blanks to rewrite $f(x) = x^2 + 6x + 7$ as $f(x) = (x + \_\_\_\_)^2 + \_\_\_\_$.*

Problem 87. *Fill in the blanks to rewrite* $f(x) = 3x^2 + 2x + 4$ *as* $f(x) =$____ $(x+$____$)^2+$____.

Problem 88. *Fill in the blanks to rewrite* $f(x) = ax^2 + bx + c$ *as* $f(x) =$____ $(x+$____$)^2+$____.

Problem 89. *Set your answer from Problem 88 equal to zero and solve for x. You should obtain the infamous **quadratic formula**,* $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Definition 32. *A **root** of a polynomial p is any number r that satisfies* $p(r) = 0$*. The point* $(r, p(r)) = (r, 0)$ *is called an **x-intercept** of the polynomial.*

Definition 33. *If p may be written as* $p(x) = (x - r)q(x)$ *for some polynomial q, then the expression **x-r** is called a **linear factor** of the polynomial p.*

Theorem 5. *Let p be a polynomial of degree n.*

1. *p has n roots and n linear factors.*
2. *If r is a root of p, then* $(x - r)$ *is a **linear factor** of the polynomial p.*
3. *Good news! Every polynomial of degree n has n roots (some of which may be repeated) and n linear factors. Bad news! Some roots may be complex (imaginary) numbers, which we will study later.*

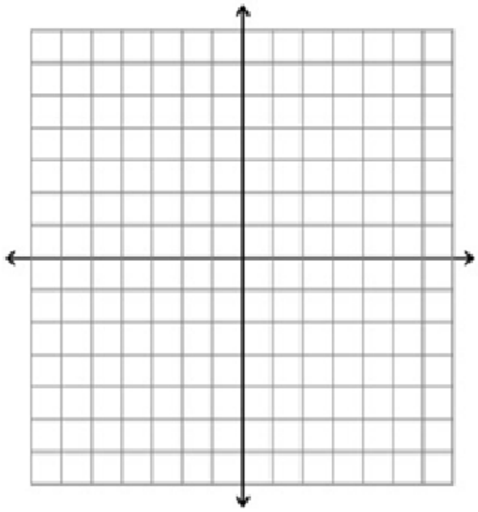
Problem 90. *Let* $p(x) = x^3 + 2x^2 - 15x$*.*

1. *Show that* 3 *is a root of p by showing that* $p(3) = 0$*.*
2. *Show that* $x - 3$ *is a linear factor of p by factoring p.*
3. *Write down two more linear factors of p.*

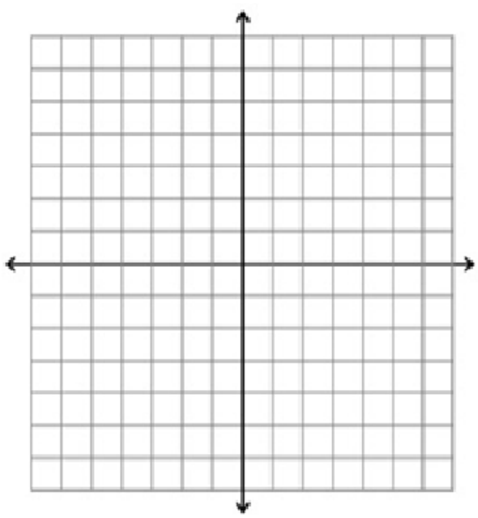
Problem 91. *Ted's wife has a* 24×40 *foot rectangular garden. She would like a sidewalk around it and Ted, being a cheapskate, estimates that his neighbor will have enough concrete left after his driveway is completed to cover 660 square feet. How wide should Ted make the sidewalk in order to use up all the concrete?*

Problem 92. *Solving and graphing.*

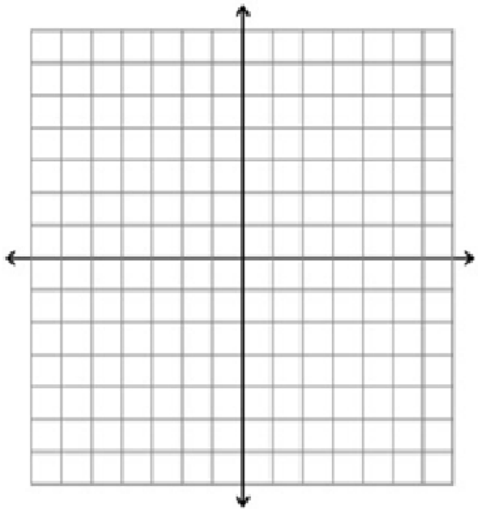
1. *Find the roots of* $p(x) = x^3 - 9x$ *by solving* $x^3 - 9x = 0$. *Graph* p *by plotting these* $x-$*intercepts and a point on either side of each root.*



2. *Find the roots of* $T(x) = \frac{1}{2}(x^2 + 6x + 9)(x^2 - 1)$ *by solving* $T(x) = 0$. *Graph* T.



3. *Find the roots of* $m(x) = 4x^4 - 64x^2$ *by solving* $4x^4 - 64x^2 = 0$. *Graph* m.

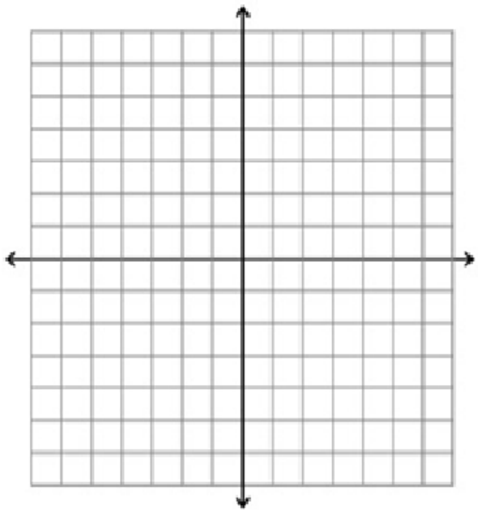


Problem 93. *Find all real numbers for which the inequality* $2 \leq 3x + 5$ *is true.*

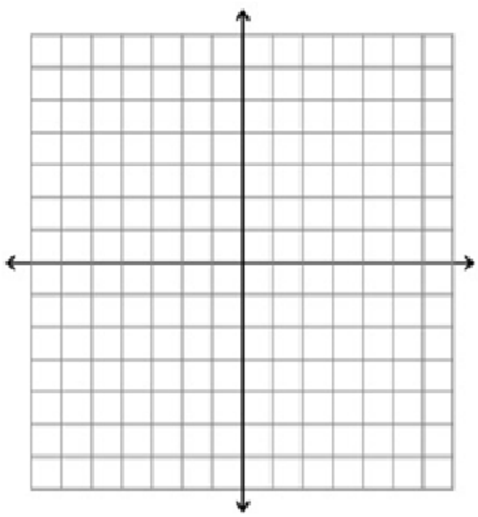
Problem 94. *Find all real numbers for which the inequality* $x^3 < x$ *is true.*

Problem 95. *Solving and graphing.*

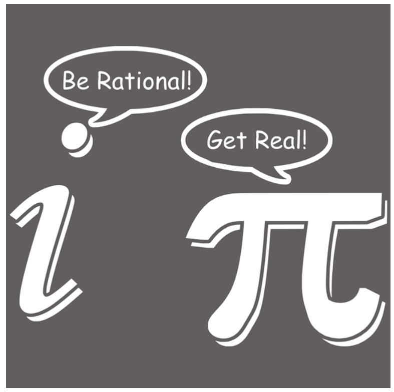
1. *Solve* $x^2 - x - 1 = 0$ *and graph* $p(x) = x^2 - x - 1$.



2. *Solve* $-2x^2 + 3x + 5 = 0$ *and graph* $q(x) = -2x^2 + 3x + 5$.



3. *Solve* $x^2 + x + 1 = 0$. *What do you notice about this solution?*



The solutions to the third part of Problem 95 contains the square root of a negative number. There is no real number whose square is negative, thus the solutions cannot be real numbers. To resolve this issue, mathematicians created a new set of numbers called complex (imaginary) numbers which depend entirely on this next simple definition.

Definition 34. *Let* $i = \sqrt{-1}$.

It follows that $i^2 = -1$.

Problem 96. *Show that any number written as* $a + \sqrt{-b}$ *where* a *and* b *are real numbers can be written as* $a + \sqrt{b}\, i$*. Write the solutions from the third part of Problem 95 in the form* $a + bi$ *where* a *and* b *are real numbers.*

Definition 35. *The* ***complex numbers*** *is the set of all numbers* $a + bi$ *where* a *and* b *are real numbers. For the complex number* $z = a + bi$*,* a *is called the* ***real part*** *of* z *and* b *is called the* ***imaginary part*** *of* z*. A complex number is in* ***standard form*** *when it is written in the form* $a + bi$*.*

Definition 36. *For any complex number* $z = a + bi$*, the* ***conjugate*** *of* z *is the complex number* $a - bi$*.*

Definition 37. *Complex addition, subtraction, and multiplication.*

1. ***Addition*** $(a+bi)+(c+di) = (a+c)+(b+d)i$
2. ***Subtraction*** $(a+bi)-(c+di) = (a-c)+(b-d)i$
3. ***Multiplication*** $(a+bi)\cdot(c+di) = ac+adi+bci+bdi^2 = (ac-bd)+(ad+bc)i$

Problem 97. ***Divide*** *this expression by multiplying out both of the numerator and denominator.*
$\dfrac{a+bi}{c+di} = \dfrac{a+bi}{c+di} \cdot \dfrac{c-di}{c-di} = \dots$

Problem 98. *Show that if $z_1 = a+bi$ and $z_2 = a-bi$, then z_1z_2 is a real number. z_1 and z_2 are called a conjugate pair, since each is the conjugate of the other.*

Problem 99. *Rewrite the following in standard form $a+bi$, where a and b are real numbers.*

1. $(1-i)+(3+5i)$

2. $(1-i)-2(8-2i)$

3. $3(2-2i)(3-i)$

4. $\dfrac{3i}{1+i}$

5. $\dfrac{3+3i}{9(2-3i)}$

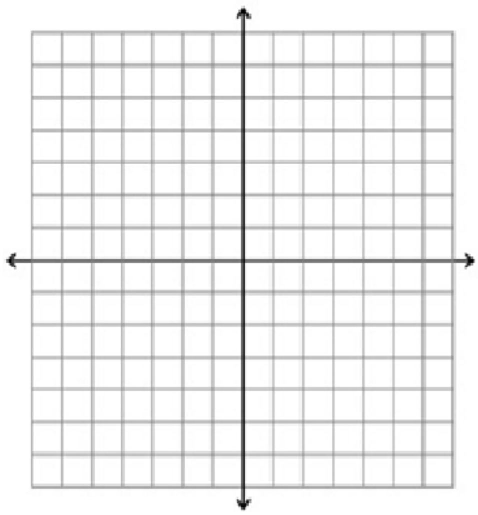
Essay 2. Write an essay on any mathematician's contribution to society. Explain mathematical ideas in terms a classmate could understand. Cite your sources.

Problem 100. *Taterhead Ted BoxCo takes* 20×20 *inch square metal* $1/4$ *inch thick plate and folds it into boxes by cutting out square corners and bending the sides up. Write down a function that represents the volume of the box if the corners we cut out each are x inch squares. Recall the volume of a box is length* $\times$ *width* $\times$ *height.*

Problem 101. *Taterhead Ted FenceCo has 3000 feet of fencing left over from a government job. He plans to fence in a rectangular dog pen for his LabraDork puppy. What is the largest area he can fence in? What is the smallest area he can fence in?*

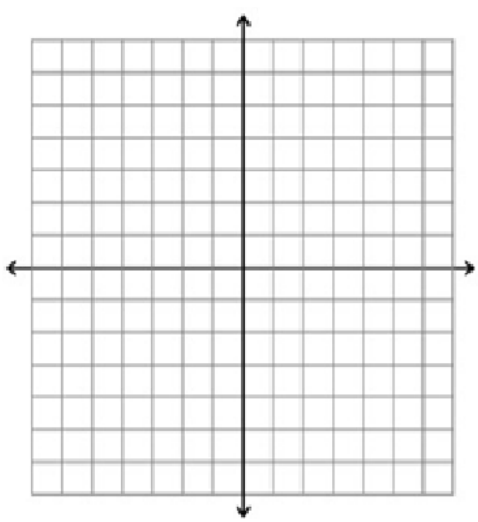
Problem 102. *Draw any feasible graph for polynomial p that satisfies the following requirements.*

1. *The degree of p is 5.*
2. *p has exactly three real zeros.*



Problem 103. *Draw any feasible graph for polynomial p that satisfies the following requirements.*

1. *The degree of p is 7.*
2. *p has exactly two non-real complex zeros.*
3. *The leading coefficient is $a_7 < 0$.*



Example 6. *Long division for polynomials parallels long division for numbers.* [11]

Let's divide 343 by 5. Does 5 go into 3? No, but it does go 6 times into 34. Multiply 60 times 5 and write 300 below 343. Subtract to obtain 43. Does 5 go into 4? No, but it does go 8 times into 43. Multiply 8 times 5 and write 40 below 43. Subtract to obtain 3. Does 5 go into 3? No, so we are left with the remainder $\frac{3}{5}$. We conclude, $343/5 = 68\frac{3}{5}$. Written another way, $343 = 5 \times 68\frac{3}{5}$.

Let's divide $x^3 - x^2 - 14x + 24$ by $x - 3$. Does $x - 3$ go into x^3? No, but $x - 3$ times x^2 times is $x^3 - 3x^2$. Subtract $x^3 - 3x^2$ from $x^3 - x^2 - 14x + 24$ to obtain $2x^2 - 14x + 24$. Does $x - 3$ go into $2x^2$? No, but $x - 3$ times x is $2x^2 - 3x$. Subtract $2x^2 - 3x$ from $2x^2 - 14x + 24$ and repeat this process until it terminates. Conclude that $x^3 - x^2 - 14x + 24$ divided by $x - 3$ equals $x^2 + 2x - 8$. Written another way, $x^3 - x^2 - 14x + 24 = (x-3)(x^2 + 2x - 8)$. This process illustrates Theorem 5, the Fundamental Theorem of Algebra, which states every polynomial of degree n may be factored into n linear roots.

