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ORIGINAL ARTICLE

Electronic vs. paper textbook presentations of the various aspects of mathematics

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Abstract

Based in part on our work in adapting existing paper textbooks for secondary schools for a digital format, this paper discusses paper form and the various electronic platforms with regard to the presentation of five aspects of mathematics that have roles in mathematics learning in all the grades kindergarten-12: symbolization, deduction, modeling, algorithms, and representations. In moving to digital platforms, each of these aspects of mathematics presents its own challenges and opportunities for both curriculum and instruction, that is, for the content goals and how they connect with students for learning. A combination of paper and electronic presentations may be an optimal solution but some difficulties with such a complex solution are presented.

Keywords Textbooks · Aspects of mathematics · Mathematics textbooks · Digital presentations · Schoolbooks · Electronic publishing

The emergence of digital platforms for instruction in the past two decades and their popularity has generated great attention throughout the mathematics education research community, reflected not only in many dozens of individual papers but also full volumes (e.g., Gueudet, Pepin, & Trouche 2012; Clark-Wilson, Robuti, and Sinclair 2014; Bates and Usiskin 2016; Pepin and Choppin 2018). This paper analyzes paper and digital presentations of mathematics with respect to aspects of mathematics closely related to a parsing of the mathematics curriculum that is found in the materials of the University of Chicago School Mathematics Project (Usiskin 2015).

The analysis in this paper was motivated by headlines that have appeared in many places over the past few years, such as this recent headline from the *Philadelphia Inquirer* 19 Mar 2017: "Textbooks could be history as schools switch to free online learning".[1]

This statement, like many others of its kind, is meant to apply to textbooks in all subjects. Here we examine its applicability to mathematics. Years from now will there be no paper mathematics textbooks? If so, why? If not, why not?

This paper is split into three major sections. The first section is background, about the significance that paper textbooks have had in mathematics and the motivations for moving to electronic formats. The second section, occupying the majority of the paper, is a consideration of five broad aspects of mathematics with respect to their suitability for electronic and paper textbook formats. The third section contains a summary that tries to look to the future.

1 Background

What is a "textbook"? If we take the word "textbook" literally, a textbook is "text in a book form, that is, writing that is gathered together in a single volume or set of volumes". But that is too general a definition, for it would include all sorts of writing, including novels, encyclopedias, diaries, and so

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[1] Source: http://www.philly.com/philly/education/History-books-could-soon-be-history-as-schools-switch-to-free-online-learning.html, retrieved 25 Feb 2018.

Table 1 Some textbooks that have defined parts of the mathematics we teach

Text (with current English availability)	When written (language)
Euclid, *Elements* (Heath 1956)	Third century, B.C.E. (Greek)
Jiuzhang suanshu (*Nine Chapters on the Mathematical Art*) (Kangshen, Crossley, and Lun 1999)	Tenth to second century, B.C.E. (Chinese)
Larte de Labaccho (Art of the Abacus—the "Treviso Arithmetic") (Swetz 1989)	1478 (Italian)
Leonhard Euler (1972), *Vollständige anleitung zur algebra* (*Complete Instruction in Algebra*)	1770 (German)
Adrien-Marie Legendre (1825), *Eléments de géométrie*	1794 (French)
Tukey, *Exploratory Data Analysis*	1977 (English)

on. For us and for most people, a textbook is "a book used as a standard work for the study of a particular subject".[2]

Our interest here lies particularly in mathematics textbooks. In simplest terms, mathematics itself has two major constituents: concepts and problems, which build on each other. Mathematicians have created and continue to create and use concepts in order to solve problems. Mathematicians put forth and solve problems in order to better understand concepts. Textbooks historically have treated concepts or problems, or both.

1.1 Evolution of the printed textbook

Over the centuries, some textbooks have had more impact than merely learning our subject. They have helped to *define* mathematics, detailed the concepts and problems that together comprise what we call mathematics.

All of the texts in Table 1 are devoted only to mathematical exposition and/or problems, and all existed at a time when the mathematics in them was studied by a small minority of the population. However, as there became a need for textbooks to appeal to larger percents of an age group, features other than exposition and problems began to appear. In the United States, the most popular series of secondary school textbooks from the 1960s into the 1980s, the Dolciani series (Dolciani, Berman, & Freilich 1963; Jurgensen, Donnelly, & Dolciani 1963; Dolciani, Berman, & Wooten 1963) featured "extra for experts" sections, sample tests, and historical snapshots. Current textbooks now tend to include test preparation and review questions and often have links to online resources where more questions can be found.

In many places, textbooks have also became more colorful, not just with pictures, but with the use of color to help explain mathematical concepts. I remember being awed by the use of color as an integral part of the exposition in the text *Mathematique Moderne*, by the Belgium mathematician Papy (1963), for two reasons. First, at the time it was extremely costly to print a textbook in full color in the USA and I wondered how it could be produced so economically that it could be used by individual students. Second, Papy used color in inventive ways to introduce all sorts of theoretical concepts, from the underpinnings and operations on sets, to basic concepts of geometry and linear algebra, and all this for students around the age of 7th graders, 12–13-olds in the United States. A good text can be a revelation.

Teachers' editions of today's textbooks in the United States tend to contain commentary on the purposes of each lesson, suggestions for teaching the lesson, additional examples to present in class, extension activities for better students, remedial activities for slower students. Also, for each lesson there may be practice sheets available either in paper or online. And, for the teacher, publishers will provide a wide variety of assessments and possibly assessment tools to help teachers create their own assessments.

Publishers in the United States are afraid to publish books without these features because they feel they will lose sales to books that have these features. As a result of including all of these features, over time US textbooks have come to include more and more pages, and the Teacher's Editions are even larger.

As an example of the length of texts in the United States, I examined texts for the year-long introductory course in algebra that is typically taken in 8th or 9th grade by students who are 13–15 years old. My personal library contains 70 student texts for this course, published from 1907 to 2009. Table 2 lists the average number of pages in these books for

Table 2 Average length of texts in algebra books in the author's collection, over time

Time period	Number of texts	Average length (number of pages)
1907–1949	10	389
1950–1959	6	455
1960–1969	19	541
1970–1979	16	582
1980–1989	7	596
1990–1999	4	702
2000–2009	8	823

[2] Accessed at https://www.google.com/search?q=textbook%2C+definition&ie=utf-8&oe=utf-8&client=firefox-b-1.

various time periods. After the grouping of older texts in the first row, the other rows show the average length of the texts by decade. The data show that the length of textbooks has increased markedly over time.

