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3. RESEARCH ON THE LEARNING AND TEACHING OF ALGEBRA

INTRODUCTION

For many years, the mathematics education community has investigated the difficulties students have with algebra. Different aspects of algebraic thinking, considered to be fundamental to overcome those difficulties, have been analyzed by researchers using a variety of theoretical frameworks. Different proposals to redress those difficulties, based on research results, have been suggested and their effectiveness has been investigated. Results of these research studies have signalled that in order to foster students' development of algebraic thinking it is necessary to help them: improve their sense and dynamic conception of variables (i.e., variables as changing entities); develop their capability to generalize and to express their generalization; become aware of the dynamic relation that exist between variables; and, identify algebraic structure (e.g., Kieran, 2006; Radford, 2008, 2011; Mason, Stephens, & Watson, 2009; Ursini & Trigueros, 2011; Cooper & Warren, 2008). Within the mathematical topics studied in high school, algebra plays an important role. Researchers have found that the higher level of mathematics courses students take in high school, the greater their chance of attending and graduating from college and finding better paid employment in the future (Carnevale & Desrochers, 2003).

Algebra still remains an area of interest for the PME community. There have been major shifts in the focus of this research since it first came to the attention of PME researchers (Kieran, 2006). The themes that have witnessed a growth in interest by the PME community over the last ten years are: algebraic thinking in the elementary years (early algebra); generalization; structure sense; advanced algebra; and, the use of technological tools to support the development of algebraic understanding of secondary school students. One of the major gains that early algebra research has seen over the past ten years is the inclusion of researchers whose previous focus of research has been predominantly 8th and 9th students (aged 13 to 14 years) (e.g., Becker & Rivera, 2005, 2008; Radford, 2006; Sabena, Radford, & Bardini, 2005). Recent findings have indicated that many of the difficulties young students experience with algebraic thinking mirror the difficulties students' exhibit as they begin formal algebra (e.g., Becker & Rivera, 2007; Warren, 2006). Thus research with 13–14 year-old students is not only informing research regarding upper secondary and tertiary students but also informing research involving elementary students. Additionally there has been an increase in the emphasis on theorizing

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related to the nature of algebraic thinking of 5–14 year old students (e.g., Radford, 2006, 2010; Rivera, 2013).

Algebraic thought rests on some basic pillars and fundamental among them are the two aspects: variables as general abstract entities that can be represented in different ways; and, the 'structure sense' of algebra. These two aspects are involved in any algebraic activity (generalization, equations, functions, etc.). Research conducted this last decade on the teaching and learning of algebra, in secondary and university level, has used a variety of theoretical frameworks to analyze the complexity of working with these two aspects, paying attention to the particularities of their specific instantiations. In particular, PME researchers have been concerned with them through the analysis of students' and teachers' work with equations, relation between variables, pattern recognition, and generalization. In addition, this last decade has witnessed a rapid development of technologies. Its influence on mathematics' learning and in particular, in algebra learning has demanded further research. The possibility to use technologies to develop students' algebraic knowledge and skills, and the obstacles involved in their use by teachers has strongly attracted the attention of researchers.

This report is organised into four sections: The development of algebraic thinking among students aged from 5–14 years; the development of algebraic thinking among upper secondary and tertiary students; the use of technology; and a final section which integrates these themes, identifies the gaps in these themes and presents possible directions for future research in the domain of the learning and teaching of algebra. The examples presented in these sections have been drawn from the research that has occurred over the ten-year period from 2005 to 2015, with a particular focus on PME conference proceedings.

THE DEVELOPMENT OF ALGEBRAIC THINKING AMONG STUDENTS AGED FROM 5–14 YEARS

Early algebraic thinking refers to thinking about algebra early and looking at number from a more structural perspective. Thus, its focus is on developing in students the awareness of the structure of patterns and the structure of arithmetic (Mason, Stephens, & Watson, 2009). Early algebraic reasoning entails encouraging young students to become naturally aware of generalizations in numerical and non-numerical contexts and expressing these generalizations using a variety of semiotic signs (Radford, 2006). Traditionally algebra has only been taught after students have acquired a substantial amount of arithmetic knowledge, with the assumption that arithmetic provides the grounding on which to build algebraic knowledge. But as evidenced by the current research conducted by PME researchers, many early secondary students (13–14 years) struggle with this domain, and most of these difficulties can be traced back to their prior experiences in arithmetic. Thus a research focus for the future is how (and should) arithmetic and algebraic thinking be intertwined with each supporting the other across the first 10 years of school.

Underpinning the research classified as early algebra is the perspective that for many students the meaning of algebra is derived from its numerical foundations. In particular the focus of research over this last ten year period has been on investigating the challenges of the well-documented discontinuities that arithmetic has created for students beginning to formally explore algebraic concepts (Kieran, 2006). For example, students' prior arithmetical use of letters in formulas and as labels can negatively impact on their understanding of the concept of a variable (e.g., Kuchemann, 1981; Clement, 1982); and, many students experience difficulties when (a) interpreting equations with several numerical terms and unknowns (e.g., Linchevski & Linveh, 1999), (b) manipulating algebraic expressions (e.g., Kirshner, 1989), and (c) articulating the structure of a pattern or relationship in ordinary language (Macgregor & Stacey, 1993). Traditionally early algebra has tended to be associated with the elementary years of schooling. Given that the recent findings have indicated that many of the difficulties young students experience mirror the difficulties students exhibit as they begin formal algebra, this section has been extended to include both of these groups of students (i.e., elementary school students – 5 to 12 year old – and students beginning formal algebra – 13 to 14 year old).

In all 62 PME full research reports informed this section of the review. No short communications were included in this analysis. However, if the PME research report was further elaborated in a journal article, book section or book this source was also utilized to inform this section of the review. This section reflects the types of research that have occurred, and the particular themes that researchers have investigated with regard students aged 5–14 years. Initially the papers were classified according to their focus and the methodology utilized to explore this focus. The number of students included in quantitative studies ranged from 50 to 1300. Table 1 presents the frequency of PME papers according to their focus and their data collection method.

Table 1. PME papers – Algebraic thinking 5–14 year old students

Focus	*Method*				
	Qualitative	*Quantitative*	*Mixed method*	*Theory*	*Total*
Student learning	25	8	3		36
Teaching algebra	6	4	1		11
Teaching & learning	5	4	4	2	15
Total	36	16	8	2	62

Thus in the last ten years research has predominantly occurred in the area of student learning with a focus on the use of qualitative methods such as interviewing individual students or videoing small groups of students as they engage in algebraic tasks. Secondary to this interest is the concern with the act of teaching algebra, and the relationship between what is taught and what is learnt. A meta-analysis of these research papers revealed that the focus of PME algebra researchers investigating algebraic thinking amongst 5–14 year old students over this period were: noticing and representing pattern structure (35 papers); and, working with equations, expressions and variables – the influence of arithmetic thinking (27 papers). This analysis was driven by the understanding that research in early algebra in the elementary school arose from the many difficulties beginning high school students exhibited as they studied algebra, difficulties that emanated from 'operating from an arithmetic frame of reference' with a focus on calculating (Kieran, 2006, p. 25). Thus the fundamental purpose of early algebra research has focused on further investigating and theorising about redressing this issue (e.g., thinking about numeracy equalities as relational, symbolizing relationships between qualities, developing functional thinking). In addition, the categories reported in this section of the review are also utilised in the second section of the chapter, allowing for a more coherent analysis of the issues pertaining to learning algebra to occur. The next section presents a synopsis of the research findings related to these themes.