Summarizing, if we looked at our first polynomial $p(x) = x^3 - x^2 - 14x + 24$ and noted that $x = 3$ is a root (i.e. $p(3) = 0$), then we would know that $x - 3$ is a factor. This means that if we divide the cubic polynomial p by $x - 3$ we will get a quadratic polynomial and maybe we can factor it, or find another root and long divide again.

Problem 104. *Use long division to divide $x^3 - x^2 - 14x + 24$ by $x + 4$. Verify via multiplication.*

Now that we know that finding a single root can help us factor, here is a nice tool to find roots.

Theorem 7. *Suppose p is a polynomial with:*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0$$

Then all rational zeros of p are of the form $\frac{r}{s}$ satisfying:

1. *r is an integer factor of the constant coefficient a_0.*
2. *s is an integer factor of leading coefficient a_n.*

Problem 105. *Find all rational zeros of $p(x) = x^4 - 2x^3 - 5x^2 + 4x + 6$ and write it as the product of its four linear factors.*

Chapter 6

Trigonometric Equations

Problem 106. *At what minimum height above ground level must a satellite dish be placed so that at a* 30° *angle it will be able to "see" over a building that is 40 feet tall and 50 feet away from the satellite dish?*

Problem 107. *Solve* $2\sin(x) = 1$ *for all possible values of x.*

Problem 108. *Solve* $2\cos(\theta) = -\sqrt{3}$ *for all possible values for* θ.

Problem 109. *Solve* $\tan(\theta) = 1$ *for all possible values for* θ.

Problem 110. *Solve* $\cos^2(x) - 1 = 0$ *for all possible values for x.*

Problem 111. *Solve* $\sin^2(x) - \sin(x) = 0$ *for all possible values for x.*

Problem 112. *Solve* $2\sin^2(x) - 3sin(x) + 1 = 0$ *for all possible values for x.*

Problem 113. *Solve* $\cot(x) > 0$ *for all possible values for x.*

Problem 114. *Solve* $\tan(x) < 0$ *for all possible values for x.*

Problem 115. *Let S be a square. Let M and N be the midpoints on two adjacent sides. Let V be the corner point that is opposite both points M and N. Let θ be the angle measure of the angle formed by two lines connecting the point V with the points M and N. Compute* $\sin(\theta)$.[12]

Problem 116. *Solve* $\csc(x) = -\frac{2\sqrt{3}}{3}$ *for all possible values for x.*

Problem 117. *Ted plants an Arizona Ash tree in his yard. While studying for calculus 100' from the base of the tree, a fire ant notes that the angle of elevation (the angle between the ground and the ant's line of sight to the top of the tree) is exactly* $\pi/4$*. Two months later, the ant returns to the same spot and notes that the angle is now* $\pi/3$*. How fast is the tree growing in feet per month? In radians per day?*

Problem 118. *Solve* $2\sin(4x) - \sqrt{3} = 0$ *for all possible values for x.*

Problem 119. *A crop duster plane passes directly over your head at an altitude of* $500 feet$. *Two seconds later you observe the plane with the same altitude and an angle of elevation of* 42° *from ground level. Find the plane's average speed over the two seconds.*

Problem 120. *Solve* $2\sin(\frac{x}{3})+1=0$ *for all possible values for x.*

Problem 121. *Solve* $\cos^2(2x) - 2\cos(2x) + 1 = 0$ *for all possible values of x.*

Problem 122. *Assume* θ *is any real number such that neither* $\sin(\theta)$ *nor* $\cos(\theta)$ *is zero. Simplify* $\tan(\theta)\cot(\theta)$. *Explain why we made the assumption. To simplify means to write an expression using fewer functions or algebraic operations.*

Chapter 7

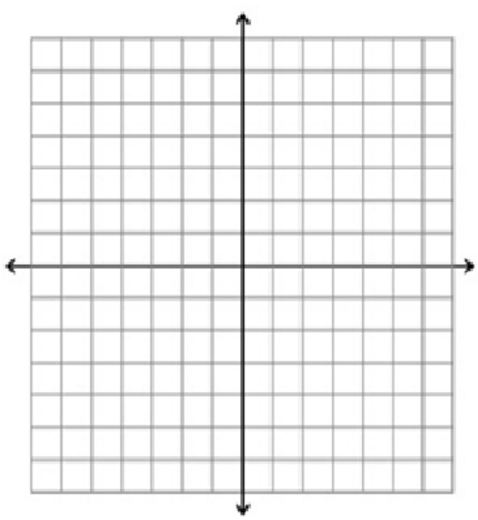
Rationals

Definition 38. *A **rational** number has the form* $r = \frac{p}{q}$ *where both* p *and* q *are integers and* $q \neq 0$.

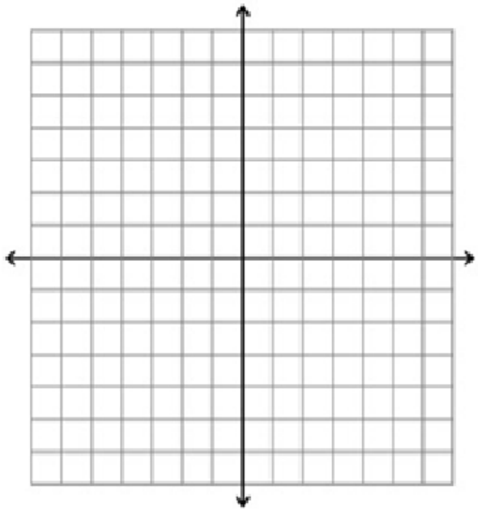
Definition 39. *A **rational** function has the form* $r(x) = \frac{p(x)}{q(x)}$ *where both* p *and* q *are polynomials.*

Definition 40. *An **asymptote** of a function is a line that the graph of the function approaches. The asymptote is **vertical** if its equation is* $x = c$ *for some number* c*; **horizontal** if its equation is* $y = c$ *for some number* c*; and **slant** if it is neither horizontal nor vertical.*

Problem 123. *Sketch a graph of* $r(x) = \frac{2x-1}{x-4}$ *by plotting the points associated with* $x = -100, -10, 0, 3.99, 4.01, 10$ & 100. *List the vertical asymptote,* $x =$____, *and the horizontal asymptote, and* $y =$____.

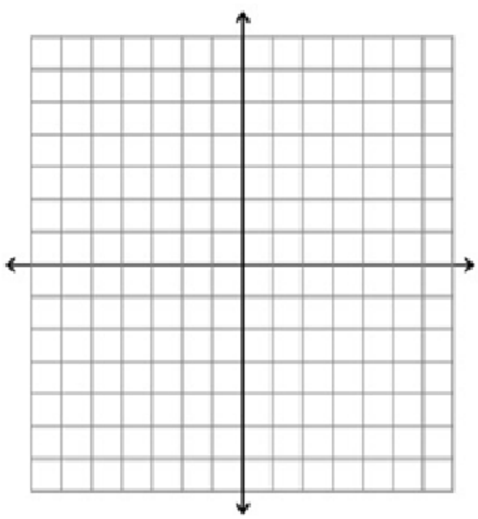


Problem 124. *Sketch* $s(x) = \dfrac{x^2-9}{x^2-4}$. *List the domain and asymptotes.*



Definition 41. *Given the rational function* $r(x) = \dfrac{p(x)}{q(x)}$ *if the degree of* p *is greater than the degree of* q*, then we refer to this as an* ***improper*** *rational function.*

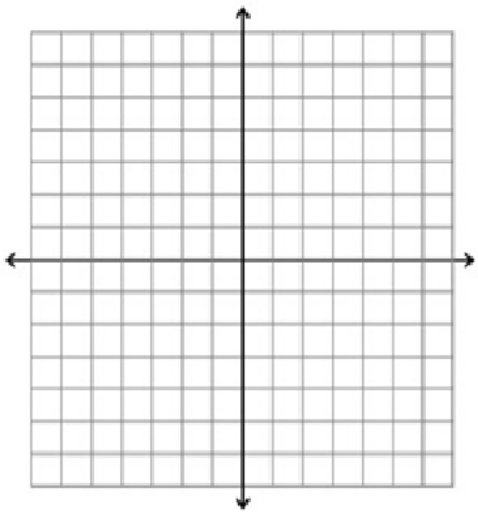
Problem 125. *Graph of the rational function* $h(x) = \dfrac{x^2-9}{x-3}$. *Does* h *have a vertical asymptote at* $x = 3$*?*



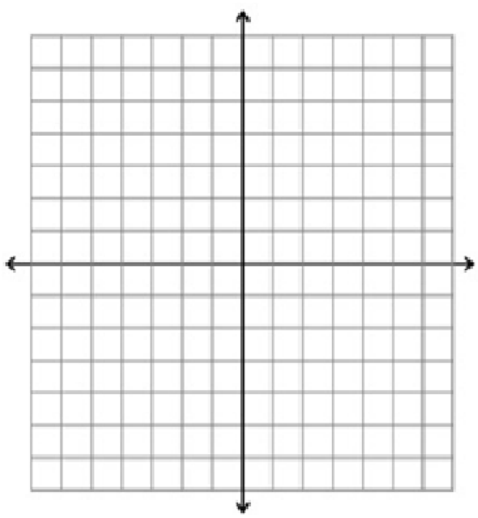
Problem 126. *You've just graduated and are assigned as an analyst at FoMoCo. Your first boss hands you the equation* $v(t) = \frac{90t}{t+10}$*, which he plans to present to his boss in 20 minutes. But he doesn't understand it because he failed pre-calculus. He says the equation models the velocity in miles per hour of a new hydrogen powered car they are about to produce and needs to know if it "seems believable." Is it a believable model, why or why not?*

Problem 127. *Taterhead Ted SnackCo sells JunkieMunchies which cost \$.50 per bag to produce in addition to the \$500 per day it costs to keep the shop open. Using the Texas Academy Consulting Company, it is determined that if x is the number of bags produced and the price per bag is set at* $\$1.95 - \frac{x}{2000}$*, then all the bags produced will be sold. How many bags per day need to be produced and sold to turn a profit? What is the possible range of prices?*

Problem 128. *Sketch* $h(x) = \frac{x^2 - 2x - 3}{x^2 - 1}$. *List all vertical and horizontal asymptotes.*



Problem 129. *Sketch* $h(x) = \frac{x^2 - 1}{x^3 - 1}$. *List all vertical and horizontal asymptotes.*



Problem 130. *Let* $h(x) = \dfrac{f(x)}{g(x)}$ *be any rational function such that* f *and* g *have no common roots.*

1. *If the degree of* f *is less than the degree of* g*, what is the horizontal asymptote?*

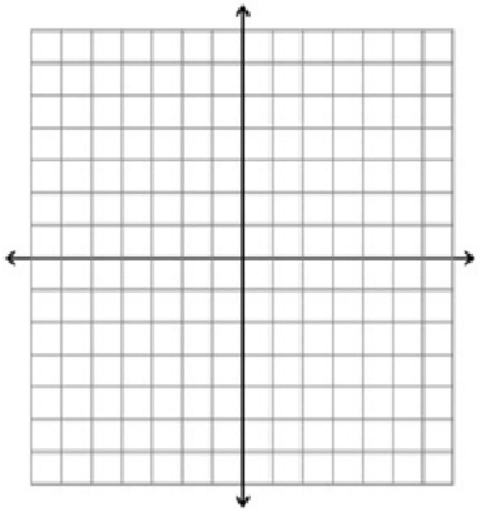
2. *If the degree of* f *is greater than the degree of* g*, why is there no horizontal asymptote?*

3. *If the degrees of* f *and* g *are equal, what is the horizontal asymptote?*

4. *Give an example where they share a common root. Does that change anything?*

Problem 131. *The earth has an approximate radius of* 6.4×10^6 *meters and the gravitational acceleration of an object relative to earth is approximately* $g(r) = \dfrac{4 \times 10^{14}}{r^2}$ *where r is the distance from the object to the center of the earth.*

a. *Graph our function g.*



b. *What is gravitational acceleration at the surface of the earth?*

c. *When is gravity zero? At what distance do you escape the earth's gravitational pull?*

Problem 132. *Recall Definition 40. Write an equation for the slant asymptote for* $h(x) = \dfrac{x^2 + 2x + 1}{x - 1}$.

Chapter 8

Trigonometric Identities

Problem 133. *Review definitions 24 and 25. Show that* $\sin^2(x)+\cos^2(x)=1$ *for all values of x.*

Problem 134. *Show that* $\tan^2(x)+1=\sec^2(x)$ *for all values of x.*

Problem 135. *Show that* $\cot^2(x)+1=\csc^2(x)$ *for all values of x.*

The three identities above are called the **Pythagorean Identities**.

Problem 136. *Solve* $1+\sin(x)=\cos(x)$ *for all possible values of x.* [13]

Problem 137. *Solve* $\csc(x)+\cot(x)=1$ *for all possible values of x.*

Problem 138. *Solve* $\cos^2(x)=\cos(x)+\sin^2(x)$ *for all possible values of x.*

Problem 139. *Simplify* $\dfrac{\tan(x)}{\sec(x)\csc(x)}$. *For what values of x is this expression defined? For what values of x is your simplified expression defined? Are the two expressions equal?*

Problem 140. *Simplify* $\left(\cos^2(x)-1\right)\left(\tan^2(x)+1\right)$. *For what values of x is this expression meaningful? For what values of x is your simplified expression meaningful?*

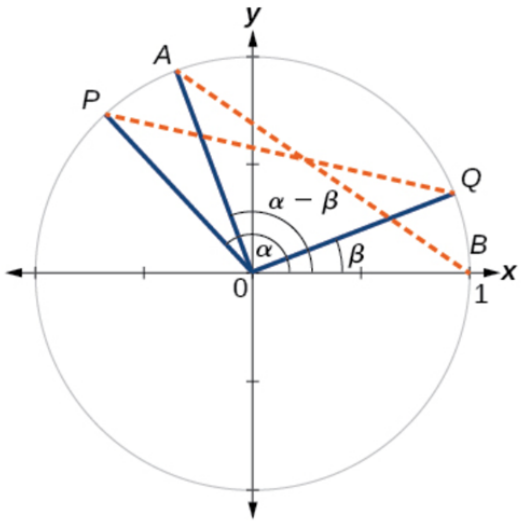
Problem 141. *Simplify* $\tan(x)\cos(x)$.

Problem 142. *Simplify* $\dfrac{\sec(x)}{\csc(x)}$.

Problem 143. *Simplify* $\dfrac{\sec(x)+\csc(x)}{1+\tan(x)}$.