Textbooks for the second year of algebra, usually taken by students 2 years later, are even longer. The 6 second-year algebra student texts in my library published since 2000 have an average length of 926 pages.

Greater length means greater weight. My own project's books are slightly shorter than the average but still are quite heavy: 1875 g (4 lb 2 oz) for UCSMP Algebra; 2072 g (4 lb 9 oz) for UCSMP *Advanced Algebra*. It may be accurate to assert that United States printed textbooks have the most pages, and are the heaviest of textbooks in the world. The sizes of the textbooks for other subject areas such as science, history, and the social sciences are also quite large, often weighing over 2 kg each even for subjects in the lower secondary grades.

Regardless of factors such as cost and features of the materials, the weight and convenience of digital platforms is a major factor in the US causing some schools to move away from paper textbooks in all subjects.

1.2 From printed to digital textbooks

The trend to move from printed to digital textbooks is worldwide and has been with us for a while. It caused headlines such as the headline that appeared 12 Nov 2010 in The Telegraph, a British newspaper: "Textbooks becoming obsolete due to rise of computers in the classroom, claim teachers". The article began, "Textbooks are becoming redundant in schools as computers and electronic teaching aids take their place, according to a survey of schoolteachers."[3]

At about the same time there was this headline reported by Foxnews.com, about a development in the United States. "Education chief Duncan wants textbooks to become obsolete"[4].

The article that followed began, "Education Secretary Arne Duncan called Tuesday for the nation to move as fast as possible away from printed textbooks and toward digital ones. "Over the next few years, textbooks should be obsolete," he declared." (Foxnews.com, 2 October 2012).

Given these developments world-wide, it is ironic that the first recent conference devoted to mathematics textbooks, held in Shanghai, was in 2011. Were I and others meeting about something that was about to die? Are we just trying to hold on to traditional practices even though new practices are better? Or do we believe, either through research or experience, that there is something about paper textbooks that makes them special? Or are we willing to convert our thinking from a paper textbook to a digital textbook and thus also aim our textbook research in the direction of digital textbooks.

The Foxnews.com report went on.

"It's not just a matter of keeping up with the times", Duncan said in remarks to the National Press Club. "It's about keeping up with other countries whose students are leaving their American counterparts in the dust."

South Korea, which consistently outperforms the US when it comes to educational outcomes, is moving far faster than the US in adopting digital learning environments. One of the most wired countries in the world, South Korea has set a goal to go fully digital with its textbooks by 2015.

"'The world is changing', Duncan said. 'This has to be where we go as a country'".[5]

In November 2014, 2 years after Duncan's statements, Hee-chan Lew, a leader in Korean mathematics education, gave the following assessment of the Korean government's move towards digital curricula in all subjects:

"When the Korean government proposed "An Agenda for Action" (2011), the development of a digital textbook was one of the main strategic goals to drive forward smart education reform. However, *in the case of mathematics* it seems to have failed to reach the goal." [Lew (2016), emphasis mine].

In Korea, the content and even the sequence of content in textbooks is strictly controlled by the education ministry, to such a point that authors have complained they have no freedom. The digital textbooks in mathematics that were created were the paper textbooks with one additional feature—a student could click on a link by each problem in the text and get an answer. There was no attempt to take additional advantage of the power of the technology the student was using to access the text. Lew bemoaned this development and was happy that the Korean ministry did not consider this adaptation to make sufficient use of the digital medium.

In the United States, and I think in many other countries, the move to curricula on digital platforms is often done as it was in Korea, with all subjects simultaneously. The experience of Korea supports the notion that, among the school subjects, mathematics has special qualities that cause policies that might be easy to adopt and beneficial for other subjects to be inconsequential for the learning of mathematics, or perhaps even harmful.

[3] Source: http://www.telegraph.co.uk/education/educationnews/8127270/Textbooks-becoming-obsolete-due-to-rise-of-computers-in-the-classroom-claim-teachers.html, retrieved 25 Feb 2018.

[4] Source: http://www.foxnews.com/politics/2012/10/02/education-chief-duncan-wants-textbooks-to-become-obsolete.html, retrieved 25 Feb 2018.

[5] Ibid.

1.3 Features of digital textbooks

For over 30 years the University of Chicago School Mathematics Project (UCSMP) has been involved in the development and production of textbooks and accompanying materials for mathematics in schools from pre-kindergarten through grade 12.[6] My particular aspect of the project has been with materials for secondary schools, that is, for grades 6–12 or 7–12. For the past couple of years, we have been working at the University of Chicago to create an interactive digital version of the print textbooks that we have developed over the past three decades and are in their third edition.[7] Specifically, we have been working to adapt our well-tested materials so that they can be dealt with easily by students who are working on computers, tablets, or smartphones. Our work has generally taken the approach of what might be thought as the minimal characteristics of a digital textbook in mathematics.

Such a textbook, since it is a *mathematics* textbook, needs to include explanations of concepts and problems for students to solve, Also, since it is a *textbook*, it should be aligned with a course of study, organized by lessons or topics, including a table of contents and a recommended order of coverage of topics.

Since it is *digital*, the content must be able to be read on a variety of platforms: a desktop computer, a laptop, a tablet, or a smartphone; the visual presentation should scroll, be able to be enlarged or reduced in size, and—for the smaller-screen devices—be able to be changed between landscape and portrait views; and the presentation should take advantage of the digital platform and have links to a glossary and to other places in the text where students can find explanations.

Within these parameters, electronic textbooks take three general forms: minimal, hybrid, or exclusive (see also Adler 2000; Remillard 2005; Pepin, Choppin, Ruthven & Sinclair 2017). In the *minimal platform*, the electronic form may simply be an e-reader of a print textbook, or it may be a redesigned but not reorganized version of a print textbook. This form is particularly appropriate for students who do not wish to carry a heavy textbook in school or between school and home.