Noticing and Representing Pattern Structure

Pattern activities have been considered to be one of the main ways for introducing students to algebra (e.g., Ainley, Wilson, & Bills, 2003; Mason, 1996). From this perspective, algebra is about generalizing (Radford, 2006). Previous research has evidenced that visual approaches generated in tasks involving the generalization of geometric figures and numeric sequences can provide strong support for the development of algebraic expressions, variables, and the conceptual framework for functions (Healy & Hoyles, 1999). However, not all activities lead to algebraic thinking. For example, placing the emphasis on the construction of tables of values from pattern sequences can result in the development of closed-form formulas, formulas that students cannot relate to the actual physical situation from which the pattern and tables of values have been generated (e.g., Amit & Neria, 2008; Hino, 2011; Warren, 2005). This impacts on students' ability to identify the range of equivalent expressions that can be represented by the physical situation.

The patterns utilised in the 2005–2015 research encompassed both linear and quadratic functions that were represented as a string of visual figures or numbers. The activities students engaged in involved searching for the relationship between the discernable related units of the pattern (commonly called terms), and the terms' position in the pattern. These reflect the types of activities predominantly used in current curricula to introduce young adolescent students to the notion of a variable and equivalence.

Students noticing and representing the pattern structure. Fundamental to patterning activities is the search for mathematical regularities and structures. In this search, Rivera (2013) suggests that students are required to coordinate two abilities, their perceptual ability and their symbolic inferential ability. This coordination involves firstly noticing the commonalities in some given terms, and secondly forming a general concept by noticing the commonality to all terms (Radford, 2006; Rivera, 2013). Finally, students are required to construct and justify their inferred algebraic structure that explains a replicable regularity that could be conveyed as a formula (Rivera, 2013). At this stage the focus is no longer on the terms themselves but rather on the relations across and among them (Kaput, 1995).

While the 2005–2015 research involved the exploration of the notion of variables, the findings suggest that signs other than the conventional alphanumeric symbols of algebra can be used to express variables (Radford, 2010). Given this caveat, the findings of this research exhibit that algebraic thinking can appear in students at an early age (e.g., Radford, 2010; Anthony & Hunter, 2008; Rivera, 2011; Warren, Miller, & Cooper, 2011). Having students engage in *quasi-generalization* processing using *quasi-variables,* that is expressing generalizations in terms of specific large numbers as examples of 'any number' significantly assists students to noticing and representing pattern structure, and providing a generalization in language and other signs including alphanumeric symbols (Cooper & Warren, 2011).

Difficulties students experience in noticing pattern structure. Emerging from the findings of this current research is that while young students are capable of noticing pattern structure and engaging in pattern generalization, they exhibit many of the difficulties found in past research with older students. As revealed in the findings of this research: young students have difficulties moving from one representational system to another such as from the figures themselves to an algebraic form that conveys the relationships between the figures (Becker & Rivera, 2007); students tend to be answer driven as they search for pattern structure (Ma, 2007); they engage in single variational thinking or recursive thinking (Becker & Rivera, 2008; Warren, 2005); they fail to understand algebraic formula (Warren, 2006; Radford, 2006); and, they have difficulties expressing the structure in everyday language (Warren, 2005). In addition, initial representations of the pattern (e.g., pictorial, verbal and symbolic) can influence students' performance. This is particularly evident as 139 10 and 11 year-old students explored more complex patterns (Stalo, Elia, Gagatsis, Teoklitou, & Savva, 2006), with pictorial representations of patterns proving easier for students to predict terms in further positions and articulate the generalization.

Capabilities that assist students to notice structure. Adding to the research is a delineation of the types of capabilities that assist young students to reach generalizations. The ability to see the invariant relationship between the figural cues is paramount to success (Becker & Rivera, 2006; Stalo et al., 2006). The development of specific language that assists students to describe the pattern (e.g., position, ordinal language, rows) (Warren, Miller, & Cooper, 2011; Warren, 2006) and fluency with using variables (Becker & Rivera, 2006) help students to express and justify their generalization. In addition, Becker and Rivera (2006) found that students who had facility with both figural ability and variable fluency were more capable of noticing the structure, and developing and justifying generalizations. By contrast, students who fail to generalize tend to begin with numerical strategies (e.g., guess and check) as they search for generalizations and lack the flexibility to try other approaches (Becker & Rivera, 2005). This has implications for the types of instructional practices that occur in classroom contexts. It is suggested that instruction that includes verbal, figural and numerical representations of patterns, and emphasises the connections among these representations assists students to reach generalizations (Becker & Rivera, 2006). An ability to think multiplicatively has also been shown to assist students generalize figural representations of linear patterns (Rivera, 2013).

Theories pertaining to noticing structure and reaching generalisations. Results from Radford's longitudinal study of 120 8th grade (typically 13–14 year-olds) students over a three year period delineated three types of generalization that emerged from the exploration of figural pattern tasks: *factual; contextual;* and *symbolic* (Radford, 2006). The first structural layer is factual: 'it does not go beyond particular figures, like Figure 1000'. The generalization remains bound at the numerical level. Expressing a generalization as factual does not necessary mean that that is the extent of student's capability. It may simply be that this level can answer the question posed by others or the context in which algebra is needed (Lozano, 2008). The second layer is contextual; 'they are contextual in that they refer to contextual embodied objects' and use language such as *the* figure and the *next* figure. Finally, symbolic generalization involves expressing a generalization through alphanumeric symbols. The suggested criteria that can be used to assist teachers to distinguish these levels of early algebraic reasoning are: the presence of entities which have the character of generality; the type of language used; and, the treatment that is applied to these objects based on the application of structural properties (Aké, Godino, Gonzato, & Wilhelmi, 2013). The latter refers to how students express this generality. Aké et al. (2013) suggest that algebraic practice involves two crucial aspects, namely, being able to use literal symbols as a general expression and relate this expression to the visual context from which it is derived. In addition, with growing patterns gesturing between the variables (e.g., pattern term, pattern quantity) in conjunction with having iconic signs to represent both variables (e.g., counters for pattern term and cards for pattern quantity) helped 7–9 year old Indigenous students to identify the pattern structure (Miller & Warren, 2015).

Rivera (2013) from the results of his longitudinal study with 2nd grade to 7th grade students begins to provide insights into how these shifts in thinking occur, from the figural representation, to the factual, contextual and symbolic generalizations. He theorises that these shifts involve toing and froing between thoughts and pattern, and fundamental to this movement is the role of abduction and induction. At the initial stage, from a limited sample set, a generalization is abduced or inferred and a hypothesis constructed. Induction involves testing this hypothesis through intensive experimentation. As students obtain terms for larger positions (e.g., step 10 and step 100), they review their generalization and make necessary adjustments. Thus a combined abduction-induction process allows students to state their generalization. Rivera adds rigour to this process by suggesting the conditions required to be in place in order to help students and teachers evaluate their generalizations. These are: The generalization must 1) Be *Non-monotonic*: the generalization that offers the best explanation can still be shown to be false if additional or different assumptions are made. 2) Deal with the *cut off point:* the generalization can explain why the stated generalization that is based on a few examples hold for the whole population. 3) Allow for *vertical extrapolation*: the generalization must support conclusions for sequences of values beyond the values that are already known. 4) Accommodate *the eliminative dimension*: the generalisation has be chosen from several plausible ones and provides the best understanding of the pattern beyond what is superficially evident. These condition stem from the research of Psillos (1996) and Peirce (1960).

Reaching generalization from figural patterns is a complex and difficult process for many young students. From the results of interviews conducted with 19 7 year-old students, Rivera (2011) explored the use and implications of parallel distributing processing to begin to explain the differences between their ability to generalize the structure inherent in figural patterns. Underpinning this theory of cognition of learning is that ‘knowledge emerges and is stored in connections among neuron-like processing units with experiences and learning altering, strengthening and continuously making adjustments in connections among units’ (Rivera, 2013, p. 100). The complexity of the model reflects the complexity entailed in students thinking as they search for generalizations. One advantage of the model is that it begins to take into account the notion of context (and prior learning/connections) as we explain the differences between students’ capabilities. In addition, it has been shown that students activate and coordinate a number of different semiotic systems when exploring figural patterns. They engage in oral speech (utterances), draw figures, construct patterns, and use iconic gestures (e.g., Chen & Leung, 2012; Sabena, Radford, & Bardini, 2005). The specific role this synchronization of these systems plays in the objectification of knowledge, and in particular as students move through the three types of generalization needs further investigation.