Problem 144. *Simplify* $\big(\csc(x)+1\big)\big(\csc(x)-1\big)$.

Problem 145. *Simplify* $\dfrac{1+\sec(x)}{\tan(x)+\sin(x)}$.



Problem 146. *For the unit circle in the figure above, use the distance formula to determine the distance from* $P = \big(\cos(\alpha), \sin(\alpha)\big)$ *to* $Q = \big(\cos(\beta), \sin(\beta)\big)$.

Problem 147. *For the unit circle in the figure above, use the distance formula to determine the distance from* $A = \big(\cos(\alpha - \beta), \sin(\alpha - \beta)\big)$ *to* $B = (1, 0)$.

Problem 148. *Why are the distances in Problems 146 and 147 equal? Set them equal and simplify.*

The identity found in Problem 148 is called the **Cosine Difference Identity**. This identity leads to a variety of identities which will now be developed.

Definition 42. *If f is a function and $f(-x) = f(x)$ for every x in the domain of f, then we say that f is an* ***even*** *function.*

Definition 43. *If f is a function and $f(-x) = -f(x)$ for every x in the domain of f, then we say that f is an* ***odd*** *function.*

Even and odd functions are symmetric about the $y-$axis and the origin respectively.

Problem 149. *Let $f(x) = x^2$, $g(x) = x^3$, and $h(x) = f(x) + g(x)$. Show that f is an even function, g is an odd function, but h is neither an even function nor an odd function.*

Problem 150. *Determine which of the six trigonometric functions are even functions and which are odd functions.*

Problem 151. *Prove that if f is an even function and g is an odd function, then $h(x) = f(x)g(x)$ is an odd function.*

Problem 152. *Replace β with $-\beta$ in the cosine difference identity and develop the cosine sum identity.*

Problem 153. *Replace α with $\frac{\pi}{2}$ in the cosine difference identity and develop one of the co-function translation identities. What does this co-function identity say about the graphs of* sin *and* cos*? List at least three additional co-function identities.*

Problem 154. *Replace β with $\beta - \frac{\pi}{2}$ in the cosine sum identity and develop another identity. What would you call this identity?*

Problem 155. *There are four **Sum and Difference Identities** for cosine and sine functions. Write down the three we have derived and derive the fourth.*

Problem 156. *Develop the **Tangent Difference Identity** by simplifying the quotient of the sine difference identity and the cosine difference identity.*

Problem 157. *Let f be any function with domain all real numbers. Show that f can be written as the sum of an even function and an odd function.*

Problem 158. *Develop the tangent sum identity.*

Problem 159. *Prove or disprove:* $\cot(x)+\cot(y)=\dfrac{\cos(x-y)}{\cos(x)\sin(y)}$.

Problem 160. *Use identities to prove* $\cot(\frac{\pi}{2} - x) = \tan(x)$.

Problem 161. *Compute an exact value for* $\cos(\frac{\pi}{12})$ *and use this to construct a right triangle to compute all six trigonometric values for both* $\frac{\pi}{12}$ *and* $\frac{5\pi}{12}$.

Problem 162. *Prove* $\sin(2x) = 2\sin(x)\cos(x)$ *is valid for all values of x.*

Problem 163. *Prove* $\cos(2x) = \cos^2(x) - \sin^2(x)$ *is valid for all values of x. Use a Pythagorean Identity to develop two other identities for* $\cos(2x)$ *that are valid for all values of x.*

Problem 164. *Prove* $\tan(2x) = \dfrac{2\tan(x)}{1-\tan^2(x)}$ *is valid for all values of x for which both sides of the identity are defined.*

The last few problems have developed identities that are referred to as the **Double Angle Identities**. They are quite useful in calculus. The next two are referred to as the **Half Angle Identities** and can be developed quite easily from the double angle identities once you see what substitutions to make.

Problem 165. *Prove* $\sin^2(\frac{a}{2}) = \frac{1-\cos(a)}{2}$ *is valid for all values of a.*

Problem 166. *Prove* $\cos^2(\frac{a}{2}) = \frac{1+\cos(a)}{2}$ *is valid for all values of a.*

Problem 167. *Solve* $\sin(2x) = 2\sin(x)\cos(x)$ *for all possible values of x.*

Chapter 9

Inverses

Definition 44. *Given two sets, X and Y, the* ***Cartesian product*** *of* $\mathbf{X}$ *and* $\mathbf{Y}$ *is the set* $\mathbf{X} \times \mathbf{Y} = \{(x,y) \mid x \in X \text{ and } y \in Y\}$. *A* ***relation*** *on* $\mathbf{X} \times \mathbf{Y}$ *is any subset of* $\mathbf{X} \times \mathbf{Y}$. *A* ***function*** *on* $\mathbf{X} \times \mathbf{Y}$ *is a relation on* $\mathbf{X} \times \mathbf{Y}$ *where no two elements have the same first coordinates. The set of all first coordinates of a relation is called the* ***domain*** *and the set of all second coordinates of a relation is called the* ***range***.

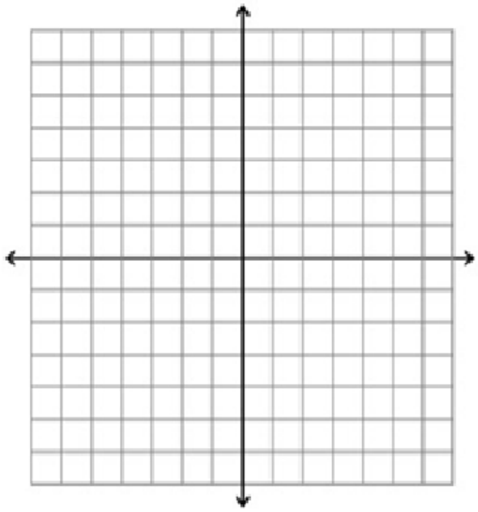
Problem 168. *Let* $\mathbf{A} = \{1,2,3\}$ *and* $\mathbf{B} = \{\square, \diamond, \triangle\}$. *Which of the following are relations on* $\mathbf{A} \times \mathbf{B}$*? Which are functions?*

1. $\big\{(1,\square),(1,\triangle),(2,\diamond)\big\}$

2. $\big\{(3,\square),(1,\triangle),(2,\triangle)\big\}$

3. $\big\{\big((1,1),\square\big),\big((1,2),\triangle\big),\big((2,1),\diamond\big),\big((2,2),\diamond\big)\big\}$

Problem 169. *Let* $f = \left\{(x,y) \mid x \in \mathbb{R} \sim \{1\} \text{ and } y = \dfrac{2x}{x-1}\right\}$ *where* $\mathbb{R} \sim \{1\}$ *means all real numbers except the number one. List five elements of the set (function)* f*. Sketch and state the domain and range of the function.*



Problem 170. *Are* f *and* g *below the same function?*
$f = \left\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x \in \mathbb{R} \text{ and } y = \sin(2x) + \sin^2(x) + \cos^2(x)\right\}$
$g = \left\{(x,y) \mid x \in \mathbb{R} \text{ and } y = 2\sin(x)\cos(x) + 1\right\}$

Because we tend to think of a function f on $\mathbf{X}\times\mathbf{Y}$ as a rule assigning elements of $\mathbf{X}$ to elements of $\mathbf{Y}$, we often write $f:\mathbf{X}\to\mathbf{Y}$. When (x,y) is an element of f we write, $f(x)=y$ and say that f **maps** x to y. When we use this notation, it means that $\mathbf{X}$ is the domain of f, while $\mathbf{Y}$ merely *contains* the range of f. That is, there might be elements of $\mathbf{Y}$ so that no $x\in\mathbf{X}$ satisfies $f(x)=y$. Restated, no element of $\mathbf{X}$ maps to y. This concept motivates the next definition.

Definition 45. *If $f:\mathbf{X}\to\mathbf{Y}$ is a function, then f is **onto the set** Y if for each element $y\in\mathbf{Y}$ there is some element $x\in\mathbf{X}$ such that $f(x)=y$. We say that f is **one-to-one** if no two elements of f have different first coordinates and the same second coordinate. Restated, no two elements in $a,b\in\mathbf{X}$ satisfy $f(a)=f(b)$. A function $f:X\to Y$ that is both one-to-one and onto is called a **bijection**.*

Problem 171. *Suppose $f(x)=\dfrac{x^2-5x+6}{x-3}$ and $g(x)=x-2$. What is the largest subset of $\mathbb{R}$ that is an allowable domain for f? For g? Does $f=g$? Does graphing with technology help?* [14]

Problem 172. *Let $\mathbf{A}=\{1,2,3,4,5,6,7,8,9\}$. Table 9.1 below defines a function $f:\mathbf{A}\to\mathbf{A}$. For each $x\in\mathbf{A}$, the value of $f(x)$ is written below x. Is f one-to-one? Is f onto $\mathbf{A}$? Is f a bijection?*

x	1	2	3	4	5	6	7	8	9
f(x)	5	7	9	3	1	2	6	4	8

Table 9.1: A function, f

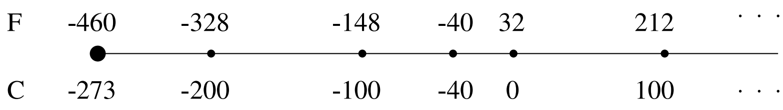


Figure 9.1: Thermometer showing both Fahrenheit and Celsius temperature scales

By Definition 44, every function $f : \mathbf{X} \to \mathbf{Y}$ is onto its range since the range of f is the set of all y such that $(x,y) \in f$ for some $x \in \mathbf{X}$. Restated, the range of f is $\{y \mid (x,y) \in f\} = \{f(x) \mid x \in \mathbf{X}\}$

Should you ever board an airplane (a function) that "maps" you from Houston to Chicago, at some point in the future you will definitely want to board another airplane (the inverse function) that "maps" you back home! Countless people have been lost because they built a time travel machine but forgot to build the inverse time machine!

Problem 173. *Figure 9.1 above shows the relationship between the Fahrenheit and Celsius temperature scales. Write a formula (equation) for a function f that converts Celsius to Fahrenheit, and a function c that converts Fahrenheit back to Celsius. Verify that $f(c(F)) = F$ for every $F \in \mathbb{R}$ and that $c(f(C)) = C$ for every $C \in \mathbb{R}$.*

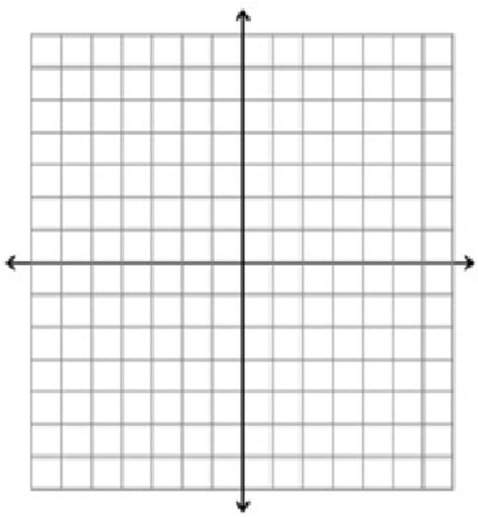
Problem 174. *For each function, determine if it is one-to-one. Determine if it is onto the set $\mathbb{R}$.*

1. $f = \left\{(x,y) \mid x \in \mathbb{R} \sim \{2\} \text{ and } y = \dfrac{x}{x-2}\right\}$.
2. $f : \mathbb{R} \to \mathbb{R}$ *where* $f(x) = \sqrt[3]{x-1}$.

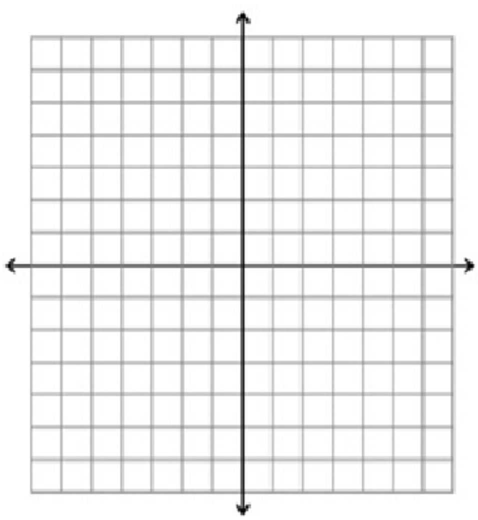
Definition 46. *Given a function $f : X \to Y$, the relation f^{-1} is defined by $f^{-1} = \{(y,x) \mid (x,y) \in f\}$.*

Problem 175. *Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 3x - 7$. Find a function $g : \mathbb{R} \to \mathbb{R}$ so that if $f(x) = y$, then $g(y) = x$.*

Problem 176. *Graph $f = \{(x,y) \mid x \in \mathbb{R} \text{ and } y = x(x-1)\}$. Swap all the coordinates to graph f^{-1}. This is the same process as flipping f across the line $y = x$. Is f^{-1} a function?*



Problem 177. *Graph $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt[3]{x-1}$. Swap all the coordinates to graph f^{-1}. This is the same process as flipping f across the line $y = x$. Is f^{-1} a function?*



Problem 178. *Let f be a function from X onto Y. Show that if f is one-to-one, then f^{-1} is a function.* [15]

Problem 179. *Let f be a function from X onto Y. Show that if f^{-1} is a function, then f is one-to-one.*

Together Problems 178 and 179 show that a function f is one-to-one if and only if f^{-1} is a function.

Chapter 10

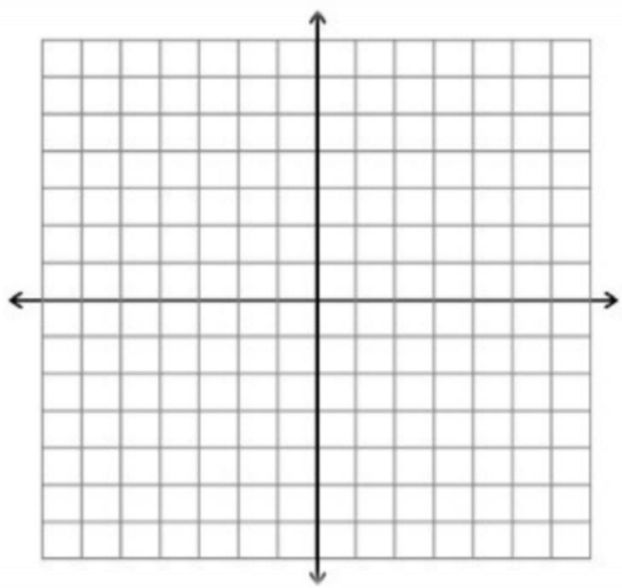
Inverse Trigonometrics

By reversing the coordinates of each of our six trigonometric functions, we can find the inverse relations for each. Some of these relations are not functions, as they fail the vertical line test. This is why some definitions state the domain, which must be memorized.