In the *hybrid platform*, the course materials exist in both paper and electronic form, giving schools a choice of how far they wish to go in incorporating digital enhancements. These enhancements may include interactive resources that could never be part of a paper textbook. These materials may have built-in features or links to

- allow students to write in their texts;
- video explanations of the main ideas in each lesson or unit;
- hints or worked out answers to many or all questions;
- additional exercises and problems for extension or remediation, possibly produced adaptively (based on responses to existing exercises)
- practice evaluation instruments, or to the actual evaluations, again possibly adaptive;
- mathematical software for some or all of the following: manipulating objects, graphing, drawing, solving equations, statistical analyses, etc.;
- social networks of students in the same class, same school, or more broadly to discuss and compare mathematical work.

The materials from our project for grades 6–12 have some of these features and those for pre-kindergarten through grade 6, *Everyday Mathematics* (UCSMP 2016), have almost all of these features, as do materials from all the other major publishers in the United States.

The third broad platform is the *exclusively electronic* platform, which takes one of two forms. One form is open source, available free on the internet and is often tied to specific objectives. A second form is available only by subscription through a publisher. The subscription is typically paid for each student by the school the student attends.

In all these forms, I am speaking of materials that are designed for a classroom of students. A student who is being schooled at home or in a special environment, or a person who wishes to learn mathematics on his or her own may have difficulty obtaining access to materials that are available only by subscription.

Many additional features are designed for the teacher or for parents and might be viewed as electronic management. For instance:

- Content selection: being able to add or subtract whole lessons or parts of lessons or individual questions for entire classes or for individual students.
- Ongoing performance evaluation: the ability to know how students have answered questions in the text and performed on any of the evaluations found in the materials, with data organized either by question or by student.
- Home links: materials designed for parents to help their children.
- Links to internet sites for data, exploration, background, etc..

[6] For information on the K-12 materials of UCSMP, see http://ucsmp.uchicago.edu.

[7] For a description of these materials, see http://uchicagosolutions.com.

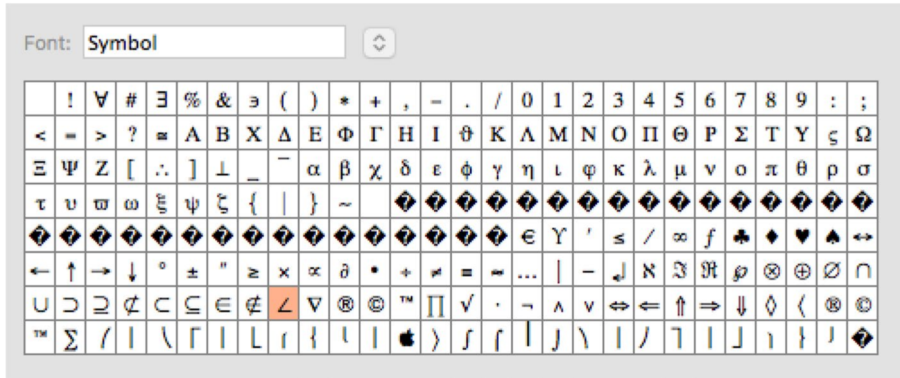

Fig. 1 The symbol font in Microsoft Word

The result of having all of these features is that the electronic versions of today's mainstream textbooks in the United States contain a dazzling array of features for both the teacher and the student.

For instance, consider the digital version of a Grade 4 lesson from *Everyday Mathematics*. When we open the student e-book, we see, in a column at the left, a table of contents with the 8 chapters. We scroll to Lesson 11 in Chap. 6, titled Angle Measures as Additive. In this lesson as in others, there is a sequence of six boxes that continues on another screen. The student is to go through these boxes in order. The third box is titled Unknown Angle Measures; it contains a nice dynamic picture of a linear pair of angles. I decide to go to the next page where there are three more boxes. The home link box tells me what is for homework. Certainly, there is a structure to the lesson but it is hidden inside the boxes. Perhaps the desire to know what is in each box helps to motivate students. Now I go to the bottom of the home link page and look for e-tools. There are a very large number of these tools, and they show that the possible manifestations of this lesson are endless.

A nice characteristic of electronic text is that it can be changed by the publisher at a moment's notice. This enables errors to be corrected immediately, but it also means that students using these materials may find themselves using a version that is slightly different from the version adopted by the school. In the United States, publishers need to keep track of the versions that have been adopted so that they do not violate contracts with school districts, even if the changes made are corrections of errors.

2 Aspects of mathematics in a digital world

It is significant that, with the exceptions of mathematics tools and links to mathematics software, features of electronic texts are essentially the same for all academic subjects, and for all branches of mathematics. However, with these characteristics and constraints on electronic platforms, some aspects of mathematics present unique difficulties, and other aspects present unique opportunities.

Here we consider five major aspects of mathematics, each having manifestations in all of the grades, and examine the impact of the digital world on each of these aspects. Four of these aspects parallel the SPUR (Skills, Properties, Uses, and Representations) characterization of understanding found in the UCSMP materials for grades 6–12 (Usiskin 2015). The fifth, and the first discussed here, is basic to the language of mathematics.

2.1 Symbolization: vocabulary and notation

The written language of mathematics calls for special symbols, not just from the Greek alphabet for those of us not from Greece(!), but also symbols for binary and unary operations; derivative, integral and summation notations, arrows, brackets, subscripts, superscripts, vectors, matrices, perhaps symbols for triangle and parallelogram, perhaps special symbols to identify important sets of numbers such as the integers and reals.

No other subject has so many symbols not in the everyday written language; evidence for this is in the number of mathematical symbols among all symbols in word processing programs. Consider Fig. 1. Aside from the Greek letters, this palette is dominated by mathematical symbols found in arithmetic, algebra, geometry, and logic.