The transition to noticing the structure of quadratic patterns. The difficulties that students exhibit when generalizing figural linear patterns appear to be compounded as they move into figural quadratic patterns. In an empirical study involving 50 talented students aged 12–14 years Amit and Neria (2008) found that 23 students used an additive strategy when finding successive terms in the pattern. These strategies encompassed drawing other terms in the pattern and counting or using tables and lists, and tended to result in the generation of recursive generalizations.

Aligning with the findings of research incorporating linear figural patterns, the 14(28%) students who were successful in reaching a global generalization used a visual-based approach (i.e., visualised the growth in the pattern). By contrast, Chua and Hoyles (2012, 2013, 2014) reported that 93(56%) of similar aged students investigating a similar quadratic figural pattern reached a correct global generalization. They conjectured that this is the result of students' prior experiences in algebra, specifically the teaching of number patterns followed by an introduction to the concept of a variable. This conjecture aligns with Sigley, Maher and Wilkison's (2013) results showing that being introduced to the technical language of algebra in conjunction with formal notation assisted the 11 year old student, who was the focus of their study, to correctly articulate his global generalization and link it to symbolic notation.

Concluding comments. The findings of the 2005–2015 research with regard to noticing and representing pattern structure are significant for four reasons. First, while noticing and representing the structure of patterns is a complex process, these findings show that young students are capable of engaging in pattern activities and expressing the pattern structure as generalizations. This practice involves two crucial aspects, namely, generating a general expression for the pattern and relating this expression to the visual context from which the pattern is derived. Initial representations of the pattern (e.g., tables of a values) and the visual cues inherent in the pattern can influence this practice. Additionally, the development of specific language to describe the pattern and an increase in fluency in using variables can help students to express their generalizations. Second, this research has produced a number of theoretical frameworks that will guide the research that occurs in the future. The two main dimensions further elaborated in this research pertain to (a) the levels students pass through as they notice pattern structures and express these structures in a symbolic form (*factual, contextual, symbolic*) and (b) the role combined abduction-induction processes play in helping students move through these levels. Both these dimensions not only assist teachers to evaluate what students know but also inform the types of instructional practices that occur within the classroom context. Third, this research has reaffirmed the finding that expressing generalizations in symbolic notation is not a necessary condition of thinking algebraically. Algebra can be 'practiced by resorting to other semiotic systems and signs' (Radford, 2006, p. 3). The findings also suggest that prior experiences with less complex patterns (e.g., number and linear patterns) influence students' ability to generalise the structure of more complex patterns (e.g., quadratic patterns). Fourth, this research has generated a number of directions for future research in this area: How do we help young students develop their mathematical language and visual capabilities as they progress through the elementary years? How do we help older students transfer this knowledge to more complex patterns including quadratic patterns? What representations and pattern sequences assist these processes?

Working with Variables, Expressions and Equations – Arithmetic Thinking

Although it is now well recognized that algebraic thinking in the early grades can occur without the need to use letter-symbolic algebra (Kieran, 2006), students' understanding of the structure of arithmetic and associated use of arithmetic symbols still impacts on their ability to effectively engage in letter-symbolic algebra. For example, past research has presented many examples of how adolescent students hold a persistent belief that the equal sign is a syntactic indicator for a place to put the answer. Additionally, many of the misconceptions of the meaning of a variable persist (e.g., Lim, 2007; Trigueros & Ursini, 2008). It is also recognized that this could be due to the types of activities that are occurring in the early grades. Thus in the last ten years attention has been drawn to this issue, particularly in terms of investigating students' ability to form and manipulate equations, and solve inequalities. The two types of equations utilised in the 2005–2015 research were equations containing only numeric symbols (arithmetic equations) (e.g., 5+ = +7) and equations containing alpha-numeric symbols (algebraic equations) (e.g., 2x + 3 = y + x).

Working with variables. The 2005–2015 research with regard to variables has evidenced that young students can engage in the concept of a variable without the use of letters. The inclusion of visual-gestural cues, such as the sign for 'secret' proved important for deaf students (Fernandes & Healy, 2014) understanding of a variable. Additionally, Khosroshani and Asghari (2013) showed that the notion of a *specular* number (a number that is *spe*cific to the user but is treated as a partic*ular* non-specific number) helped pre-schoolers to engage in algebraic thinking. It also needs to be acknowledged that the symbols and letters commonly used in representing variables have emerged from particular historical and cultural contexts. For Australian Indigenous students, allowing them to create symbols that are culturally appropriate and personal appears to be an effective way to introduce them to working with variables (Matthews, Cooper, & Baturo, 2007).

However, evidence also suggests that the use of letters and symbols does not necessarily mean that students understand the notion of a variable (Hewitt, 2014). In his study with 12–13 year old students, Hewitt (2014) found, through probing students' statement that letter/symbol could be 'any number', that many responses exhibited the 'natural number bias', interpreting a letter as a natural number. In addition, for some, the natural number value it could be was mitigated by the ease of calculation that could occur, for example, 'it can't be 572' or 'it is even'. Christou and Vosniadou (2009) suggested that a reorganisation of students' initial knowledge of number needs to produce a conceptual change so that students can recognize a variable as a symbol that can stand for any real number.

Interpreting, manipulating and generating expressions. The difficulties that many students experience in both arithmetic and algebra in interpreting and manipulating expressions involving more than one operation are well documented. Van Hoof, Vandewalle and Van Dooren (2013) claimed that secondary students in their second year (14 years of age) also interpreted a letter as a natural number rather than a rational number which led them to make consistent errors when determining whether expressions such as $2 \times m > m$ are always true. Additionally, students continued to exhibit the propensity to calculate all arithmetic expressions from left to right, a problem that Gunnarsson, Hernell and Soonerhed (2012) found in a large proportion of 169 students aged 12 to 13. However, intervention involving tasks where brackets were used to emphasise the precedence of the operations (e.g., $5 + (3 \times 2)$) did not result in a significant number of these students transferring from a left to right strategy to using precedence rules when computing arithmetic expressions without brackets (Gunnarsson et al., 2012).

How students manipulate and generate equivalent algebraic expressions is also guided by student's structural sense of arithmetic. Geraniou, Mavrikis, Hoyles and Noss (2011) showed that the use of pattern-based activities involving figural growing patterns helped 11 and 12 year-old students generate and justify equivalent expressions. They also identified three main categories that students used to justify the equivalence of their expressions: Structural Justification for Equivalence (focussing on structural aspects of the figural pattern with little reference to its symbolic rule); Symbolic Justification (focussing on both the symbolic rule and the figural pattern); and Empirical Justification (focussing solely on the numerical aspect of the rule).

Finally, Meyer (2014) conjectured that, when 12 year-old students manipulated expressions, they utilized two different processes: Giving relevance (relating the certain parts of an algebraic expression to each other, while neglecting other parts); and, Basic structure (recognizing the basic structure of the expression together with how it is represented symbolically). For example, in an expression like ab + ac + ab a student may *give relevance* to the two ab's while ignoring ac and reformulate the expression as 2ab + ac. By contrast, another student may recognise the basic structure of the expression (each term is a multiple of a) which might result in the transformation of the expression to a(b + c + b).

Understanding equality and inequality. The 2005–2015 research with regard to equality and inequality resulted in two broad findings. First, the types of representations students experience can influence their ability to form an equation and recognise the equivalence between the equations. For example, five year-old students successfully used the balance scales to model arithmetic problems in real world contexts as equations with more than one value on each side (Warren, 2007). Additionally, Carlo and Ioannis (2011) found that using brackets to encompass each side of an arithmetic equation (e.g., (5+) = (+7)) with 2nd and 3rd grade students helped them to 'see' the unity of different terms connected by a sign or operation, that is, the equivalence between the equations expressions.