Definition 47. *We define the* ***inverse sine function****, denoted by* $\sin^{-1}(x)$ *or* $arc\sin(x)$*, to be the function satisfying* $\sin^{-1}(x) = y$ *whenever* $\sin(y) = x$ *and having domain* $-1 \leq x \leq 1$ *and range* $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

Problem 180. *Create a table and graph for the inverse sine function.*

x	y

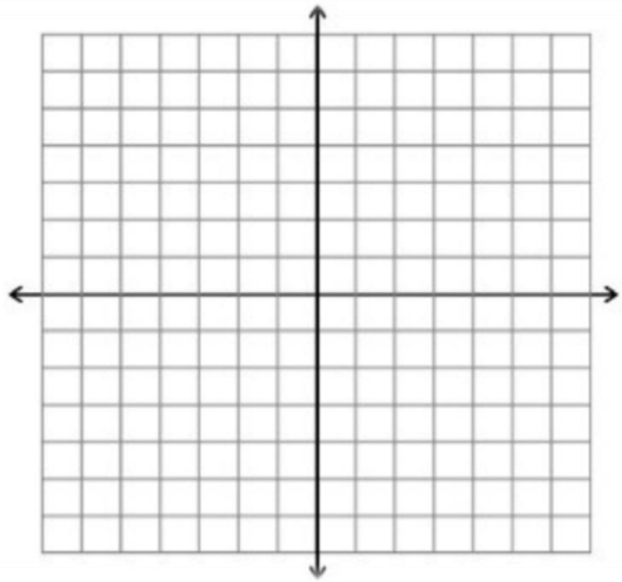


Essay 3. Write a two or more page essay describing how you expect to use mathematics after you graduate from college. Cite any sources.

Definition 48. *We define the* ***inverse cosine function****, denoted by* $\cos^{-1}(x)$ *or* $\arccos(x)$*, to be the function satisfying* $\cos^{-1}(x) = y$ *whenever* $\cos(y) = x$ *and having domain* $-1 \leq x \leq 1$ *and range* $0 \leq y \leq \pi$*.*

Problem 181. *Create a table and graph for the inverse cosine function.*

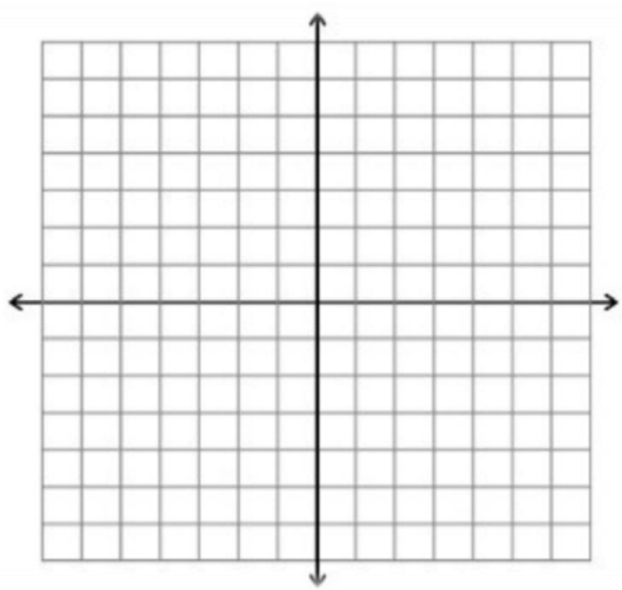
x	y



Definition 49. *We define the* ***inverse tangent function****, denoted by* $\tan^{-1}(x)$ *or* $\arctan(x)$*, to be the function satisfying* $\tan^{-1}(x) = y$ *whenever* $\tan(y) = x$ *and having domain* $-\infty < x < \infty$ *and range* $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$*.*

Problem 182. *Create a table and graph for the inverse tangent function.*

x	y



Warning. The $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$ functions are often written respectively as $\sin^{-1}(x)$, $\cos^{-1}(x)$, and $\tan^{-1}(x)$. This is an abuse of notation since $\sin^{-1}(x) \neq \frac{1}{sin(x)}$. Just remember that $\sin^{-1}$ means *invsin* and not the reciprocal of *sin*. To add to the confusion, while $\sin^{-1}(x) \neq \big(\sin(x)\big)^{-1}$, $\sin^{n}(x)$ does mean $\big(\sin(x)\big)^{n}$ for every number n except -1.

Problem 183. *What is the value for* $arcsin(\frac{-\sqrt{3}}{2})$*?*

Problem 184. *Compute* $arctan(1)$.

Problem 185. *Solve and state the difference between the solutions to the next two problems.*

1. Solve $cos(\theta) = \frac{1}{2}$.

2. Compute $\arccos(\frac{1}{2})$.

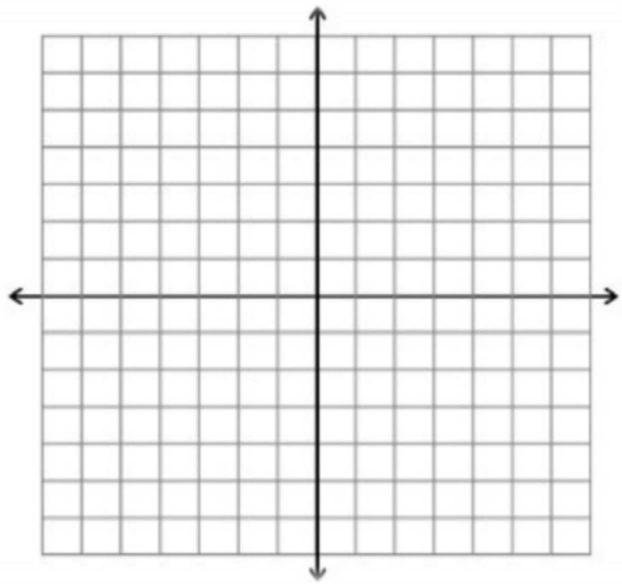
Problem 186. *Compute* $\arccos(-\frac{\sqrt{2}}{2})$.

Problem 187. *Compute* $\sin(\arctan(\frac{3}{4}))$.

Problem 188. *Compute* $\cot(\cos^{-1}(-\frac{2}{3}))$.

Problem 189. *Create a table and graph for* $y = 3 - 2\cos^{-1}(\frac{x}{3})$ *for* $-3 \leq x \leq 3$.

x	y



Problem 190. *Solve* $\tan^2(x) = 2\tan(x) + 1$ *for all values of x. Use your calculator and approximate your answers to four digits.*

Problem 191. *Solve* $\sin^2(x) + 2 = 4\sin(x)$ *for all values of x. Use your calculator and approximate your answers to four digits.*

Chapter 11

Exponentials and Logarithmics

We will not develop the theory for exponential functions. While it is easy enough to define b^n to be the product of b with itself n times when b is a real number and n is a positive integer, it is not as obvious what this means when n is a real number but not an integer, for example 4^{π} or $3^{\sqrt{7}}$. Yes, you have been cheated and really, the only solution is to take more math.

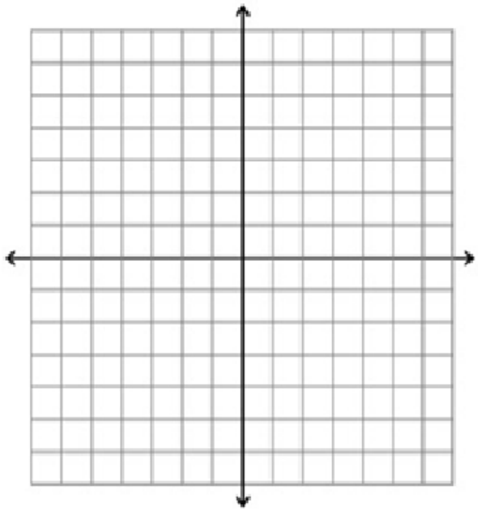
Definition 50. *If $b \in \mathbb{R} \sim \{1\}$, then the function $f(x) = b^x$ is called an **exponential function** and has domain $\mathbb{R}$. The inverses of exponential functions are called the **logarmithmic** functions.*

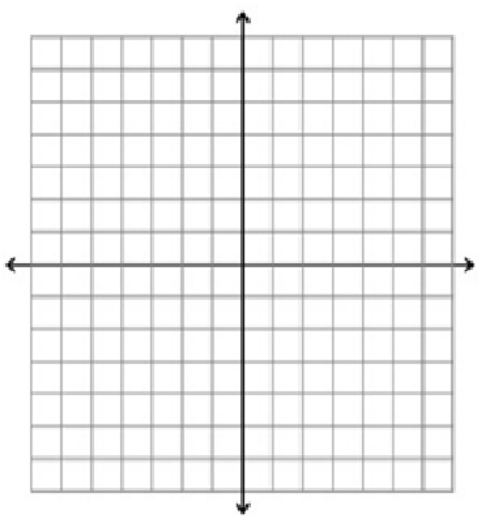
Problem 192. *From scrunchy sales, Merry puts \$300 into her savings account on the first day of each month. She earns 2% annually in her savings account. The interest is paid monthly, so she earns $\frac{2}{12}$% interest each month.*

1. *How much will she have at the end of the first, second, third and sixth month?*

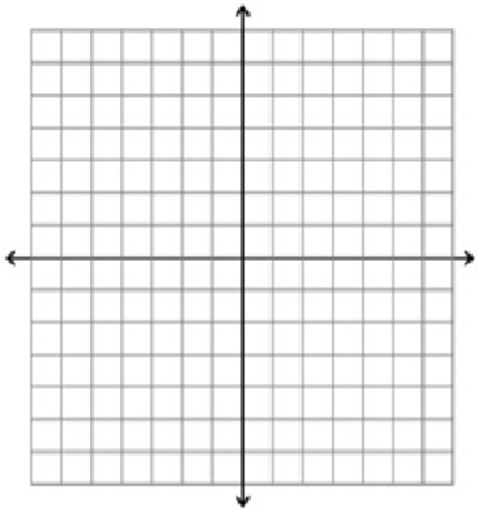
2. *Write down a polynomial that gives the total amount Merry has in the bank at the end of month n.*

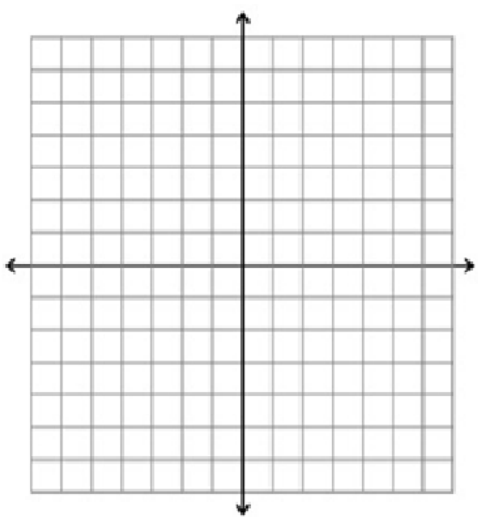
Problem 193. *Graph $f(x) = 2^x$ and $g(x) = (\frac{1}{2})^x$ listing domain range and asymptotes.*





Problem 194. *Graph* $h(x) = 3^x$ *and* $k(x) = (\frac{1}{3})^x$ *listing domain range and asymptotes.*



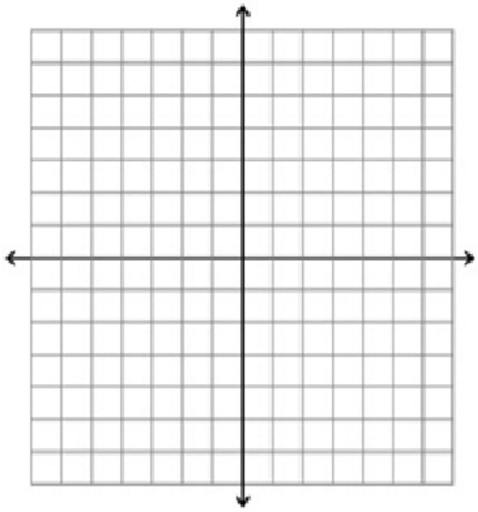


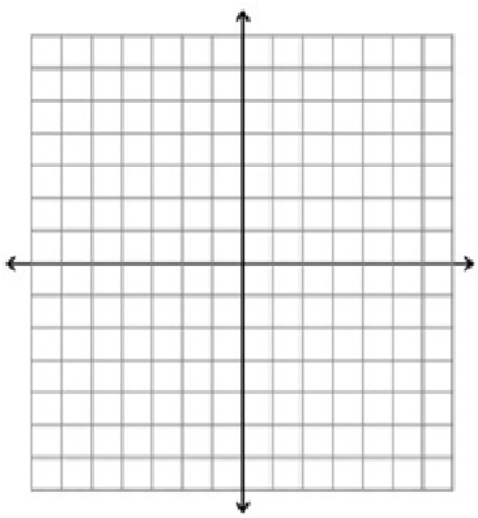
Problem 195. *Merry's business thrives. She now puts \$600 in her account each month. By savvy investing, she averages* 8% *annually with the interest paid monthly. She would like to retire in 30 years. How much will she have in the bank?*

Problem 196. *Compute* $f(n) = (1+\frac{1}{n})^n$ *for* $n = 10, n = 100, n = 1000$ *and* $n = 10000$*. Approximate these values to two decimal places.*

Definition 51. *We define the real number* $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$.