Fig. 2 The equation editor menu in Microsoft Word and PowerPoint

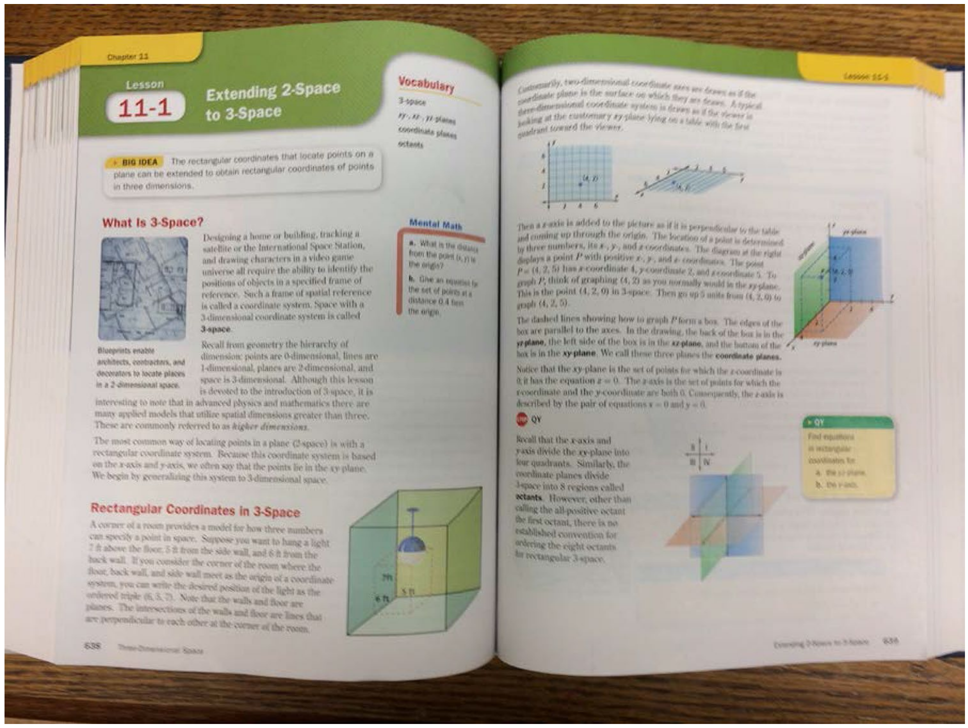

Fig. 3 The start of Lesson 11-1 from the paper version of UCSMP *Precalculus and Discrete Mathematics* (Peressini et al. 2016, pp 638–639)

The symbols of Fig. 1 are restricted to those mathematical symbols that can take the place of normal letters. A separate menu, seen in Fig. 2, exists for fractions, matrices, vectors, and all sorts of expressions that require parentheses, subscripts or superscripts.

In transferring our paper textbook to an electronic form that could be read on all sorts of devices, we were often quite frustrated. Each symbol was originally written in a particular font, but that font might not be available in the electronic format. We could not be satisfied with symbols that were close but could be misinterpreted, because we knew that many students use the paper version when in school and an electronic version on their mobile devices.

We in the field learn to adapt to different fonts, but students do not always adapt as easily. In the drop-down menu shown at the left in Fig. 2 are the choices that are available for matrices. Which symbol might not stand for a matrix but for its determinant? Are brackets the same as parentheses? We could not take for granted that when we moved from text on paper to digital text, our symbols would translate as we wanted them to. They all needed to be re-examined.

Line breaks are another issue. An equation that can fit on one line on the printed page may be too long for a smartphone. An expression or equation that is written within a paragraph may be broken where you do not want it broken. If you make the expression or equation something that will not break, then you need to scroll the page, which is hard to read when you cannot see the entire expression or equation at any one time. The software that was used to create our books automatically made line breaks, but everything had to be proofread to ensure that the line breaks worked on devices of all sizes.

To exhibit this, Fig. 3 shows the first two pages of the first lesson, "Extending 2-Space into 3-Space" of Chap. 11 of the UCSMP text *Precalculus and Discrete Mathematics* (Peressini et al. 2016). This lesson, as do all other lessons of the book, begins with Vocabuary, a Big Idea, and Mental Math. There are pictures of a 2-space graph, then it is flattened, and then a couple of 3-space graphs.

In the digital version (Fig. 3), the features are the same but the organization of the page is quite different. And so are the fonts. The lesson has exactly the same content as in the paper version but organized differently and there are links to a glossary in boldface. However, even on a large screen, the student cannot see the equivalent of two of the pages of the paper version, so the student does not see as easily how the content is organized.

In the digital version, the shape of the page can be changed to fit any device, such as a tablet or a smartphone. When this is done, all the lines move nicely with nice line breaks. Even when we move from portrait to landscape version, the line breaks are still nice. Our editors checked as many screen sizes as they could in the time they had to do the conversion (Fig. 4).

If we were starting from scratch, we would have linked the 3-space graphs to a 3-D grapher. But we were committed to making digital versions of our books that would be compatible with our print texts.

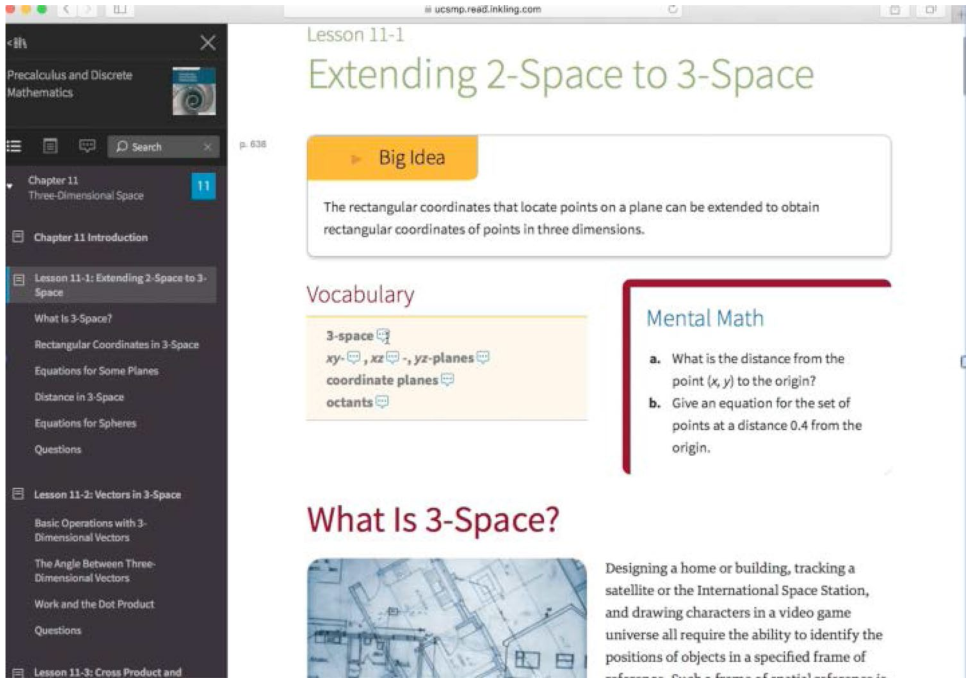

Fig. 4 The start of Lesson 11-1 from the digital version of UCSMP *Precalculus and Discrete Mathematics* (Peressini et al. 2016)