Second, even if students are successful in representing and manipulating equations, this knowledge does not necessarily positively impact on their ability to understand and handle inequality (Verikios & Farmaki, 2006). The findings of this later research suggest that this is due to the fact that the manipulations required to represent and solve inequalities do not align with those used for equalities. Finally, teachers' perceptions of their students' ability with regard to the algebraic concept of an equation do not necessarily align with their capabilities (Alexandrou-Leonidou & Philippou, 2005). Students are often more capable than some teachers imagined.

The influence of teaching. Teaching arithmetic for algebraic purposes can have a positive impact on students' growth in mathematics. Pittalis, Pitta-Pantazi and Christou's (2014) empirical study of 204 6-year-old students showed that these students' growth in algebraic arithmetic (understanding of patterns, equations and functions) over an eight-month period had a direct effect on their growth in conventional arithmetic, and an indirect effect on their growth in elementary number sense. Teaching arithmetic for algebraic purposes can also assist at risk students to transfer their arithmetic knowledge to algebra contexts. Livneh and Linchevski (2007) in an empirical study with at risk 7th grade students showed that intervention focussing on developing an understanding of the structure of algebraic expressions in arithmetic contexts entailing arithmetic expressions prevents students from making structural mistakes in compatible algebraic expressions. The results of a post-test at the completion of one years teaching without intervention indicated that the students at risk were unable to meet the requirements of a basic algebra course by the end of their first year of algebra. In the second year of the study, 7th grade students (aged 12 years) at risk participated in a purposely designed intervention consisting of items, such as, "Is 75 – 25 + 25 equal or not equal to 75 – 50?" This was considered to be compatible with the algebraic task: "Is 16 – 4x + 3x equal to or not equal to 16 – 7x?" The results of the post-test at the end of this intervention evidenced that these students could successfully engage with compatible-algebra tasks (tasks that mirrored the arithmetic tasks utilized in the intervention). However, these students failed to show significant progress in algebra tasks that were not compatible with the numerical tasks used in the intervention.

If instruction is appropriate, young students can learn to understand powerful mathematics structures such as the backtracking (unwinding) principle and the balance principle (Cooper & Warren, 2008). In their five-year longitudinal study with 7 to 11 year-old students, Cooper and Warren (2008) showed that the combination of balance and number line models was powerful in assisting these young students to determine that change resulting from addition–subtraction requires the performance of the opposite change (subtraction-addition respectively of the same amount) if one wants to return the expression to its original state. This mathematical structure underpins solving equations using the backtracking (unwinding) principle and balance principle.

However, Eisenmann and Even (2008) found that the same teacher does not necessarily enact the same curriculum materials in a different classroom. The results of an intense study on a 7th grade teacher working in two different classrooms showed that the discipline problems in one of these classrooms resulted in them engaging in fewer global/meta level activities (e.g., activities involving generalizing, problems solving, proving and justifying; see Kieran, 2004), activities that are seen as at the very heart of algebraic thinking.

With regard to texts and the importance they place on developing students understanding of the structure of arithmetic, Demosthenous and Stylianides (2014) in their intensive study involving 2814 tasks from a series of 4th grade to 6th grade Cypriot texts discovered that only 10.7% of these tasks were algebra related and less than 12% of these attended to exploring the structure of arithmetic and generalizing its arithmetic relations. By contrast, in their study involving 60 10–11 year-old students, Slovin and Venenciano (2008) reported that the 19 students who had prior experience with their Measure Up Curriculum, a program built on the theoretical framework developed by Elkonin and Davydov (1966) and focussing on relationships among quantities and the use of literal symbols from the first grade, were more capable at working with variables.

Concluding comments. The findings of the 2005–2015 research with regard to working with variables, expressions and equations are significant for four reasons. First, young students can engage in the concept of a variable provided they are allowed to use signs and symbols that are culturally and developmentally appropriate (e.g., own invented signs, spectacular or secret numbers). Additionally, it seems that focussing on relationships amongst quantities (e.g., lengths and volumes) and using literal symbols from the first grade, helps these students later understanding of the concept of a variable. Second, students' lack of understanding of the structure of arithmetic and associated use of symbols persists in negatively impacting on their understanding of equations. While the types of representations that students experience at a young age can help them 'see' the equivalence between each side of an equation (e.g., balance scales to model equations, brackets to show the different expressions on each side of an equation), many students still exhibit a natural number bias when assigning meaning to letters and symbols. In addition, it seems that, as students grow older, the more resilient these misunderstandings become. Third, the knowledge students gain with regard to successfully representing and manipulating equations does not necessarily transfer to nor is applicable for inequality contexts. Fourth, teaching arithmetic for algebra purposes can positively impact directly on students' understanding of algebra and arithmetic, and indirectly on their growth in elementary number sense. In particular, intervention that focuses on developing an understanding of the structure of algebraic expressions in arithmetic contexts with at risk students who are beginning formal algebra study can assist them in successfully engaging in compatible algebraic tasks.

This research, and in particular the fourth point, has generated a number of directions for future research. Although the impact that students' understanding of the structure of arithmetic remains problematic with regard to their ability to transfer to formal algebra, there has been little research particularly relating to the arithmetic curricula and teaching interventions that can assist this transition to occur smoothly. The little that has transpired clearly shows that teaching arithmetic for algebra purposes can positively impact on both arithmetic and algebra.

THE DEVELOPMENT OF ALGEBRAIC THINKING IN STUDENTS 15-YEAR-OLD AND UP

In this section we provide a panoramic view of the research on algebraic thinking and on the learning and teaching of algebra in students 15-year-old and up, with special attention on that developed by researchers in the PME community in the last ten years. As in previous section, PME papers were classified according to their focus and the methodology employed. Table 2 presents the frequency of the PME papers according to their focus and their data collection method. As can be observed, the great majority focus on students' learning using a qualitative approach.

Table 2. PME papers – Algebraic thinking 15 year old students and up

Focus	*Method*				
	Qualitative	*Quantitative*	*Mixed method*	*Theory*	*Total*
Student learning	21	4	1	3	29
Teaching algebra	2			1	3
Teaching & learning	2		4		6
Total	25	4	5	4	38

The research studies informing this section are grouped according to the topics mentioned in the introduction: Focus on variables; Generalization and skills for generalizing; Equations: Solutions and meanings; Related variables and functions; and Structure sense.

Focus on Variables

Algebra is the basis of all other fields of mathematics, and also of natural and social sciences and engineering. Solving problems in algebra involves abstraction and the capability to interpret and use symbols, together with the possibility to generalize, model different situations, and use rules to perform symbol manipulation.

Although it is true that the introduction of early algebraic thinking is not necessarily linked to the use of literals (Kieran, 2004) and, that "using letters does not amount to doing algebra" (Radford, 2006, p. 3), using and interpreting symbols to designate the objects of algebra is fundamental to the development of algebraic thinking and to the possibility of using them outside the classroom. At upper school levels this implies working with literal symbols to represent variables. Otherwise, the solution of problems would become troublesome. This is why this conceptual study area has been and continues to be of interest to researchers.

According to many studies, the development of algebraic thinking goes hand in hand with the development of the concept of variable, a multifaceted concept linked to the different facets of algebra (generalizing, problem solving, structure analysis, modelling, analysing related quantities). For more than forty years researchers have pointed out the many difficulties students encounter in grasping the essentials of the notion of variable and in working flexibly with its multiple uses at different levels of abstraction. When solving problems in school algebra the different uses of variable (unknown, general numbers, related variables, parameters) very often appear together, and the same symbols are used to represent them. Students are expected to grasp the essence of each use, work with each of them and shift fluently from one to the other as required by a specific task. Some researchers have insisted on the need to distinguish the different uses and aspects characterizing algebraic variables arguing that a more explicit distinction of the diverse meanings associated with the word 'variable' would help students make sense of the symbols used to represent them.