Problem 197. *Graph $w(x) = e^x$ and $s(x) = 3 - e^{x-2}$ listing domain range and asymptotes.*





Definition 52. ***P*** *is the* ***principal****, the amount you invest or borrow.* ***S*** *is the* ***future value*** *of your money after* ***P*** *was invested or borrowed over time.* ***r*** *is the* ***interest rate*** *that you earn or pay depending on whether you invest or borrow.* ***k*** *is the* ***compounding periods per year****, the number of times per year you earn interest or pay back a portion of your loan.* ***t*** *is the number of years you invest or borrow.*

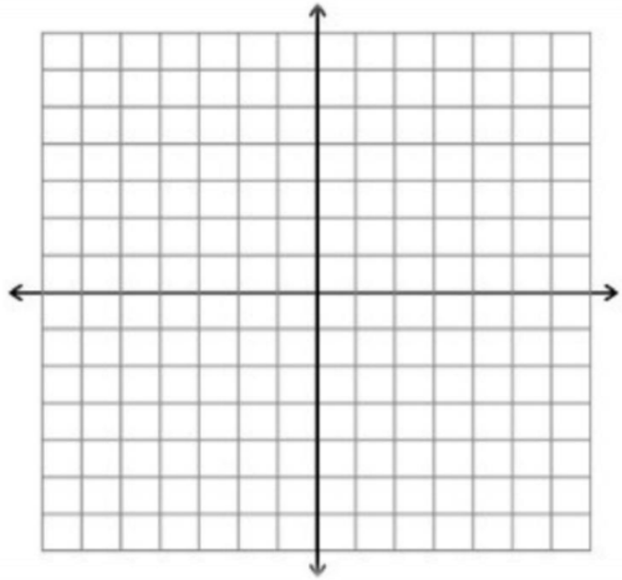
Problem 198. *Suppose you wish to be a millionaire by the time you are 50 years old and you graduate from college and start earning money at 20. Write down how much you might want to invest each month to meet your goal.*

Problem 199. $S = P(1+\frac{r}{k})^{kt}$ *is the amount of money you will have if you invest \$P for t years compounded k times per year. Suppose I invest \$10,000 for 20 years at 3% compounded quarterly, which would be typical of a bond at today's rates. How much will I have? How much should I invest to be worth \$20,000 under the same assumptions?*

Problem 200. $S = Pe^{rt}$ *yields the amount of money you have if you invest \$P for t years compounded continuously. Suppose I invest \$10,000 for 20 years at 3%. How much will I have? How much should I invest to be worth \$20,000 under the same assumptions?*

Problem 201. *Graph the inverse of $f(x) = 2^x$ from Problem 193 by interchanging x and y. List the domain, range and any asymptotes.*

x	y



You just graphed the inverse of $f(x) = 2^x$ which we refer to as $f^{-1}(x) = \log_2(x)$. All the inverses of exponentials are defined in this same way.

Definition 53. *The* ***inverse*** *of $f(x) = b^x$ is denoted by $f^{-1}(x) = log_b(x)$. Since they are inverses:*

1. $\log_b(b^x) = x$ *for any* $x \in \mathbb{R}$
2. $b^{\log_b(x)} = x$ *for any* $x > 0$

Theorem 8. ***Properties of Exponentials*** *(assumed without proof)*

1. $a^0 = 1$ *for any* $a \in \mathbb{R} \sim \{0\}$
2. $b^{-n} = \dfrac{1}{b^n}$ *for any* $b, n \in \mathbb{R}$
3. $b^x b^y = b^{x+y}$ *for any* $b, x, y \in \mathbb{R}$
4. $\dfrac{b^x}{b^y} = b^{x-y}$ *for any* $b, x, y \in \mathbb{R}$
5. $(b^x)^y = b^{xy}$ *for any* $b, x, y \in \mathbb{R}$
6. $b^{\frac{x}{y}} = (\sqrt[y]{b})^x = \sqrt[y]{b^x}$ *for any* $b, x, y \in \mathbb{R}$

Theorem 9. ***Properties of Logarithms*** *(assumed without proof)*

1. $\ln(xy) = \ln(x) + \ln(y)$ *for any* $x, y > 0$
2. $\ln(\dfrac{x}{y}) = \ln(x) - \ln(y)$ *for any* $x, y > 0$
3. $\ln(x^y) = y\ln(x)$ *for any* $x > 0, y \in \mathbb{R}$
4. $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$ *for any* $b, x > 0$

The function *log* means $\log_{10}$ and ln means $\log_e$, which is called the natural log. *The* exponential function means $f(x) = e^x$, and *the* logarithmic function means $g(x) = \ln(x)$. Some people use log for ln. They are bad people and should be avoided.

Problem 202. *Solve* $2^x = 100$ *by applying* $\log_2$ *to both sides of the equation and then applying Definition 53 Part 1. Rewrite your answer using either Theorem 9 Part 3 or Part 4.*

Problem 203. *Solve* $\log_3(2x) = 4$ *by applying* 3^x *to both sides and then applying Definition 53 Part 2. Give an exact and approximate solution.*

Problem 204. *Solve* $5^{(x-2)} = 500$ *for x giving exact and approximate solutions.*

Problem 205. *Rewrite* $\ln(x) - \ln(y) + \ln(z)$ *as a single logarithmic function using Theorem 9.*

Problem 206. *Solve* $2^x = 100$ *by applying* ln *to both sides of the equation and applying Theorem 9 Part 3. Is your answer the same as the answer to Problem 202?*

Problem 207. *Solve* $e^{.05t} = 2500$ *for* t *and approximate using a calculator.*

Problem 208. *Use Theorem 9 to solve* $\ln\left(\frac{xy^2}{z^3}\right) = 0$ *for each of* $\ln(x)$, $\ln(y)$ *and* $\ln(z)$.

Problem 209. *Solve* $3\ln(3x-5)-1=0$ *for x giving an exact and approximate solution.*

Problem 210. *Solve* $\ln(x)+\ln(x-1)=0$ *for x giving an exact and approximate solution.*

Problem 211. $S = P(1+\frac{r}{k})^{kt}$ *yields the amount of money you have if you invest \$P for t years compounded k times per year. How long will it take \$5,000 to double at 3% interest compounded quarterly? How long will it take to double at 7%?*

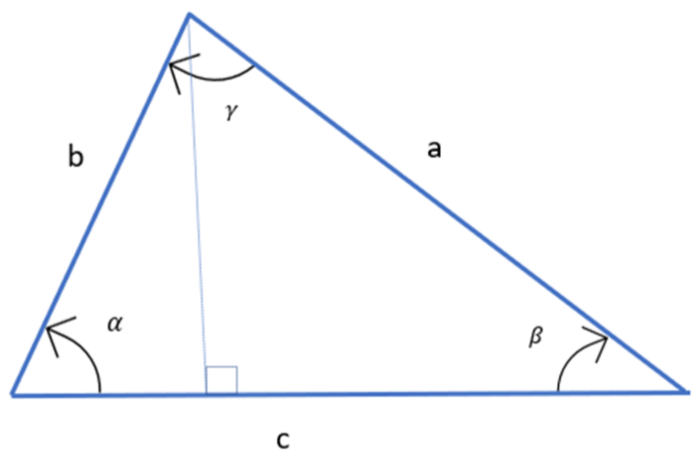
Problem 212. $m = \frac{P(r/k)(1+r/k)^{kt}}{(1+r/k)^{kt}-1}$ *represents your payments at the beginning of each period if you borrow \$P for t years making payments k times per year. You buy a \$35,000 car with a 6 year loan and pay 7.9% interest compounded monthly. What are your payments? What is the total interest you have paid when the car is paid off?*

Problem 213. $S = m(k/r)((1+r/k)^{kt}-1))$ *is the amount of money you will have if you invest \$m for each of k periods per year for t years at interest rate r. How much will you have if you put \$300 per month into an account earning 6% for 30 years? How much did you earn in interest the first year? The last year? In total?*

Problem 214. $S = Pe^{rt}$ *yields the amount of money you have if you invest \$P for t years compounded continuously. How long will it take \$10,000 to double at 3%? At 7%? What rate will guarantee the money doubles in 10 years?*

Chapter 12

Law of Sines and Law of Cosines



Problem 215. *Use the figure above to show that* $\frac{a}{sin(\alpha)} = \frac{b}{sin(\beta)}$.

Problem 216. *Use the figure above to show that* $a^2 + b^2 - 2\,a\,b\,cos(\gamma) = c^2$.

In the next few problems you will note the phrase "Draw and solve **every** triangle . . ." This is because for some of the problems the given data is insufficient to guarantee a unique triangle. There may indeed be more than one triangle satisfying the given data. Be sure to draw a picture to help you identify why some problems may have multiple solutions.

Problem 217. *Draw and solve every triangle satisfying:* $\alpha = 32^\circ$, $\beta = 41^\circ$ *and* $a = 2.5$ *cm.*

Problem 218. *Draw and solve every triangle satisfying:* $\alpha = 29^\circ$, $a = 7$ *cm, and* $c = 14$ *cm.*

Problem 219. *Draw and solve every triangle satisfying:* $\alpha = 43^\circ$, $b = 6$ *cm and* $c = 10$ *cm.*

Problem 220. *Draw and solve every triangle satisfying:* $\beta = 72.2^\circ$,$b = 78.3$ *cm, and* $c = 145$ *cm.*

Problem 221. *Find the length of one side of a pentagon inscribed in a circle of radius* 12.5 *cm.*

Problem 222. *Draw and solve every triangle satisfying:* $a = 6$ *cm,* $b = 8$ *cm and* $c = 9$ *cm.*

Problem 223. *Draw and solve every triangle satisfying:* $\alpha = 26^\circ$, $a = 11$ *cm and* $b = 18$ *cm.*

Problem 224. *Two bird watchers located at points A and B are* 12.5 *miles apart. A Yellow-Bellied Sap Sucker is seen at point C by both bird watchers. Careful measurements indicate that* $m\angle BAC = 14^\circ$ *while* $m\angle ABC = 82^\circ$. *Which bird watcher is closest to the Sap Sucker? How far away is the bird from this bird watcher?*

Problem 225. *Draw and solve every triangle satisfying:* $a = 8$ *cm,* $\beta = 60^\circ$ *and* $c = 11$ *cm.*

Problem 226. *Draw and solve every triangle satisfying:* $\alpha = 20^\circ$, $a = 10$ *cm and* $b = 16$ *cm.*

Problem 227. *Find the length of one side of a nine-sided regular polygon inscribed in a circle with a radius of* 8.32 *cm.*

Problem 228. *An observer stands at a point A at ground level that is some unknown distance from the base of a building. The angle of elevation from point A to the top of the building is* $63°$*. After taking this measurement, the observer walks directly away from the building for* 140 $feet$ *to a point B also at ground level. The observer now measures the angle of elevation from the point B to the top of the building as* $55°$*. What is the height of the building?*

To solve the following problems requires a bit of an introduction to navigation. A direction such as $N30°E$ is read as the direction $30°$ East of true North. In other words, from true North, rotate clockwise $30°$ to get the direction of travel, also called a heading. As another example, the heading $S40°E$ means from due South, rotate counterclockwise $40°$ to get direction of travel.

Problem 229. *A sailor spots a lighthouse at a heading of* $N28°E$*. The sailor then sails due East* 7.5 *miles where he then spots the lighthouse at a heading of* $N16°E$*. At this point, how far is the sailor from the lighthouse? Continuing to sail due East, how far must he sail to reach a point closest to the lighthouse?*

Problem 230. *In order to seal an oil pipeline we must make a plate of* $\frac{1}{4}$ *inch steel to place on a flange at the open end of an open pipe. Our plate must be circular with a radius of* 6 *inches and must have* 7 *holes drilled into it spaced equally apart. The holes must have diameter* $\frac{3}{16}$ *inch and their centers must be* 1 *inch from the perimeter of the plate. What will be the distance between the centers of two adjacent holes? Industry standards require an answer accurate to* 6 *decimal places.*

Problem 231. *Jack and Jill take off from the same airport at the same time in their Cessna airplanes. Jack travels* $N35°W$ *at* 160 *mph while Jill travels* $S70°W$ *at* 170 *mph. How far apart are their planes after* 2 *hours? Determine a function that will give the distance between the planes after* t *hours. Use the function to determine the distance after* 5 *hours.*

Problem 232. *Two joggers in Central Park are resting on park benches at points A and B. Point A is* 1.2 *miles directly south of point B. At midnight both joggers spot a walker traveling in a straight line path. The jogger resting at point A observes the walker at a heading of* $N20°E$*, while the jogger resting at point B observes the same walker at a heading of* $S70°E$*. Ten minutes later the jogger resting at point A observes the walker at a heading of* $N34°E$*, while the jogger resting at point B observes the walker at a heading of* $S55°E$*. Find the walker's average walking speed.*

Chapter 13

Polar Coordinates

When you plotted points to graph an equation like $y = x^2$, you used the **Cartesian coordinate system**. The **polar coordinate system** is another way to specify the location of points in the plane. Some curves have a simpler equation in polar coordinates while other curves have a simpler equation in Cartesian coordinates. Depending on the application, one coordinate system may be preferable to the other.

Definition 54. *Given a point (x,y) in the plane, we may associate (x,y) with another ordered pair (r,θ) where r is the distance from $(0,0)$ to (x,y) and θ is the angle, measured in the counter-clockwise direction, between the positive $x-$axis and the line segment from $(0,0)$ to (x,y). The pair (r,θ) is called a **polar coordinate representation** for the point (x,y).*

Problem 233. *Convert rectangular coordinates point $(x,y) = (1,1)$ into polar coordinates (r,θ).*

Problem 234. *Convert the point with rectangular coordinates $(x,y) = (-2,-5)$ into polar coordinates (r,θ). Give a second, different polar representation for the same point.*

Problem 235. *Establish formulas for r and θ in terms of x and y that will convert any point from rectangular coordinates (x, y) to polar coordinates (r, θ).*

Problem 236. *Convert the point with polar coordinates $(5, 30^\circ)$ to rectangular coordinates (x, y).*

Problem 237. *Convert the point with polar coordinates $(\frac{8}{3}, \frac{3\pi}{5})$ to rectangular coordinates (x, y).*

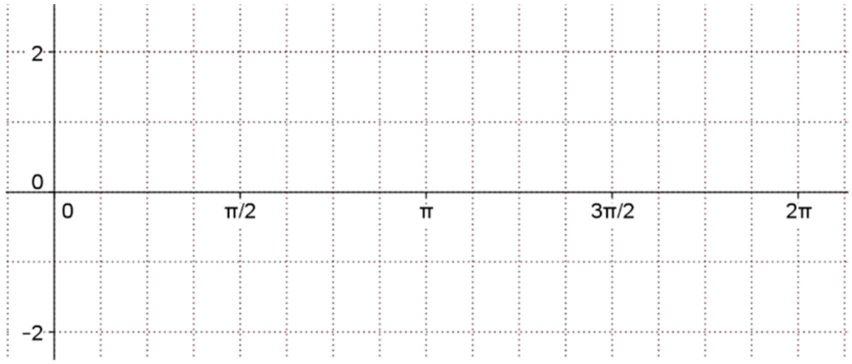
Essay 4. Write a two page essay on trigonometry. What is it? Cite your sources.