2.2 Representation: changing modes of thinking

Symbolization may be considered as a special case of the representation of mathematics, but I differentiate these two aspects of mathematics because there is a sense, as the German mathematician David Hilbert felt in his formalist approach to mathematics, that symbolization *is* mathematics. Representation is different in that we take one mode of mathematics and show it in a different way.

For me, a *representation* is the result of the move from one mode of describing a piece of mathematics to another mode, where the following are some possible modes: concrete, symbolic, pictoral, diagrammatic, tabular. Representations are particularly common in teaching because often a mathematical idea is easier to understand in one mode than another. Representations allow us to show students two or more views of the same idea.

There are very many kinds of representations. There are:

- geometric representations of whole number;
- concrete representations with actual objects
- geometric representations of real number on a number line or as the length of a segment;
- geometric representations of operations that we use with proofs of the Pythagorean theorem or showing the distributive property;
- graphs of functions;
- statistical displays of numerical data.

All those represent the move from a symbolic mode to a geometric or pictoral mode. But then there is the reverse:

- algebraic representations of curves and figures;
- matrix and algebraic representations of geometric transformations and vectors.

And there are representations of relationships among sets:

- Venn diagrams
- Networks.

A representation often occupies considerable space in a textbook, for we want the reader to make the connections between the original mathematics and its representation. For this we need the original object, the representation of that object, and an explanation or some sort of how the two are related. Consider the network in Fig. 5.

On a small screen there is room to show little more than the names of the quadrilaterals and the connections, or perhaps the names with tiny figures appended. There is no room for any explanation. A paper textbook or a digital textbook on a laptop provides a canvas large enough to show the network, the figures the network represents, and some detail. Many representations need room.

There is an important kind of representation for which the digital presentation is particularly suited: the dynamic representation. Dynamic representations display:

- change over time;
- changes in range values of functions as the domain values change;
- transformations of figures;

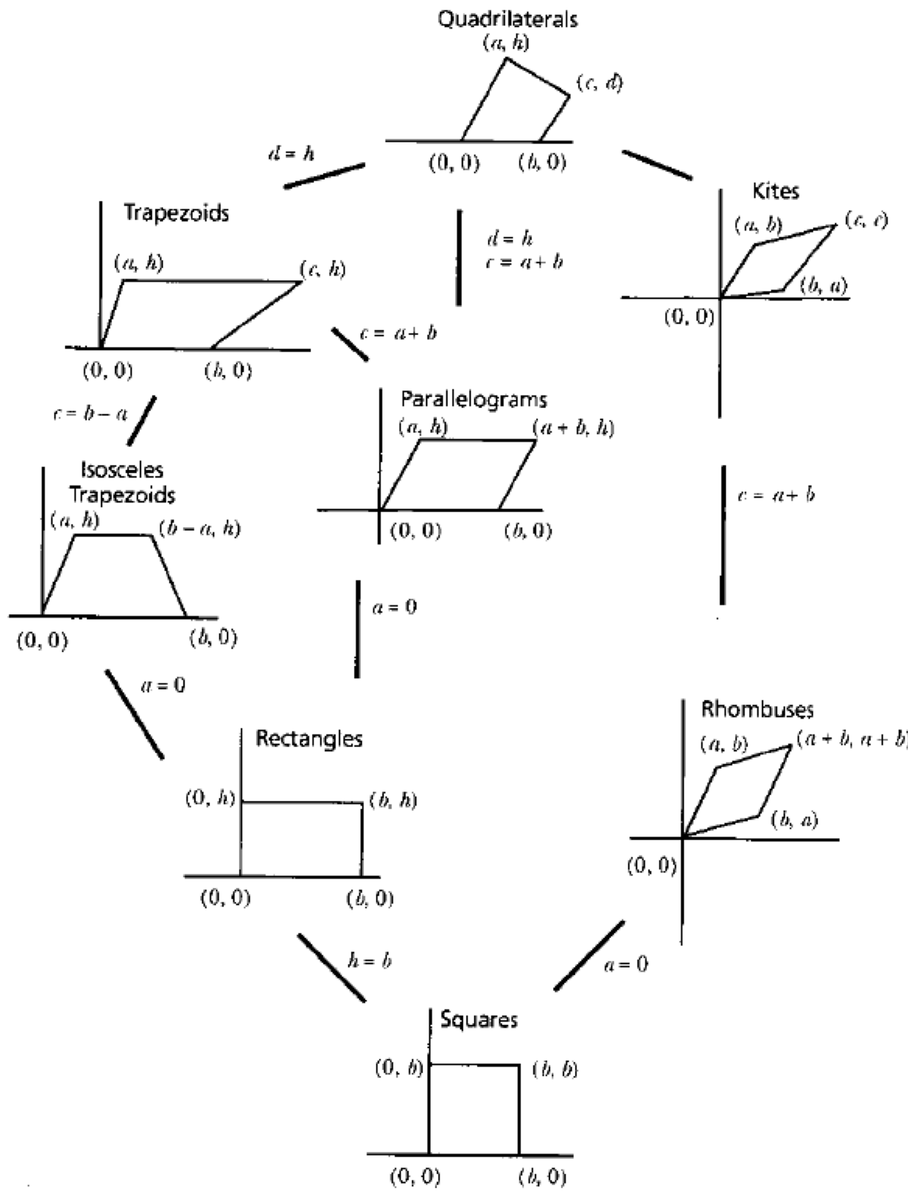

Fig. 5 A hierarchy of quadrilaterals in convenient first quadrant positions of the coordinate plane (Usiskin and Griffin 2008, p. 77)

- effects of parameters;
- sequences of events.