Research developed in the last thirty years has shown that each use of variable is linked to specific epistemological and didactical obstacles. It has been suggested as well that when algebra is taught taking only one specific use of variable as the starting point and central focus, the possibility of flexibly moving between its different uses and the richness derived from the relationships between them is lost or obscured and students' understanding of algebra remains limited (see Kieran, 2006). These difficulties continued to be studied in the last ten years and recent studies have focused on how students' can be helped to give meaning to variables. In spite of the important role played by parameters, these seem to have been neglected by algebra researchers. Since in almost any problem situation involving a variable its multifaceted character is present, a deep understanding of this concept becomes a source of richer comprehension of algebra and mathematics in general.

Several studies explored the appropriateness of different environments and approaches to promote the creation of meaning and better understanding of variables. Lim (2007), for example, created opportunities for students to attend to meaning and to use numbers as a platform to investigate algebraic expressions and structures. Through a case study this researcher illustrates the feasibility of helping 11th grade students improve their algebraic thinking, in particular, moving from manipulating symbols in a non-referential symbolic manner to reasoning with symbols in a goal-oriented manner, from association-based prediction to coordination-based prediction, and from impulsive anticipation to analytic anticipation. Wille (2008) signalled that the versatility of students' thinking about variable was enhanced when they experienced its different aspects. The possibilities of understanding the different uses of variable in parallel with each other, in real contexts and multi-representational environments was underlined by Tahir, Cavanagh and Mitchelmore (2009). They found that studying variable as a function in parallel with variable as a generalized number using multiple representations and real contexts helped students to come to a more complete meaning of the term 'variable'. Their results also showed a reduction in students' misconceptions and an improvement of their performance when using this approach. In a study about an effective strategy to help students develop ideas about the solution set of systems of linear equations, using modelling, Trigueros, Possani, Lozano and Sandoval (2009) suggested that students' strategies were strongly related to their flexibility in moving between the different uses of variable. Those students who showed proficiency when working with variables and those who developed this flexibility during the course employed richer strategies; were able to use them to model and work with different types of problems; and, were able to interpret different types of solution sets, including those containing free variables and restrictions (that is, variables in solution sets that can take any value in a given set of real numbers and that are associated to systems that have an infinity of solutions, for example t in $S = \{\mathbf{x} \in R^3 / \ x = 2 - 3t,\ y = 2t,\ z = 1-t,\ t \in [0, 25]\}$), which have proven to be difficult for most studies, and to use them to analyze both real situations and models' solutions.

Little research has been conducted internationally using the same tools to explore students' capability to work with variables in order to establish similarities and differences in achievements and difficulties across different contexts and locations. The aim of these international studies is primarily to assist countries understand their own education systems by setting their strengths and weaknesses against the backdrop of those of other countries. This was the purpose of Alvarez, Gómez-Chacón and Ursini (2015) research. They analyzed 8th and 11th grade students' responses to a questionnaire testing their understanding of algebraic variables. The results provide evidences that might help the participating countries to revisit their curricula, focusing on their strengths and weaknesses and the support provided to students to develop the ability to think in algebraic terms.

Researchers in the last ten years have been less interested in students' use and understanding of parameters in spite of previous studies reporting that students have many difficulties when they encounter parameters in algebraic expressions (see Kieran, 2006). Bardini, Radford and Sabena (2005) investigated 11th graders' cognitive difficulties when working with parameters in the context of the generalization of patterns and showed how the semiotic problem of indeterminacy, a central element of the concepts of variable and parameter, reveals students' weak understanding of letters and algebraic formulas. Moreover, the many difficulties students of different school levels have with the interpretation, manipulation and symbolisation of parameters were documented by Ursini and Trigueros (2011). Students in this study conceived parameters as products of second order generalizations, that is, product of generalising first order general statements (for example, the equation $3x^2 + px + 7 = 0$ involves a family of quadratic equations). Their results also showed that students need a clear referent or statement that gives meaning to parameters to be able to work with them; otherwise, they perceive parameters as general numbers and have many difficulties handling parameters when they encounter them in any type of algebraic expression.

Generalization and Skills for Generalizing

Algebraic thinking is characterized by the capability to generalize and express generalization. Many researchers have stressed that generalization is one of the paths to algebra (e.g. Mason, Graham, & Johnston-Wilder, 2005). According to Carraher, Martinez and Schliemann (2008, p. 3):

> Mathematical generalization involves a claim that some property or technique holds for a large set of mathematical objects or conditions. The *scope* of the claim is always larger than the set of individually verified cases; typically, it involves an infinite number of cases (e.g., ''for all integers''). To understand how an assertion can be made about ''all x'' we need to consider the grounds on which the generalization is made. (p. 3)

Fundamental to the act of generalizing is the learner. A context that has gained the greatest attention by PME researchers in the last ten years is the patterning context. In this particular context, Radford (2006) stated that:

> Generalizing a pattern *algebraically* rests on the capability of *grasping* a commonality noticed on some elements of the sequence S, being aware that this commonality applies to *all* terms of S and being able to use it to provide a direct *expression* of whatever term of S. (p. 5)

As delineated in the first section of this chapter, many studies have shown that both algebra beginners and more advanced students can deal successfully with particular cases of patterns, but have serious difficulties in generalizing and expressing the relationships in terms of algebraic language. Some of the reported dimensions that contribute to these difficulties for young 4th grade students are: the lack of spatial visualisation techniques (Warren, 2005); the lack of appropriate generalizing strategies (Moss & Beatty, 2006a, 2006b); and, difficulties in using algebraic language to express generality (Warren, 2005). Many of these difficulties have also been shown to exist in secondary students (Ursini, 2014; Alvarez, Gómez-Chacón, & Ursini, 2015). These results suggest that many students across all ages tend to lack the capability to reflect on their own actions and become conscious of them. They lack metacognitive abilities and the capability to use algebraic language as a tool to communicate mathematically.

In the PME tradition of algebra as generalization to develop and express mathematical proofs and modelling situations (Kieran, 2006), Boero and Morselli (2009) considered algebraic language as a system of signs and transformation rules useful to generalize arithmetic properties. An adaptation of Habermas' construct of rational behaviour is used by Boero and Morselli (2009) to describe and interpret some of the students' difficulties and mistakes, and to provide indications for the teaching of algebraic language. Their analysis showed that goal oriented reasoning and using verbal language are necessary in order to perform the actions needed in proving, modelling and problem solving.

Regarding teacher education programs, Hallagan, Rule and Carlson (2009) considered that strategies that involve inquiry, problem solving, and critical thinking helped pre-service teachers to focus on interpreting and making sense of the role of symbols involved in the generalization of patterns. In this same line, Radford (2006) has suggested that teachers and teacher educators should be aware of students' practices in order to distinguish algebraic generalization in students' activity from other forms of work with the general, which according to his previously mentioned definition, are not truly algebraic. He also warns that teachers must be equipped with knowledge to be able to distinguish different approaches to generalization.

Stressing the importance of focussing on teachers' explanations of students' responses to mathematical tasks, El Mouhayar and Jurdak (2015) explored teachers' arguments to explain students' responses to pattern generalisation tasks. They identified four different perspectives teachers assume (student lens, teacher lens, mixed teacher, and inability to explain students' responses) stressing that the pattern generalization types mediated teachers' perspectives.

Equations: Solution and Meaning

The PME community has devoted decades of research to the identification and the analysis of students' difficulties in interpreting and manipulating algebraic expressions. Attention has been paid to students' approaches to solve equations (particularly linear, less research has been carried out on students' solving of quadratic equations), systems of equations, and inequalities. Students' procedures and strategies to deal with such tasks have also been analyzed from different perspectives. Well-grounded teaching experiments have been designed, with and without the support of technology, to help students overcome their difficulties and to help them construct meaning for algebraic expressions, equations and solution procedures (see Kieran, 2006). In spite of these efforts, and probably due to different kinds of obstacles (from bureaucratic and political, to economic, socio-cultural and cognitive, from teachers' and parents' academic preparation, attitudes and beliefs, to curricular organization) students' difficulties are still present in most classrooms around the world and at different school levels. Research efforts have continued in the same lines and concerns about teaching equations, and the influence of students' difficulties with equations in university mathematics education have emerged (Borja-Tecuatl, Trigueros & Oktaç, 2013).