Problem 238. *Establish a formula for x and y to convert any point from polar coordinates* (r,θ) *to rectangular coordinates* (x,y).

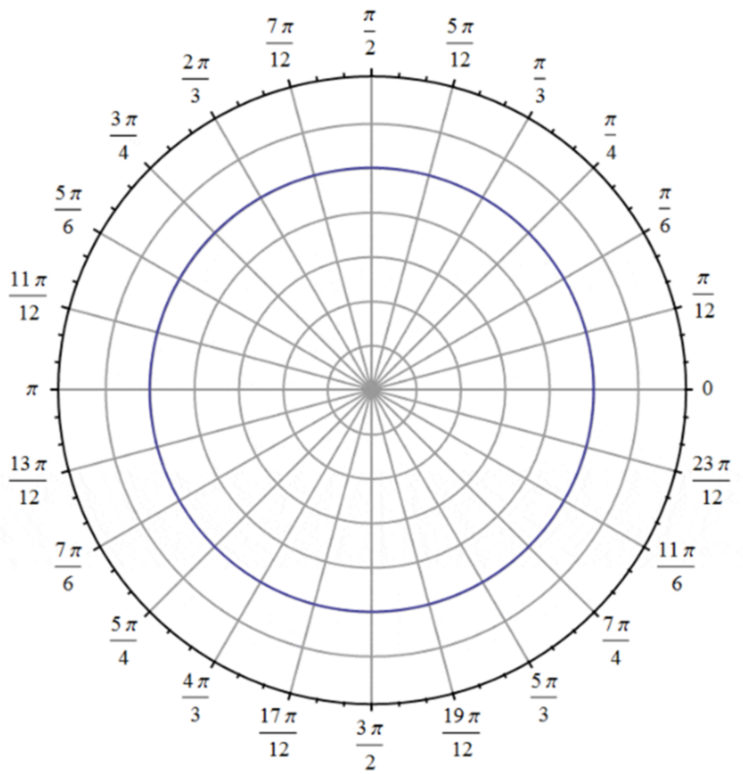
Problem 239. *A boat is launched from a point on land that is 300 yards along a straight coastline from a lighthouse. The boat sails perpendicular to the coastline for 1500 yards. From the lighthouse, what are the polar coordinates for the position of the boat?*

Problem 240. *Graphing.*

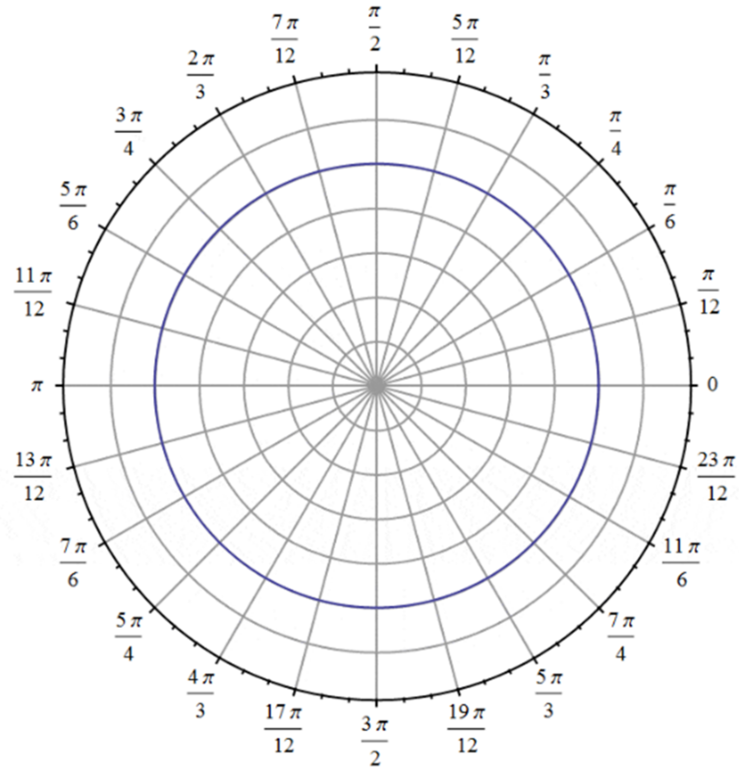
1. *Graph the curve* $y = \sin(x)$ *for* x *in* $[0, 2\pi]$ *in the Cartesian coordinate system.*



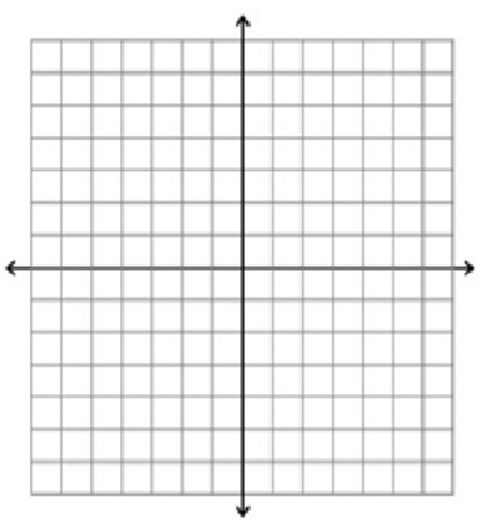
2. *Graph the curve* $r = \sin(\theta)$ *for* θ *in* $[0, 2\pi]$ *in the polar coordinate system.*



Problem 241. *Substitute multiple values of θ between 0 and 2π into $r = 9\cos(\theta)$ and plot the resulting polar coordinates to determine the graph of the equation.*

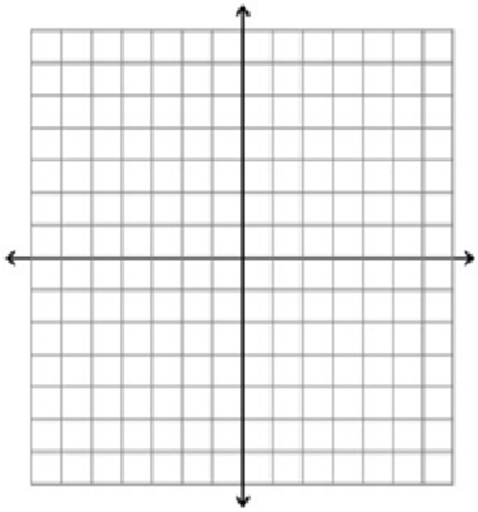


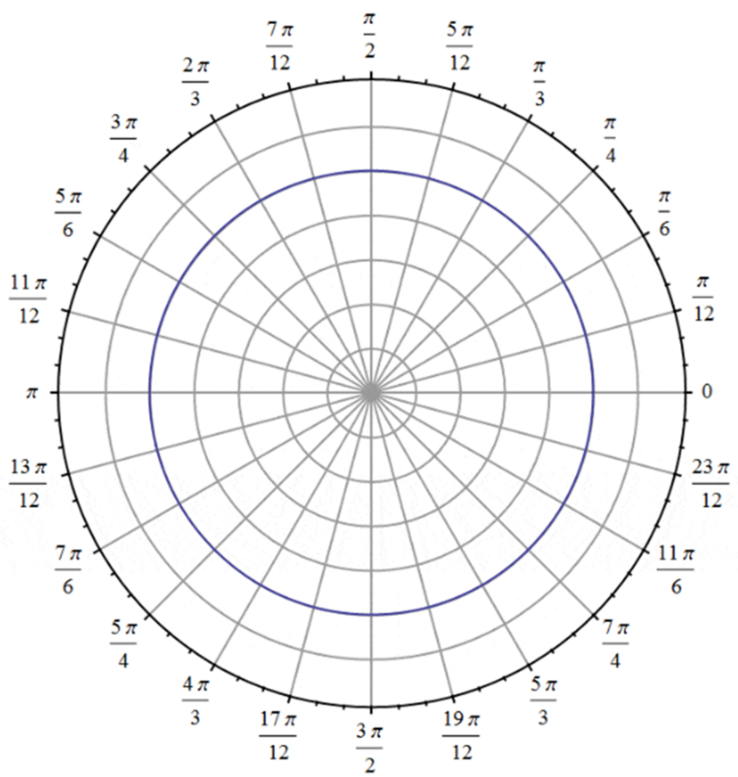
Problem 242. *Find enough pairs (x, y) that satisfy the equation $(x+2)^2 + y^2 = 16$ to sketch this equation in the Cartesian plane.*



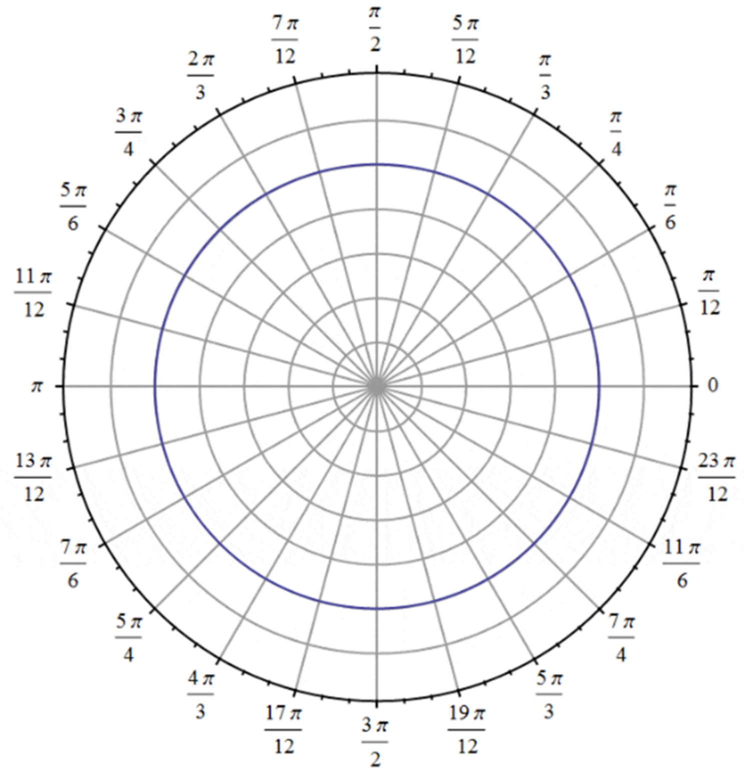
Problem 243. *Use the relationships you found in Problem 235 to write* $r = \dfrac{1}{3\cos(\theta) - 2\sin(\theta)}$ *in terms of x and y (i.e. in Cartesian coordinates).*

Problem 244. *Sketch $x = 4$ in rectangular coordinates and sketch $\theta = \pi/4$ in polar coordinates.*



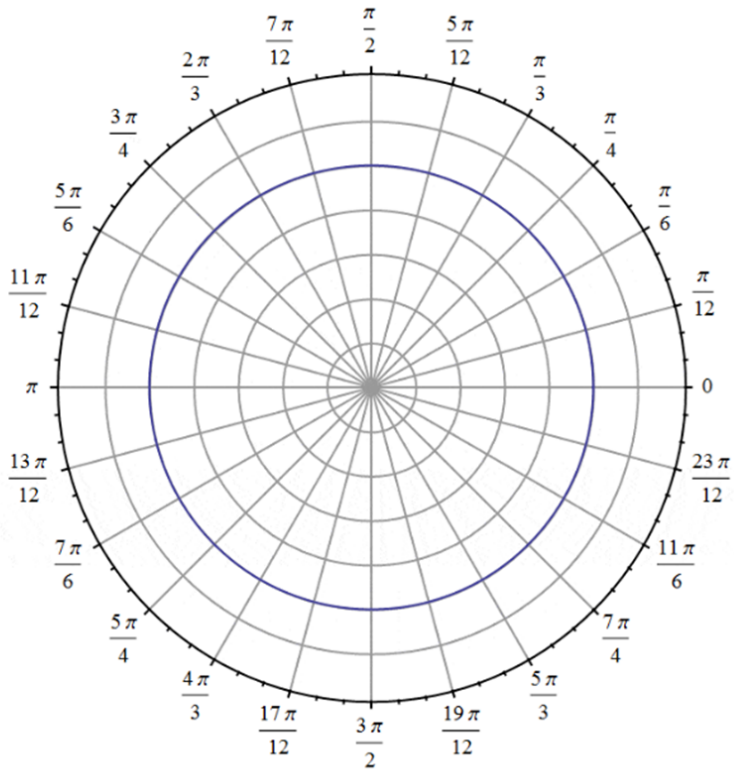


Problem 245. *Sketch $r = 5$ in polar coordinates. Convert this equation to Cartesian coordinates.*

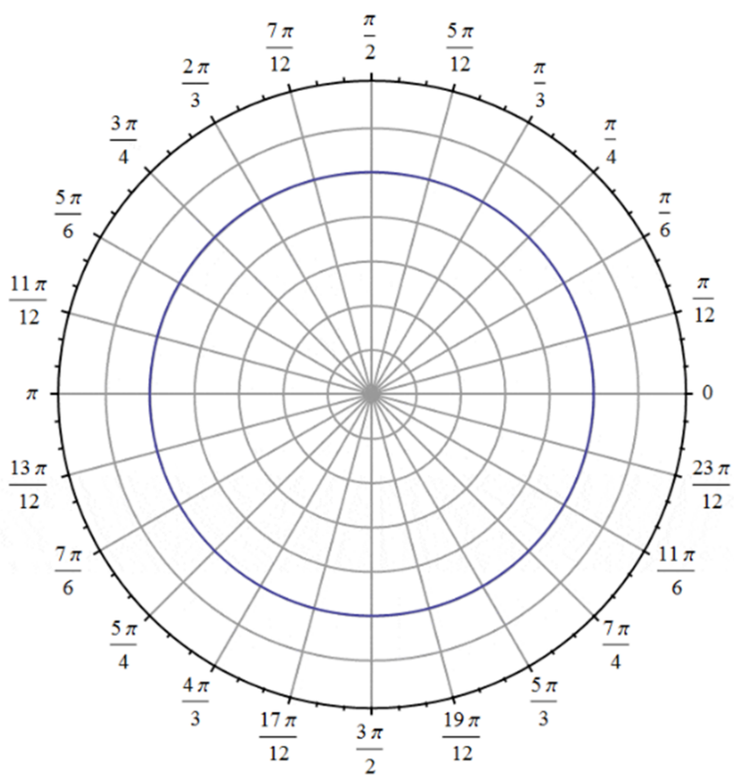


Problem 246. *In Problem 221 you determined the length of the side of a pentagon using the law of cosines. Confirm you answer with another method.*

Problem 247. *Graph $r = 1 + 2\cos(\theta)$ in polar coordinates.*



Problem 248. *Find the points where the graph of $r = 2\cos(2\theta)$ intersects the graph of $r = \sqrt{3}$.*



Chapter 14

Rate of Change

The central theme of this course has been an understanding of the properties of several classes of functions: polynomials, rationals, etc. One central theme in Calculus is the notion of rate of change. Rate of change answers the question, "how fast is some quantity, such as distance or temperature, changing at a given instant in time?" This is not terribly intuitive – if you are driving and you freeze time at 3:43 p.m., then you stop and your speed is zero. And yet, certainly if you were driving at 3:43 p.m., then you were traveling a certain speed at that instant in time and it is not the same as your average speed. This chapter gives you an introduction to this concept by solidifying the difference between average velocity and instantaneous velocity, the rate of change of distance.