For dynamic representations, print textbooks do not do the job, since they are necessarily static.

For instance, in the UCSMP book *Pretransition Mathematics* (McConnell et al. 2009, p. 65), to show increasing numbers of decimal places of coordinates of a point on a number line, three number lines are used, each of the last two a magnification of the previous, to show students successive approximations. This is about the best that could be done in this static environment. But in a dynamic environment, one can zoom in and zoom out on a number line. On a zoomable number line from (http://www.mathisfun.com)[8] a student can do much more. Clicking on the line expands the line, click-shifting contracts it. Arrows enable moving to the right or left on the lines. By repeated clicking, more and more decimal places appear. Students would certainly have fun seeing how many decimal places they can get. The website reports that the line zooms to as many as 30 decimal places.

In a College Preparatory Mathematics (CPM 2016) electronic text, the student is asked to graph the line with

[8] http://www.mathsisfun.com/flash.php?path=%2Fnumbers/images/number-line-zoom.swf&w=960&h=330&col=%23FFFFFF&title=Zoomable+Number+Line.

equation $y = 2x - 7$. Embedded in these materials is a Desmos grapher. As the student fills in the values of y in a table for various given values of x, points on the line are graphed and connected. In another place, the moves are not discrete. Sliders are used to show how the graph of a linear or quadratic equation is affected by changing one of the coefficients. None of this could be done in a paper format.

2.3 Deduction: mathematical systems

Deduction is the process by which we decide whether a mathematical statement is true or not. It is the centerpiece of our subject; it is to mathematics what the scientific method or experimental method is to science. A person who finishes school without having some experience with deduction has missed the essence of mathematics.

Deduction and proof in a digital environment raise a set of issues that increasingly surface as one goes up the grades. Deduction requires continual referrals to justifications. For young learners, the justifications in a mathematical argument are often not obvious and may not be well-understood. The student needs to know what is allowed in a proof and what is not. In a paper text, the order of lessons and of units is clear. Listing possible justifications, as some digital texts do, converts proofs into algorithmic exercises. It seems easier to flip pages of a paper text than to link back to earlier lessons.

Deduction occurs in a closed system, and the more sophisticated the electronic environment, the less it seems to be closed. The transparency that a paper textbook offers the teacher and the student is an advantage in doing deduction. In a paper text, all of the content is visible, touchable. Paper texts exhibit finiteness. We can touch the locations where terms are defined and how theorems are named. With electronic media, the hiddenness of the content makes it more difficult to understand the closed nature of deduction.

Above we noted that, in graphing $y = 2x - 7$, the Desmos grapher that is embedded in the CPM text automatically connects consecutive discrete points entered by the user. In doing so, the grapher is making mathematical assumptions that are different from the user and may not be evident to the user.

Deduction in early mathematics is very close to the postulates and definitions in that system. The paper textbook can offer a list of the postulates and definitions in use that is finite, in contrast to a learner searching the internet, which is, in practical terms, infinite and may follow a quite different deductive sequence. Goldenberg, Scher, and Feurzeig (2008) tell of a situation with the Geometer's Sketchpad where similar triangles are generated by moving a line segment to one side of a triangle. However, in the same frame, some other ratios are not preserved. Dynamic geometry software is great for exploration but there are artifacts of the software assumptions that cannot be deduced from the mathematical postulates. Consequently, to teach deduction carefully, we must step back from the digital environment.

Deduction takes space, and a small screen is a very poor window for most deductive arguments. Even the shortest of deductive arguments tends to require a number of steps, and if the justifications are included a small screen does not have sufficient space to clearly show the complete argument. Furthermore, when a student is expected to write a proof, the writing chore is often formidable, seldom done on a first attempt, and to convey this electronically is not easy.

Finally, in those cases where the text materials allow access to the internet, conflicting interpretations of the same mathematics may appear.

For instance, consider the expression 0^0. Wolfram MathWorld, a very trusted source, argues that 0^0 is indeterminate and thus undefined. "0^0 is undefined. The lack of a well-defined meaning for this quantity follows from the mutually contradictory facts that a^0 is always 1, so should equal 1, but 0^a is always 0 (for $a > 0$), so should equal 0. It could be argued that $0^0 = 1$ is a natural definition since as n approaches 0 from both the right and left, $\lim 0^n = 1$. However, the limit does not exist for general complex values of n. Therefore, the choice of definition for 0^0 is usually defined to be indeterminate."[9]

We see this easily on a graph. The graph $y = x^0$ is part of the graph of $y = 1$, an open ray in the 1st quadrant. The graph of $y = 0^x$ is part of the graph of $y = 0$ and is the positive part of the x-axis. Also, as a approaches 0 (for $a > 0$), $(a^n)^{1/n} = a$, as can be seen from the graph of $y = (x^n)^{1/n}$, which is the graph of $y = x$ for $x > 0$ and so is the open ray $y = x$. The continuity argument now sugggests 0^0 should equal 0 by a vote of 2 to 1! The conflicting limits tell us that it is impossible to define 0^0 in such a way that it is consistent with these three graphs; the conflicts supports the Wolfram Math opinion that 0^0 should be thought of as indeterminate.

However, some calculators on the web do not support the Wolfram Math opinion. Both eeweb.com and Desmos return the value 1 for the expression 0^0.

A different approach is taken in *Concrete Mathematics* by Graham, Knuth, and Patashnik, topflight mathematicians. "Some textbooks leave the quantity 0° undefined, because the functions 0^x and x^0 have different limiting values when x decreases to 0. But this is a mistake. We must define $x^0 = 1$ for all x, if the binomial theorem is to be valid when $x = 0$, $y = 0$, and/or $x = -y$. The theorem is too important to be arbitrarily restricted! By contrast, the function 0^x is quite unimportant." (Graham, Knuth, and Patashnik, 1994, p. 175).

A print textbook rarely presents conflicting views of 0^0 or any other mathematics. Do conflicting views of a piece of

[9] http://mathworld.wolfram.com/Zero.html. Retrieved 6 Oct 2017.

mathematics present a difficuty or an opportunity? The value of 0^0 is not the only place where trusted sources differ. How should we define “trapezoid”? How should the trigonometric functions of real numbers be developed? When should students be introduced to quadratic equations with non-real solutions? A paper textbook, more easily than anything electronic, can lay down the rules of the game of mathematics that it is playing. In contrast, the internet may show that the rules of mathematics are a matter of opinion. Those of us who are experienced may welcome this dissonance. But for the naïve learner or inexperienced teacher, does this freedom inhibit careful deduction or highlight the need for careful deduction?