Efforts to deepen our understanding of the possible causes of students' difficulties and to look for ways to help students interpret equations and their solutions have continued during the last ten years. Nogueira de Lima and Tall (2006) and Nogueira de Lima (2007), for example, investigated Brazilian 14- to 16-year-old students' interpretation of the concept of equation and its solution set. Their findings show the influence of previous arithmetic studies, previous algebra experience and teachers' beliefs about algebra on students' conceptions. Based on their results they stress that teaching practices focussing on a single solution procedure limits the development of students' flexibility to give meaning to equations, to the solution set, and to their procedural knowledge. The question of meaning was also tackled by Caglayan and Olive (2008) who studied 8th grade students' use of a representational metaphor for writing and solving equations in one unknown. They reported that only one of 24 students was able to construct a "family of meanings" to make sense of equations and solutions, and to connect algebraic expressions to representational metaphors when negative quantities were involved. Based on the analysis of the knowledge used by a teacher and her 8th grade students when generating and evaluating equations to model word problems, Caglayan and Olive (2008) extended previous results on meta-representational competence and stressed the importance for teachers to develop the capacity to recognize and discuss students' criteria when choosing representations.

Students' cognitive tendencies when dealing with different tasks involving unknowns has continued to attract researchers' attention. Filloy, Rojano and Solares (2008) found that the cognitive tendencies identified when students operate with a single unknown reappeared when they were learning methods to solve two-unknown linear equations systems, showing that the reference to and the sense of the different representation of unknowns must be reconstructed when facing new types of problems.

The concern for students' understanding of equations and solution sets has led researchers to focus on university students' capabilities to use equations and inequalities, to manipulate them and to interpret solution sets. Studies have focused on the development of algebraic thinking when students encounter university mathematics topics. Researchers have found that algebra is central in the learning of advanced mathematics, but it can act as a "key and a lock" at the same time in understanding advanced concepts such as limits of functions (Alcock & Simpson, 2005). They insist that when students used mathematical notation as a way to apply mathematical procedures, without making sense of expressions or of the goal of manipulations, for example to solve inequalities, algebra can become a lock to their success. By contrast, for students who were able to interpret and use the different variables involved in definitions or inequalities, algebra acted as a key to their understanding of new concepts.

Attention was also paid to teachers and teachers' development in relation to their content knowledge and the way they teach equations. Findings suggest that their poor understanding of mathematical structures and their knowledge about students influence their teaching decisions. The impact of teachers' training programmes in their teaching practice and how innovative teaching strategies can support students' learning of algebra have been explored as well. Koirala (2005), for example, found that using mathmagic, a game in which students are invited to play with numbers ("think of a number", "add 10", "multiply it by 3", and so on), can help low-performing 14 years old students develop confidence and interest in learning basic algebraic concepts and enhance their understanding of variables, expressions and equations.

Related Variables and Functions

Function is a central concept in mathematics and has a significant role in mathematics education. Literature on mathematics education has paid in the past, and in the last decade as well, a lot of attention to this concept. Studies have used different theoretical perspectives and have reported the many difficulties students, pre-service and in-service teachers face in understanding and learning the meaning of function, such as, paths to understanding functions, process-object conceptions or the use of different representations (Elia, Panpura, Eracleus, & Gagatsis, 2007; Gerson, 2008; Bayazit, 2011). Specific tools based on results of diagnosis questionnaires have been used in the design of online applications that can help and give feedback to students, both in the case of algebraic manipulation and in drawing and interpreting the graphs of functions. These applications can also help teachers to design specific strategies to help students to better understand these central concepts.

Research has shown that students can understand the correspondence between numbers, independently of the representation used, but find a dynamic conception between numbers, which includes variation, difficult (Ursini & Trigueros, 2011). To help students make sense of variation, some researchers have proposed using a method where each operation is explicated so that students can reflect on the situation and "make tacit meanings explicit" (Thompson, 2008). Other researchers stress that using an appropriate methodology, such as games, modelling situations in context or using metaphors from everyday experiences to introduce functions to students can help them develop the notion of correspondence into that of covariance (Francisco & Hähkiöniemi, 2006; Dogan-Dunlap, 2007; Pierce, 2005). Teaching strategies based in knowledge theories, such as APOS theory, have also been successful in assisting at risk students make sense of the concept of function (Dubinsky & Wilson, 2013). Students participating in this study were able to develop their algebraic thinking from beginning with their own verbal explanations to ending with writing functions in symbolic forms.

For some researchers, the main difficulty with the understanding of functions is the fact that they present different facets depending on the representations used (Nyikahadzoyi, 2006; Adu-Gyamfi, Stiff, & Bossé, 2012). All these authors suggest that flexibly working with different representations, being able to understand the characteristics of functions in each representation, and relating different representations to each other are fundamental to the learning about the concept of a function.

Some researchers have indicated that attention to the role played by symbols and the use of models has proved to be successful in fostering students' capabilities to make sense of related variables. These researchers have focused on specific types of functions, for example, quadratic, exponential or trigonometric functions (Pierce, 2005; Francisco & Hähkiöniemi, 2006; Panasuk & Beyranevand, 2010). However, after all these years there has been no consensus on how to help students understand the concept of function, and on how to overcome the difficulties inherent in its learning. Nilsen (2015) focused on the way functions are introduced by teachers in both lower and upper secondary school, and the usual gap between formal explanations provided in textbooks. This author argued that the introduction of functions is done without explicitly considering mathematical aspects like the range and domain, or the uniqueness property, and that dependent and independent variables continue to prove problematic. He concluded that examples and explanations provided should underpin and support the mathematical properties of function.

All the previously referred-to studies have analyzed one variable functions. More recently, however, this field of research has expanded and two variable functions and parametric functions have been investigated. These studies have shown that generalization of knowledge from one variable functions to other types of functions is not straightforward and that difficulties students experience with one variable function transfer to other types of functions. Results demonstrate that each generalization involves particular obstacles that need to be studied in depth and that effort is needed in order for students to construct a formal and more inclusive definition of function (Trigueros & Martinez-Planell, 2010; Martinez-Planell & Trigueros, 2012; Stalvey, 2014; Weber & Thompson, 2014).

Function is clearly one of the most important concepts in mathematics. The teaching and learning of functions has received a lot of attention. For example, strategies to help overcome students' conceptual difficulties have been investigated, and activities that promote students' reflection and formally make aspects of functions explicit have been developed. However, more effort is required both in research and in making research results available to a wider educational community in order to help a majority of students learn more deeply about functions.

Structure Sense

Students at different school levels and university levels, all have difficulties in transferring what they learnt in the context in which they first met different concepts to other unfamiliar contexts (Hoch & Dreyfus, 2006). This research emphasises the importance of making sense of manipulations, generalization, functions, and different properties of algebraic expressions. Previous studies have already called attention to the importance of making sense of the structure in algebra (see Kieran, 2006). But the meaning of "structure sense" was and continues to be under debate.

Arcavi (1994), for example, referred to symbol sense as a complex feel for symbols. He included in it an appreciation of the power of symbols, the ability to manipulate and to interpret symbolic expressions, and have a sense of the different roles symbols can play in diverse contexts. Other researchers have emphasized identifying structure in expressions or equations; identifying visually repeating characteristics of expressions; and, exhibiting versatility of thought as key dimensions of structure sense (Kirshner & Awtry, 2004, Tall & Thomas, 1991). Hoch and Dreyfus (2007) gave a precise and pragmatic definition of structure sense based on their studies with high school students. They defined structure sense as: recognising a familiar structure in its simplest form; dealing with a compound term as a single entity and through appropriate substitutions; and, choosing appropriate manipulations to make best use of a structure. They also underlined the importance of the substitution principle in this definition. In accordance with these criteria and based on their observations, Novotná and her colleagues (Novotná, Stehlíková, & Hoch, 2006; Novotná & Hoch, 2008), suggested definitions linked to university algebra. They also suggested that there is a relationship between high school and university algebra structure sense, and viewed high school structure sense as sub-components of university structure sense components. In addition, they underlined the importance of structures as part of mathematics in general, and in the learning of algebra in particular. Studies on students' structure sense have played an important role in research on algebraic thinking in the last ten years. This interest has been reflected in the PME community's work where the importance and development of structure sense for successful performance in algebra has been stressed, for example identifying the structure of equations or inequalities.