Problem 249. *A particle moves away from point A in a straight line. The distance of the particle from point A after t seconds is given by $d(t) = 2t^2 + t + 5$.*

1. *How far is the particle from the point A after* 1 *second? After* 3 *seconds?*

2. *What is the average velocity of the particle over the time interval from* 1 *second to* 3 *seconds?*

3. *What is the slope of the line through the points $(1, d(1))$ and $(3, d(3))$?*

Problem 250. *A particle moves from point A. The distance of the particle from point A after t seconds is given by $d(t)$.*

1. *How far is the particle from point A after a seconds? After b seconds?*

2. *What is the average velocity of the particle?*

Problem 251. *A car goes from San Leon, Texas to Edmond, Oklahoma (a distance of 500 miles). The trip takes 7 hours and 15 minutes. What was the average speed of the car during this trip? Discuss at what time of the trip the car moved at this speed.*

Problem 252. *An android Betty's* ***position function****,* $p(t) = t^3 + 2$*, gives her position on the x-axis at time* t*, where the time is in minutes and the units on the x-axis are in feet.*

1. *Assume c is a positive number and compute* $p(c)$ *and* $p(3)$*. What do these numbers represent about Betty?*

2. *Compute* $p(5) - p(3)$*. What is the physical meaning of this number? Assume u and v are positive numbers and compute* $p(u) - p(v)$*. What is the physical meaning of* $p(u) - p(v)$*?*

Problem 253. *Compute each of the following ratios and explain in detail their physical meaning. Assume that u and v are positive numbers.*

1. $\dfrac{p(5) - p(3)}{5 - 3}$

2. $\dfrac{p(u) - p(v)}{u - v}$

Problem 254. *Where is Betty when the time is* $t = 5$ *minutes?* $t = 6$ *minutes?* $t = 5.5$ *minutes? When does Betty arrive at the point* $x = 106$*?*

Problem 255. *The distance from San Leon, Texas to Edmond, Oklahoma is 500 miles. A car goes from San Leon to Edmond travelling at 70 mph for the first hour, accelerates to 90 mph, and remains at that speed for the rest of the trip. Estimate the average speed of the car for this trip. Is you answer exact or is it an approximation? Explain.*

Problem 256. *We suspect that Betty has violated the inter-galactic speed limit for androids at time $t = 3$. We need to compute Betty's instantaneous velocity at time exactly $t = 3$.*

1. *What is Betty's average velocity from time $t = 3$ till time $t = 4$?*

2. *What is Betty's average velocity from time $t = 3$ till time $t = 3.5$?*

3. *What is Betty's average velocity from time $t = 3$ till time $t = 3.1$?*

4. *What is Betty's velocity at the exact time, t=3?*

Problem 257. *A population P of androids (clones of Betty) is growing according to the function $P(t) = t^3 + t + 2$, where the time t is measured in days and the population P is measured in number of androids.*

1. *Compute $P(5) - P(3)$ and explain what this number says about the android population.*

2. *Compute $\dfrac{P(5) - P(3)}{5 - 3}$ and explain what this number says about the android population.*

3. *Assume that u and v represent times. Simplify and explain the meaning of $\dfrac{P(u) - P(v)}{u - v}$.*

4. *What is the average population growth over the time intervals*
   (a) *$t = 2$ to $t = 3$?*

   (b) *$t = 2$ to $t = 2.5$?*

   (c) *$t = 2$ to $t = 2.01$?*

   (d) *How fast is the android population growing at time $t = 2$?*

Definition 55. *Given any function f, the **average rate of change** of f over the interval $[a,b]$ is defined to be*

$$\text{Average rate of change } = \frac{f(b) - f(a)}{b - a}$$

*and is just the slope of the line through the points $(a, f(a))$ and $(b, f(b))$. The **instantaneous rate of change** of f at the time $t = a$ is how fast the function is changing at that moment in time.*

Chapter 15

Conic Sections

In this course we have seen circles and given a particular circle with radius r units and center (h,k) we saw that all points x,y on the circle satisfy the equation:

$$(x-h)^2+(y-k)^2=r^2$$

We have also seen parabolas and given a particular parabola with a given value for a and vertex (h,k) we saw that the all points x,y on the parabola satisfy the equation $y-k=a(x-k)^2$. These are two examples of what we will now call conic sections.

Definition 56. *Consider two identical, infinitely tall right circular cones placed vertex to vertex so that they share the same axis of symmetry. A* ***conic section*** *is the intersection of this three dimensional surface with any plane that does not pass through the vertex where the two cones intersect.*

These intersections are called circles when the plane is perpendicular to the axis of symmetry, lines when the plane is parallel and tangent to a side of the cone, parabolas when the plane is parallel to a side of the cone, but not tangent, hyperbolas when the plane is parallel to the axis of symmetry and ellipses when the plane does not meet any of the previous criteria.

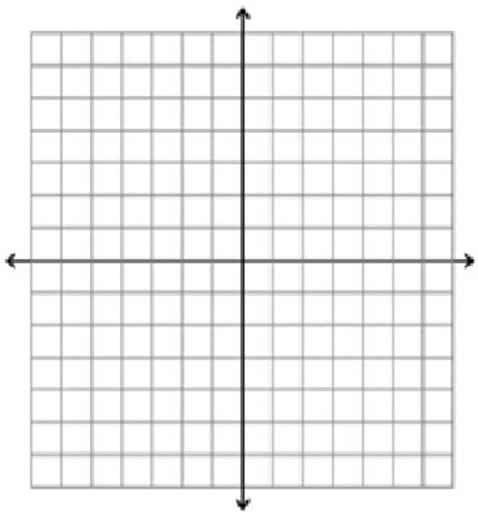
This gives a geometric description of conic sections. Any conic section may also be represented by a quadratic equation in two variables, x and y.

Every conic section may be obtained by making appropriate real value choices for A,B,C,D,E and F in the following equation:

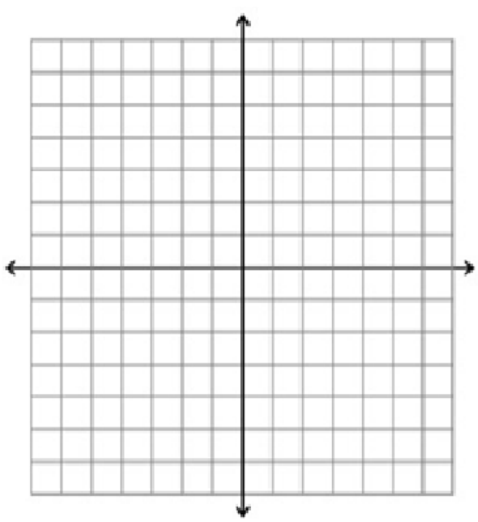
$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

Problem 258. *Graph the set of all points (x,y) in the plane satisfying $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ where:*

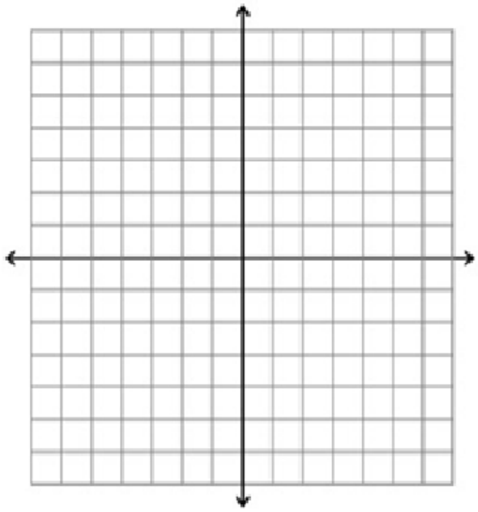
1. $A = B = C = 0, D = 1, E = 2$ *and* $F = 3$



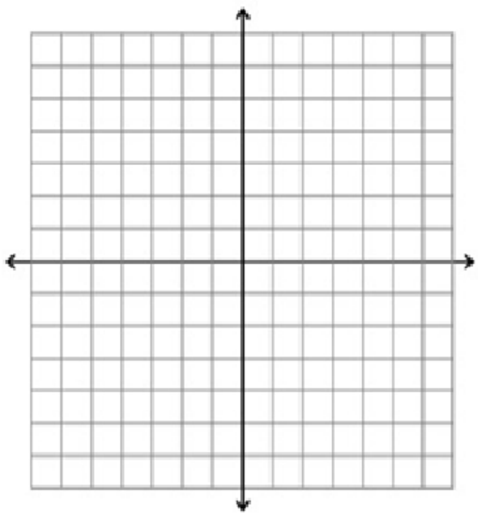
2. $A = \frac{1}{16}, C = \frac{1}{16}, B = D = E = 0$ *and* $F = -1$



3. $A = B = C = D = E = 0$ *and* $F = 1$



4. $A = B = C = D = E = F = 0$



Definition 57. *Given a plane, a point* h, k *in the plane and a positive number* r*, the circle with* ***center*** (h, k) *and* ***radius*** r *is the set of all points in the plane that are a distance* r *units from the point* (h, k).

Problem 259. *Determine values for* A, B, C, D, E *and* F *so that* $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ *represents a circle. A single point. A parabola.*

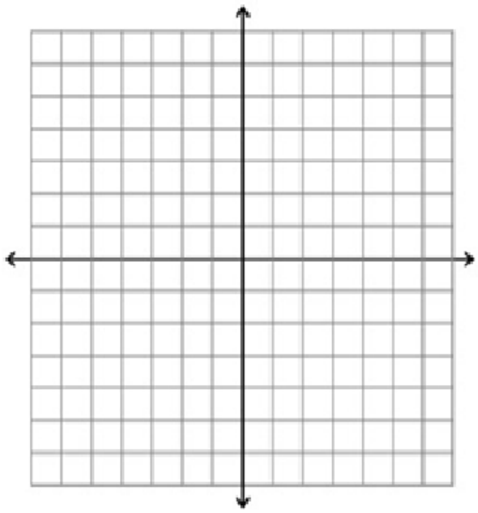
Problem 260. *Use the method of completing the square as discussed in class to rewrite the equation* $4x^2+4y^2+6x-8y-1=0$ *in the form of* $(x-h)^2+(y-k)^2=r^2$ *for some* h,k *and* r*. Now generalize your work to rewrite the equation* $x^2+y^2+Dx+Ey+F=0$ *in the same form.*

Definition 58. *The **distance between a point** P **and a line** L (denoted by* $d(P,L)$*) is defined to be the length of the line segment perpendicular to L that has one end on L and the other end on P.*

Definition 59. *Given a point P and a line L in the same plane, the **parabola** defined by P and L is the set of all points Q in the plane so that* $d(P,Q)=d(Q,L)$*. The point P is called the **focus** of the parabola. The line L is called the **directrix** of the parabola.*

Problem 261. *Parabolas:*

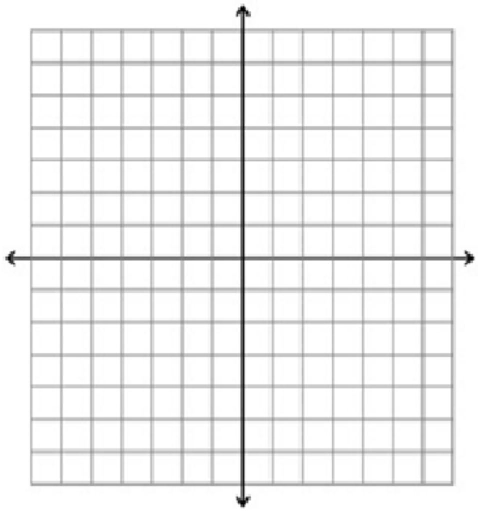
1. *Let p be any positive number. Graph the parabola that has focus $(0, p)$ and directrix $y = -p$.*



2. *Let (x, y) be an arbitrary point on the parabola with focus $(0, p)$ and directrix $y = -p$. Show that (x, y) satisfies the equation $4py = x^2$.*

3. *Let (x, y) be an arbitrary point on the parabola with focus $(p, 0)$ and directrix $x = -p$. Show that (x, y) satisfies the equation $4px = y^2$.*

4. *Graph the parabola, focus and directrix for the parabola $y = -x^2 + 3$.*



Problem 262. *Find the vertex, focus and directrix for each of the parabolas defined by the following equations:*

1. $y = 3x^2 - 12x + 27$

2. $y = x^2 + cx + d$ *where c and d are any real numbers.*

Definition 60. *Given any two points, F_1 and F_2 in a plane and a positive number d, the **ellipse** determined by F_1, F_2 and d is the set of all points Q in the plane so that $d(F_1,Q) + d(F_2,Q) = d$. F_1 and F_2 are called the **foci** of the ellipse. The **major axis** of the ellipse is the line passing through the foci points. The **vertex points** of the ellipse are the points where the ellipse intersects with the major axis. The **minor axis** is the line perpendicular to the major axis and passing through the point that is midpoint of the foci*

Problem 263. *Let $F_1 = (c,0)$, $F_2 = (-c,0)$ and $d = 2a$. Show that if (x,y) is a point of the ellipse determined by F_1, F_2 and d, then (x,y) satisfies the equation:*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ \textit{where} \ a^2 - b^2 = c^2$$

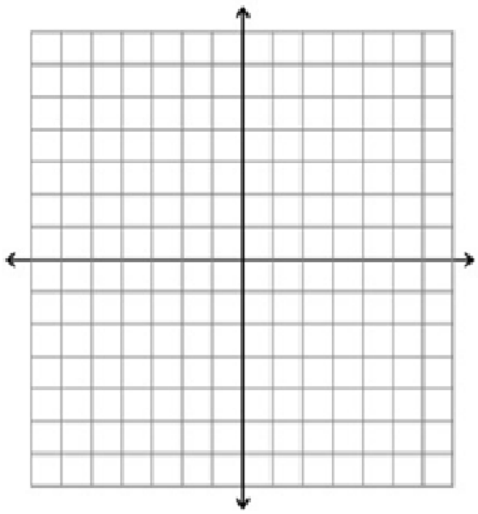
What are the restrictions on a, b and c?