2.4 Modeling: applying mathematics externally

The fourth aspect of mathematics I wish to discuss here is *modeling*, the process by which mathematics is employed to describe a real-world situation or treat a real-world question. The everyday and broad applications of modeling supply the reason that mathematics is a required subject in school systems all over the world.

Modeling contrasts with deduction in that the problems in mathematical modeling originate from outside the field, while deduction occurs inside. Thus, in contrast to deduction, which in mathematics requires a limited sphere in which a student is expected to operate, mathematical modeling is, at its best, open to any part of mathematics. In doing deduction, students often have difficulty determining what mathematical results they are allowed to use, while in modeling, students may be encouraged to use any aspect of mathematics they can find. This makes modeling quite suited to the openness of mathematics that is possible when one has access to the internet and other resources along with exposition and problems that might appear electronically or in a paper textbook.

A modeling situation in a lesson titled “Linear Functions” from the Pearson *Envision Algebra 1* digital text (Kennedy, Milou, Thomas, C.D., and Zbiek, R.M. 2018) begins with an animated film of a small cart in a grocery store proceeding to the cashier and having to make a decision as to whether to go to a line behind three large carts or the express line behind 8 small carts. There is music but no words either spoken or written. The video stops and there is to be discussion with students being asked to identify the problem that is being presented. Continuing the activity, a second film now shows how many items are in each cart. The student is thus led to the idea that the numbers of items in the carts are data to help solve the problem. It is expected that students will estimate how long it takes the cashier to slide the item over the upc reader and muliply this by the total number of items to estimate how long the small cart will have to wait in line. By numbers of items it seems that the line with the 8

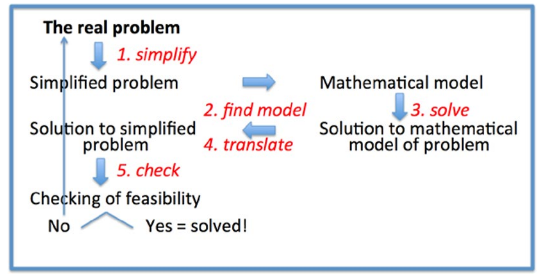

Fig. 6 The modeling process

small carts is the better choice. But a variable has not been considered—the length of time it takes to process the total for each cart. In a third film, the time to process is estimated and then it turns out that the line with the large carts is the better choice. The idea here is that the total time is a linear combination of the number of carts in front of you and the total number of items they contain, plus the product of the number of carts and a constant processing time independent of the size of carts.

This video exemplifies how real problems can be motivated by video clips created by a publisher or imported from the internet, and data can be presented or uploaded. The ability to access data, solutions to similar problems, and so on, brings it home to students that mathematics is a living and important discipline.

Introducing the learner to the variety of sources of data, the work that others have done and put on the web, and the many tools that exist for doing mathematics opens up the world to the learner. Just as pure mathematics is more suitably done in a closed and sheltered world, applied mathematics is better done in an open world.

The modeling process is often described by a diagram such as in Fig. 6.

In Fig. 5, five steps in the modeling process are identified: simplify the real problem; find a mathematical model for the simplified problem; solve the mathematical problem; translate that solution back into real terms; and check the feasibility of the solution. If the solution is not feasible, then go back to the original problem and perhaps simplify it in a different way or try a different mathematical model. The shopping cart problem in the video is treated very nicely in a way that parallels these steps.

Looking at the five steps, we see that the first and last steps take place in the real world, that the second and fourth steps are the translations from the real world into mathematics and back, and that only the third step takes place entirely within the mathematical world.

Conrad Wolfram (2013), in looking at how mathematics is used in the real world, notes that working in the mathematical model, which is sometimes called “doing the math” (step 3 in Fig. 5) can virtually always be done using calculators and computers and none of the other steps can be done that way; the other steps require thoughtful decision-making. So he asks why we spend so much of our time teaching students paper and pencil algorithms when computers are around to do them, time that could be spent on the steps in the modeling process that cannot be done using computers.

And yet the third step in the modeling process is critical, for the employment of pure mathematical techniques is the reason for the entire process in the first place. It is that step that brings us to the last of the five aspects of mathematics considered here.

2.5 Algorithm: carrying out procedures

An *algorithm* is a finite step-by-step procedure for accomplishing a task that we want to complete[10]. The algorithms students are expected to learn range from addition and subtraction of whole numbers, fractions, and decimals through the solving of linear and quadratic equations and factoring of polynomials in algebra to finding derivatives and integrals in calculus. They also include algorithms for making graphs and for geometric constructions. Some algorithms are simple to describe and easy to learn, such as how to multiply fractions. Others, such as long division, take a long time.

A large amount of time in mathematics classrooms is spent teaching paper-and-pencil algorithms. Yet, of all the aspects of mathematics, no aspect is more suitable for the electronic setting than the algorithms we teach. Computers can generate problems ad infinitum and the responses of students can be utilized to automatically generate additional exercises using built-in adaptive learning software. Adaptive learning software is being used in diagnostic tests that are part of digital textbooks and in formative evaluations of student progress so that additional practice exercises can be generated before a student takes a test for a grade.

Why do we spend so much time on algorithms that were originally developed for paper and pencil, whether or not we are using a digital textbook? There are many reasons.

- Algorithms are powerful.
- They are accurate and reliable.
- They are fast.
- They furnish a written record.
- They can establish a mental image.
- They can be instructive.
- They can be put together to be parts of more complex algorithms.
- They can be objects of study, as they are in computer science.

[10] cf. https://www.merriam-webster.com/dictionary/algorithm.

But in a digital setting, there is an irony. Virtually all the algorithms we teach can be performed more quickly and more efficiently by the same technology that the student is using to access the questions that lead to algorithms the student is being asked to perform with paper and pencil.