In the last ten years research on advanced algebraic thinking has become an important area of interest for mathematics education researchers. Aiming at finding factors that could explain students' success in the solution of complex algebraic equations and inequalities, Trigueros and Ursini (2008) analyzed in depth the approach followed by 36 university students working with equations and high achieving university students working with inequalities (Ursini & Trigueros, 2009). Their results showed that understanding the different uses of variable, including parameters, together with other factors that can be associated to structure sense, in terms of Hoch and Dreyfus' definition are crucial for success. These results led these researchers to extend the definition of structure sense to include understanding of variable. They considered this understanding as critical to successfully solving problems and as a starting point to developing the capability to work with problems requiring advanced mathematics. Using their definition, Trigueros, Ursini and Escandón (2012) analyzed the responses of 270 Pre-Calculus university students to six complex algebraic problems in order to find possible relations among all the included categories and their role. Using Implicative Statistics they found that, although all the categories appeared to be strongly interrelated, two categories: distinguishing that a problem involves the analysis of different cases and, a correct use of definitions, played significant roles in students' ability to use all the aspects considered in the definition of structure sense. Moreover, a flexible use of variables and the capability to interpret parameters played a dominant role when implication relations were studied. These were tightly linked to students' ability to identify algebraic structures and use definitions correctly. Thus, there is an urgent the need to take these implications into account in the design of activities that aim to foster structure sense in students.

Systems of linear equations are a central topic in the transition from elementary Algebra to Linear Algebra. Students are known to have difficulties interpreting the solution set of systems of equations, particularly when systems have an infinite number of solutions. Research results on this topic signal that these difficulties are related to students' difficulties in interpreting a variable as a dynamic entity, that is as a "changing entity" which can take different values (Ursini & Trigueros, 2011); with the notion of set as an object; with the distinction of the process of solution and the notion of solution set; and with the interpretation of different representations of solution sets (Afamasaga-Fuata´I, 2006; Trigueros et al., 2007). All these difficulties can be related to the lack of structure sense in terms of the definitions described above.

Focusing on students' development of structure sense, Hoch and Dreyfus (2006) used a questionnaire to analyze 165 high achieving 10th grade students' performance. Their results showed that most of these students did not use a high level of structure sense when solving exercises requiring the use of algebraic techniques, and that those students who used structure sense made fewer mistakes.

Searching for factors that could explain the lack of progress in the development of algebraic proficiency of students from 8th to 12th grade, Van Stiphout, Drijvers and Gravemeijer (2011) considered structure sense as part of what Arcavi defined as symbol sense (see above) in order to analyze test items. The analysis revealed that most students were not able to deal flexibly with the mathematical structure of more complex expressions and equations that involve, for example, the use of subtraction technique. These researchers emphasised that this is an obstacle for attaining a higher level of conceptual understanding which requires a shift of thinking. Using the anthropological theory of didactics, Chevallard and Bosch (2012) propose a structural approach that considers algebra as a tool to model different intra and extra-mathematical situations. These authors consider such an approach as a way to overcome the difficulties that the learning of algebra presents nowadays. Other authors have used the same theory, together with the ontosemiotic approach, to offer a model that takes into account the structure of algebra and that can be used in the teaching of algebra at the secondary level and to design richer teaching sequences. The emergence of structure sense in classroom interactions at the secondary level was analyzed by Janßen and Bikner-Ahsbahs (2013). Focusing on linear equations and functions' structure they studied the development of structure sense. They searched for crucial moments of objectification claiming that when it happens the object is accessible in other situations and students can identify algebraic structures.

All these researchers agree that even students, who can display proficiency when working with elementary algebra problems, may have difficulties in applying the techniques to complex problems. They also concur on the need to teach explicitly the abilities included in structure sense. Some recommendations were: using brackets to help students to "see" algebraic structure and make evident the presence of a new expression that could be considered as an entity or a new variable; working with examples where analysis or classification of problems in terms of their structural properties is the goal of the activities; making the role of variable and changes of variable explicit in classroom discourse; asking how definitions and properties can be used; asking students for the goal of the activity instead of the solution; and, stressing the importance of validation. Such approaches would lead students to engage in greater levels of reflection and analysis related to algebraic situations.

Additionally, the development of structure sense would provide students with a stronger foundation on which more complex and abstract algebraic thinking can be developed. For instance, research conducted in the last ten years has shown that many students finish Linear Algebra courses with a limited understanding of the main ideas of this discipline. Analyzing possible causes of difficulties and looking for ways to help students develop a richer understanding of concepts of this discipline has been the main concern of these studies. Researchers have underlined that most students can cope with the manipulations involved in solving a variety of Linear Algebra exercises, but that they do not develop an understanding of the concepts involved in such manipulations and are unable to apply them to problems that need competencies that go beyond rote manipulation (Hannah, Stewart, & Thomas, 2014). Other researchers have suggested that Linear Algebra difficulties can also be related to students' conception that a great effort is needed in its learning (Martinez-Sierra, García, & Dolores-Flores, 2015). Some researchers have related students' difficulties in understanding particular Linear Algebra concepts to the lack of formal thinking (Britton & Henderson, 2009; Wawro, Sweeney, & Rabin, 2011), described in the definition of structure sense as based in a complex feel of symbols and a flexible use of variables (Arcavi, 1994; Ursini & Trigueros, 2009).

However, other studies have pointed out that the need to work jointly with several algorithms also generates obstacles for students (Hannah, Stewart, & Thomas, 2014). In a study focusing on students' understanding of linear independence and dependence, Stewart and Thomas (2006) found that working with the solution algorithm for homogeneous systems of equations, together with the interpretation of the obtained solution set in terms of these concepts was an obstacle for most students. They also concluded that interpreting free variables in the solution set seems to be key to understanding linear dependence and independence as well as their relation to other Linear Algebra concepts such as span and basis.

Results of studies related to specific concepts (Plaxo & Wawro, 2015; Stewart & Thomas, 2008) underline that difficulties in Linear Algebra are related mainly to lack of understanding of definitions, and the use of properties in the solution of exercises when working with modelling or application problems. In spite of difficulties associated with the learning of abstract concepts, several research studies have shown that using models to approach abstract concepts can help students to make sense of definitions and apply their learning in the solution of complex problems (Zandieh & Rasmussen, 2010).

Algebraic thinking in Abstract Algebra courses and its teaching also received some attention (Novotná, Stehlíková, & Hoch, 2006; Hare & Sinclair, 2015). Novotná et al. (2006) were interested in students' understanding of algebraic operations and their structure. They found that while some students are able to abstract specific properties of one or more mathematical objects to form the basis of the definition of new abstract objects, other students constructed abstract concepts through logical deduction from definitions. They concluded that few students were able to reason inside a new structure spontaneously and to find a structure's properties. On this basis they developed a model for teaching binary operations and their properties as a first step to develop students' structure sense. Hare and Sinclair (2015) used semiotic theory to analyse teachers' signs in an Abstract Algebra course. They found that the act of pointing is important in 'underlining' objects and relations among them.

It is interesting to observe that the development of structure sense, both from the perspective of high school algebra and from that of university algebra plays a fundamental role in university students making sense of algebraic structures and applying concepts related to different algebraic structures. It is of fundamental importance to do more research on this topic in order to help students to develop structure sense so that they can progress in their understanding of advanced mathematics topics and apply this structure sense to the solution of formal and real problems.