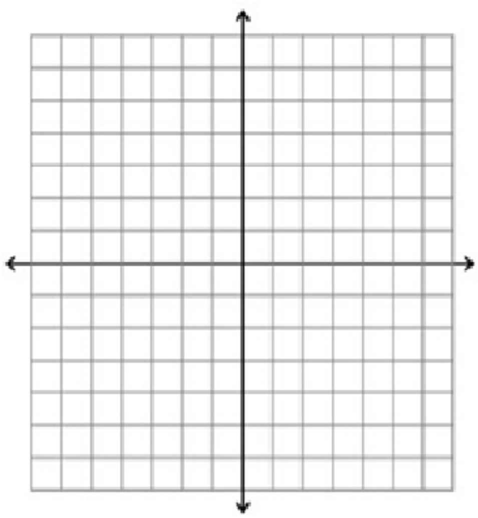
Problem 264. *Ellipses:*

Graph the ellipses defined by:

$$\frac{x^2}{25}+\frac{y^2}{9}=1 \ \ and \ \ \frac{(x+1)^2}{25}+\frac{(y-2)^2}{9}=1$$

List the foci, vertex points and major axis for each ellipse. Describe any similarities between the two graphs.





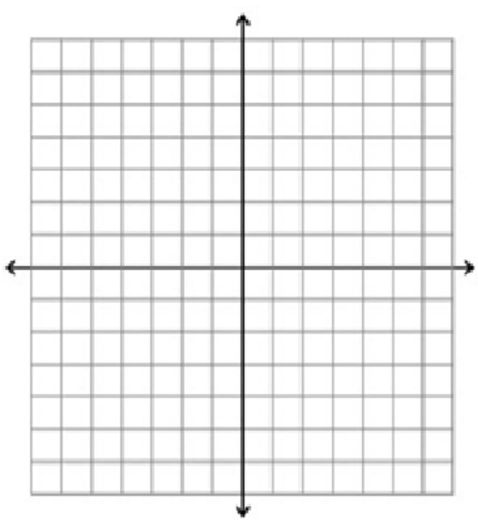
Definition 61. *Given any two points F_1 and F_2 in a plane and a positive number d, the* ***hyperbola*** *determined by F_1, F_2 and d is the set of all points Q in the plane so that $|d(F_1,Q) - d(F_2,Q)| = d$. F_1 and F_2 are called the* ***foci*** *of the hyperbola.*

Problem 265. *Let $F_1 = (c,0)$, $F_2 = (-c,0)$ and $d = 2a$. Show that the equation of the hyperbola determined by F_1, F_2 and d is given by:*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ where } c^2 = a^2 + b^2.$$

What are the restrictions on a, b and c?

Problem 266. *Graph any hyperbola of your choice. Just choose any real numbers a and b.*



Problem 267. *Show that as $x \to \infty$ the hyperbola approaches the lines:*

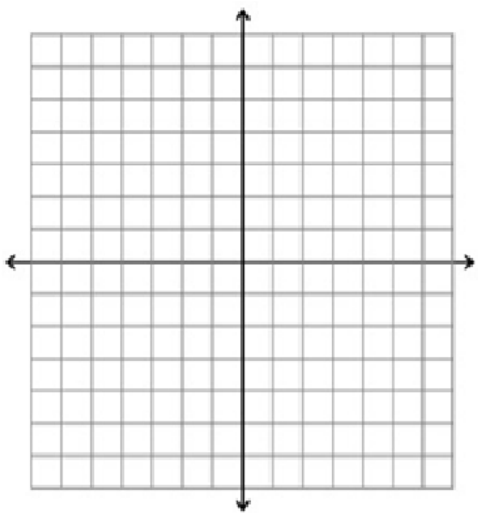
$$L = \pm \frac{b}{a} x$$

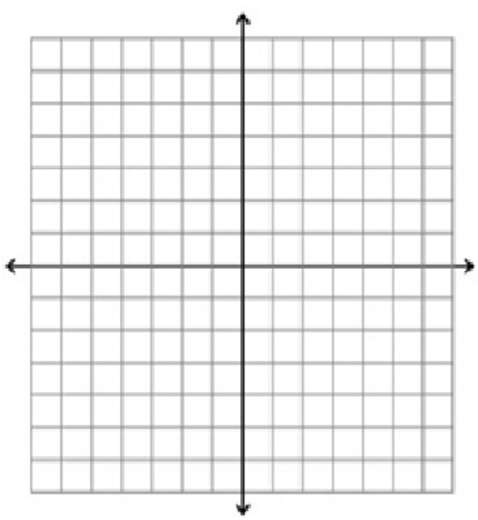
Thus the asymptotic behavior of hyperbolas is linear.

Problem 268. *Hyperbolas:*

Graph $\frac{x^2}{25} - \frac{y^2}{9} = 1$ *and* $\frac{(x+1)^2}{25} - \frac{(y-2)^2}{9} = 1.$

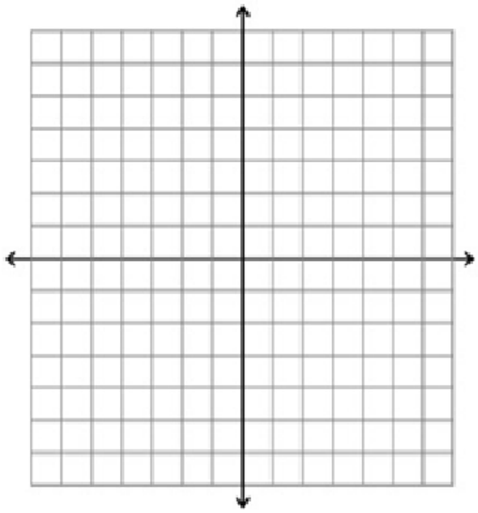
Compare the two graphs and find the similarities and differences between the two hyperbolas. Find the foci and asymptotes for both hyperbolas.



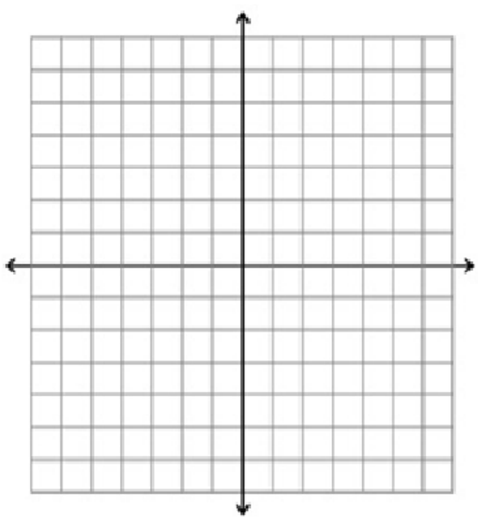


Problem 269. *Use completing the square to identify type of each conic section. Graph each conic section. List the relevant items for each from this list: center, radius, vertex (plural?), focus (plural?), directrix and asymptotes.*

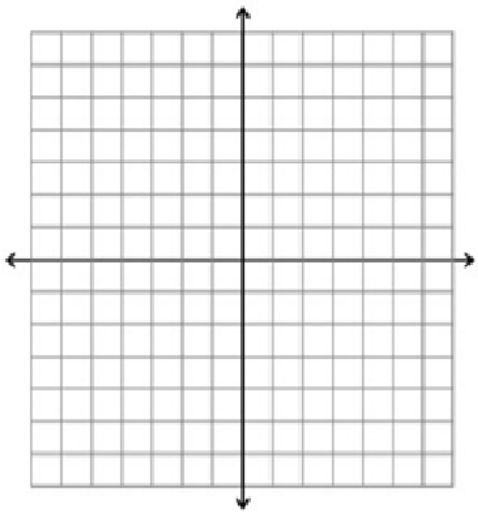
1. $x^2 - 4y^2 - 6x - 32y - 59 = 0$



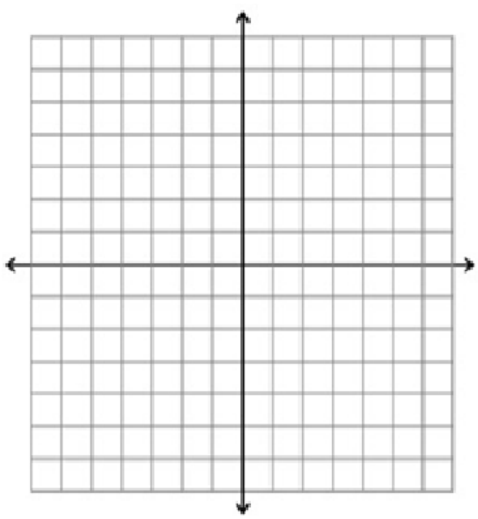
2. $9x^2 + 4y^2 + 18x - 16y - 11 = 0$



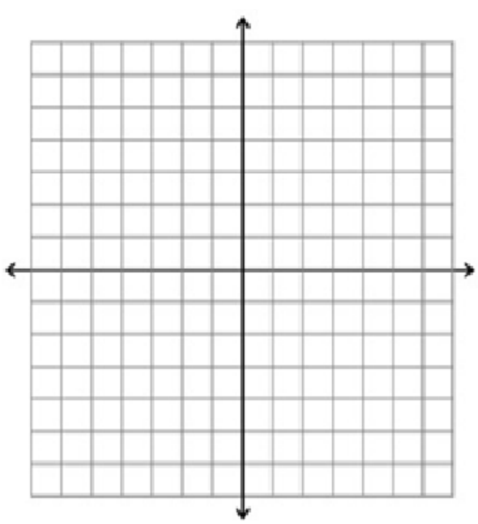
3. $x^2 + y^2 + 10 = 6y - 4 + 4x$



4. $x^2 + y^2 + 6x - 14y = 6$



5. $3x^2 - 2x + y = 5$



Chapter 16

Conclusion

By having worked through much of this book, you are prepared for Calculus at any institution. That's good. By working through the material independently, rather than being shown how to work each problem first, you have done something else. You have proven that you can do mathematics, and likely anything else, by dedicating yourself to the topic. Now you can go learn to play guitar, climb mountains, research geophysics, or build race-car engines on your own. Congratulations, you have accomplished the most important skill in life – the ability to learn on your own and not rely on others to educate you. Albert Einstein said, "Education is what remains after one has forgotten what one has learned in school." Maria Montessori said, "The greatest sign of success for a teacher is to be able to say, 'The children are now working as if I did not exist.' " And poof, we are gone and you are on your own to learn for the rest of your lives. Enjoy and don't stop. Let us know how it goes.

Notes

[1](Definition 1): I personally like to give a very brief historical talk about Rene Descartes's innovative ideas. Linking equations to geometric objects was not an obvious thing to do!

[2](Theorem 1) This theorem is done with an entire class discussion.

[3] (Theorem 3) It is best not to let this theorem sit too long. If students appear to be making no progress, a good hint is to use parallel lines cut by a transversal.

[4]The purpose of Problem 36 is to introduce them to the the point slope form for the equation of a line so after presented, I review various forms of lines: $y = mx+b$, $(y-y_1) = m(x-x_1)$, and $ax+by = c$.

[5](Problem 45) This problem reveals a usual algebraic weakness. Students often have a very difficult time coordinating units. Good idea to have a mini-lecture on proportionality and dimension analysis. Appropriate quizzes may help students gain needed insights.

[6](Note after Problem 48) I have found this to be a critical junction point in the course. This is a good point for a good comprehensive review of all topics covered, specifically mentioning which topics will be covered on exams. Students should know to read the following definitions carefully, as they are a direct link to trigonometric functions.

[7] (Definition 22) This is an appropriate time to emphasize the importance of the concept of function in mathematics. The class should discuss the various ways of defining the function concept.

[8] (Problem 56) This is a good place to have a class discussion regarding the idea of notating an infinite solution set. Two main points need to be understood:

1. The students must be held accountable to the convention they agree upon to notate an infinite solution set.
2. The students need to understand that if they enter another community of people doing mathematics, then they may need to adapt another convention to notate an infinite solution set. There are many ways to communicate mathematical ideas!

[9](Theorem 4) It is best not to let this theorem sit too long. A class discussion and perhaps even a mini lecture can be very useful here. Emphasis needs to be placed on the fact that, mathematically, this theorem is very accessible to the students. The only difficulty is the ability to leverage their knowledge to prove a very intriguing concept that the students will shortly see has many applications.

[10](Definition 29) Time well spent discussing this definition in detail.

[11]This example is always given as an interactive lecture, asking students what to do at each step and being sure the class follows each step. We emphasize the parallel between long division of numbers and of polynomials by literally working the two problems simultaneously in two columns side-by-side and one step at a time. A student may mention that there is an easier way, synthetic division, in which case I ask them to demonstrate the method. In my thirty years very few students can implement it, which is a great opportunity to point out that memorizing tricks is a poor substitute for understanding logical processes. We also emphasize the Fundamental Theorem of Algebra. If we can find one root, we can reduce the degree of the polynomial and potentially repeat the process and perhaps completely factor the polynomial.

[12](Problem 115) After this problem is done, a mini lecture discussing how a pentagon construction can yield exact value for $sin(36°)$.

[13](Problem 136) Solving this problem by squaring both sides results in unwanted solutions. This is a good time for a mini-lecture and class discussion concerning extraneous solutions and why they occur.

[14]Many problems are poorly phrased to generate discussion. This problem is an example. Many teachers would consider the definitions for f and g to be adequate, but they are not. Without explicitly stating the domain, which is why one should define functions as sets, we cannot determine whether $f = g$. They are equal if the domain is all numbers except three, but if we allow the maximum possible domain for each, then they are not equal since g then has a larger domain than f.

[15]These two problems illustrate our belief that every set of notes should have problems that allow the weakest student in the room some success, and problems that challenges the strongest students in the room. The error of "teaching to the middle" is all to common and hurts both the top and the bottom.