Algorithms in mathematics can be performed in three ways: mentally, in your head; with paper and pencil or pen; and with computer. I think almost everyone agrees that students need to be able to obtain many mathematical results in their head, and should be able to do the simple algorithms ln their head, and the most complicated with the aid of a computer or calculator. A major question at all levels of mathematics education (arithmetic, algebra, calculus, etc.) is which tasks students are traditionally taught to do with paper and pencil should no longer be in the curriculum.

This question particularly surfaces if students are taught with digital materials, because every task for which there is a paper and pencil algorithm can be accomplished by computer, usually faster and certainly more reliable. Furthermore, people are more likely to have smart phones than paper and pencil, both when they are students in school and when they are adults. It is strange and ironic to be teaching a paper and pencil algorithm to accomplish a task on a device when the same device can be employed to do the task more easily. It forces a re-examination of when we require students to use paper and pencil.

Not only do people have the technology to do mathematics on their person if they own a smartphone, but that technology goes beyond the algorithms that are normally taught in school. So a digital textbook should assume that a student has access to the technology available on the electronic device that is used. Otherwise it is like asking a person to walk even to places for which driving a car or riding a bicycle woud be more appropriate.

Many people with influence in education, from government policy makers through school personnel and classroom teachers, are so enamored with the ability of digital technology to *deliver* all the subjects that they are blind to the reality that this same technology can *do* mathematics. This blindness often comes from the fact that mathematics is the only one of the major school subjects for which a large amount of what is taught can be accomplished by technology.

For example, since 2010 the publication *Educational Leadership*, a journal "intended primarily for leaders in elementary, middle, and secondary education" in the United States, has published two special issues devoted to technology in education (ASCD 2013; ASCD 2015). Only one of the 21 technology-discussing articles in these two issues touches on mathematics in any deeper way than to acknowledge that mathematics is a school subject. A single paragraph by Marc Prensky, a writer about education, is the exception. He parallels the view of Wolfram mentioned earlier.

> "We need to start teaching our kids that technology is, in a great many cases, the best way to learn something. The best way to learn to solve real math problems is through technologies such as spreadsheets, calculators, and the software Mathematica, which force students to think about how to structure the problem. These technologies then do the calculations—the part that machines do best—enabling students to focus on whether the answer makes sense." (Prensky 2013)

On the surface, the Common Core in the United States (CCSSI 2010), meant to apply to all students in grades K-11, would seem to have an accepting view towards technology. The elucidation of the 5th of the 8 standards for mathematical practice, "Use appropriate tools strategically.", includes the following:

> Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software…Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital congent located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

However, there is no mention of calculators or any technology in the standards in grades K-8. Nor is any traditional paper-and-pencil arithmetic or algebraic skill absent in the Common Core Standards. A student who is using one of the online curricula designed specifically to cover the Common Core Standards is seeing mathematics *delivered* using technology while at the same time is not seeing how that same technology affects the algorithms people employ to do mathematics in today's world.

3 Summary and a look to the future

This paper has examined five aspects of mathematics—symbolization, representation, deduction, modeling, and algorithm—with respect to the opportunities and challenges they offer when presented in a digital learning environment. These opportunities and challenges also depend on the type of hardware the student has at hand: a full-size computer or laptop; a tablet; or a smartphone, and of course on the software and apps to which the student has access.

This examination suggests that the digital medium is better suited than the paper medium to some work with mathematical modeling and with representations. The digital medium seems more poorly suited than the paper medium when working with deduction. The jury is out regarding which medium is to be preferred when working with mathematics as a language or when learning algorithms.

This suggests that the best textbook presentations of the future should have significant print *and* electronic components. This conclusion is not surprising, in linear programming and game theory, optimal solutions are often combinations of pure simple strategies.

However, in the real world of schools, a strategy in which a teacher has to involve *both* print and electronic media in significant ways is not at all easy to implement—or, should I say, based on experiences in the United States with curriculum change, implementation of a complex strategy such as this is not easy. A number of factors seem to make it difficult.

Cost In the best of all worlds, a student has to have access both to print materials and to the electronic medium of presentation. Also, the value of the electronic medium is increased as the size of the screen increases. These place pressure on the school or the student to spend more money on materials than before.

Complexity The teacher has to juggle two modes of presentation and, if students are allowed to use their own technology, the electronic mode may have many variants. In the US, some schools have dealt with this problem by purchasing ipads or similar devices for all students, a rather expensive proposition. And because of the cost of such purchases, print textbook use is discouraged. This is in line with a third factor that markedly affects what teachers do.

The desire for simplicity This solution is to do away with print textbooks entirely. It reflects an "all-or-none" view that we have seen occur with a number of other innovations: individualized instruction, in which whole-class explanations are seldom if ever given; the use of calculators, often banned altogether even though their use at times would be extremely helpful; the movement towards the metric system in the US in the 1970s, in which some books had no discussion of any of the customary units; the Common Core standards, in which evaluation agencies penalize textbooks if they depart in any way from a literal interpretation.

These factors may make it difficult to sustain a situation in which both print and electronic delivery systems are present. This analysis suggests that mathematics textbooks of the future are destined to be electronic, with hard-copy print media existing only as reference books.

Identity A course that is totally digital would not have any sort of tangible identity. A reference book would offer identity to such a course while at the same time providing a backbone for the mathematical theory. This suggests that a future without some sort of print manifestation of a course is quite unlikely.

My analysis leads me to hope that there we will soon see a partnership of paper and digital settings. The paper partner brings the tangible identity that identifies the mathematics being covered, the transparency necessary for public approval and scrutiny of the content and approach, the specificity needed in mathematical vocabulary, the closed nature of the content needed for treatments of deduction, and shows static representations of the mathematics. The electronic partner exhibits the power of mathematics in modeling the real world and its open nature in solving real-world problems, and dynamic representations that picture variation and change, and gives students the power to utilize algorithms developed by mathematicians over centuries.

In a partnership of print and digital presentations, teachers are the intermediaries between text and student. The traditional role of the teacher has been to manage the classroom learning experience. In this partnership of digital and paper presentations, the role of the teacher includes the management of each setting and the relationships between them. The teacher's role is more difficult, but more exciting and more in tune with the world outside of school, and one that better serves the variety of aspects of mathematics that we wish students to encounter.

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