ABOUT THE USE OF TECHNOLOGY

The use of technology continues to be an interesting and important topic of research in the mathematics education community. As new software is developed these questions always arise: How do we use it in the classroom? What kind of tasks need to be designed within specific technological environments? What are the results of its use in terms of developing algebraic thinking and learning? Research in the last ten years evidences that technology has had a small but significantly positive impact on students' learning. However, this impact is dependent on teachers' use of the technology, the classroom interactions that occur (e.g., Ursini & Sacristan,

Table 3. PME papers – Use of technology in algebraic thinking

Focus	*Method*				
	Qualitative	*Quantitative*	*Mixed method*	*Theory*	*Total*
Student learning	14		1	3	18
Teaching algebra	3				3
Teaching & learning	3				3
Total	20	0	1	3	24

2006; Rakes, Valentine, McGatha, & Ronau, 2010; Guzman, Kieran, & Martinez, 2011) and, the tasks' design (Johnson, 2015). Table 3 gives a glimpse of PME papers related to technology use and clearly calls for more research in this area.

Already well-known technologies, such as CAS, continue to be studied particularly in terms of the potential of instruction design that includes their use and the development of teacher training programs. Researchers have found that high expectations on this tool can be considered as naive and that experiences show the complexities involved in their use in the classroom (Trouche & Drijvers, 2010). In spite of this complexity, it has been found that the possibilities offered by CAS can enrich and extend students' and teachers' view of algebra. CAS can for example foster the emergence of algebraic reasoning or be used to the development of novel tasks (Kieran & Guzmán, 2009; Heid, Thomas, & Zbiek, 2013; Kieran & Drijvers, 2006; Kieran & Saldanha, 2005).

PME researchers continue analyzing if and how the use of specific technologies help to improve communication and understanding in the classroom. Among others, Sacristán and Kieran (2006) analyzed a student's difficulties in understanding notation for general polynomials using CAS. They found that by making conjectures and trying them in TI-92 Plus calculator the student improved in the use of general notation and eventually could make sense of the ellipsis sign. The role of teachers in orchestrating class discussion and fostering attention of students to overcome their difficulties shows how some approaches to classroom communication can be useful for teachers (Kieran, Guzmán, Boileau, Tanguay, & Drijvers, 2008).

The impact of CAS in students' learning continues to play a role in research (Kieran & Damboise, 2007; Solares & Kieran, 2012). Researchers have demonstrated the impact of CAS in improving weak 10th grade algebra students in being able to do procedures and understand concepts, or help students' articulate different perspectives to understand equivalence of expressions. These studies found that CAS had a positive impact on the development of procedures and on concept development. In another study (Lim, 2007) technology and its graphing potential was used to teach transformation of functions and their graphs to secondary school students. The impact on students' learning was examined using APOS theory. They compared results of students who used CAS with those who didn't use it in their classroom. They found that the group that used CAS were able to apply transformations on mental objects they had constructed to construct new graphs, while the other students needed to calculate specific points to graph its transformations.

Comparing teachers' beliefs, how they are reflected in their use of CAS in secondary school algebra instruction, and how they shape CAS algebra tasks has also been studied (Kieran, Tanguay, & Solares, 2011; Özgün-Koca, 2011). Researchers concluded that technology can help to change teachers' views and to make sense of specific algebraic content by working with appropriate practical and theoretical rich experiences where they can reflect, and discuss to shape these experiences for their classroom needs.

Other technologies have been designed and tested in terms of their potential in the teaching and learning of algebra. The dynamic metaphors of change and dragging, together with the process of naming when working with spread-sheets, was studied by Wilson, Ainley and Bills (2005). Their results appeared to support the evolution of meaning and notation for variable. The ReMath European project was developed to investigate the role of representations of mathematical objects offered by different Digital Dynamic Artifacts (DDA) when used in educational contexts. The DDA that received more attention in terms of research reported at PME meetings was a microworld called Aplusix (Nicaud et al., 2006). Studies on its impact on students' learning were conducted in different settings. Maffei and Mariotti (2006) found that a specific tool of the software, called detached step, played an important role in making students conscious of their errors and in helping them to reflect on and overcome their difficulties. Exploring the potential of the feedback component of Aplusix when teachers used it during discussion in the classroom, Maffei, Sabena and Mariotti (2009) found that teachers' questioning played a pivotal role in developing a semiotic chain starting from the DDA's signs that led students to give mathematical meaning to algebraic expressions. The development of this chain of interpretations was not linear, but it helped to maintain students' interest in asking new questions and, at the same time, promoted interactions among them and with the teacher. As a result, a process of unfolding the meaning of the tool feedbacks signs was developed in the group. In a teaching experiment conducted by Maffei and Mariotti (2013) to investigate how the Graph representation provided by this DDA could become a tool of semiotic mediation, they found that the Graph representation tool helped students to carry out algebraic manipulation and allowed them to refer to the mathematical meaning of equivalence class of algebraic expressions. Based in semiotic considerations Chaachoua, Chiappini, Pedemonte, Croset and Robotti (2012) analysed and compared DDA possibilities in terms of their impact on students' learning.

In another study, Kynigos, Psycharis and Moustaki (2010) performed a design experiment to explore 17-year-old students' construction of meaning and the use of algebraic like equations. These researchers focused on students' engagement while they constructed and controlled animated models on MoPix and in their possibility to make sense of structural aspects of equations underlying the models' behaviour. Through the experiment, students edited ready-made algebraic like equations and constructed new ones to assign particular behaviours to objects. Results showed how students developed different degrees of structuring and shifted gradually from a view of equations as processes to a view of equations as objects. According to authors reification was not a one-way process of meaning making; it was a dynamic process of understanding supported by the use of the technology.

The effects of teaching algebraic solving of word problems using Hypergraph Based Problem Solver (HBPS) software were studied with 15–16 years old students (González Calero, Arnau, Puig, & Arevalillo–Herráez, 2013). Their strategies were analyzed. González et al. (2013) found the emergence of a tendency to construct equations where one of the variables appears on one side of the equation, without using all the available information, and how they tried to use it to calculate values of the related variable. They attributed this tendency to the interpretation of the equal sign in the equation as a comparison between quantities rather than as a sign to do something.

Naftaliev and Yerushalmy (2009) investigated innovative uses of technology in the domain of school algebra. They compared the contexts for mathematics learning created by printed diagrams vs. interactive diagrams and video clips vs. interactive animations, when presenting to students an activity describing a motion situation and another requiring the description of a linear function. The two activities included an interactive diagram. They argue that the process of concept construction occurred as a result of the students' decision to change the representation of the data in the activity to build a focused collection and to expand the given representations, or build new ones. The ways in which sketchy interactive diagrams were used by students transformed sketchy information into an important component of conceptual learning.

Concern about teachers' developmental process to integrate technology into their classroom practices led researchers to study how teachers use their knowledge of algebra, to teach different concepts using a Dynamic Software (GeoGebra). Studies have found that stressing interconnection between knowledge of teaching algebra and knowledge of teaching with technology is fundamental for success. Johnson, for example, analysed the transition from variation to covariation creating environments that involve non-temporal changing quantities. Her experience led her to underline the importance of activities that provide students with opportunities to attend to multiple varying quantities from the same measure space as well as the role of the teacher in promoting interaction between students who have a different conception of variation and covariation (Johnson, 2015).

Overall, the studies related with the use of technology in the teaching and learning of algebra in the last ten years have focused mainly on how technologies can help students to make sense of mathematical signs, calculations and results. Interesting results have been found on how teachers can use technologies in their classroom to promote communication and reflection. Research findings suggest that when used in an active and participative way, technology can help teachers in improving students learning. It is evident that technologies have evolved towards more dynamic and powerful designs that can be used both by teachers and students to conjecture, explore, try out and make sense of the meaning of algebraic expressions, to find solutions of equations and to work with related variables in a variety of interesting and motivating contexts. The dynamics of technology development continue to pose challenges to researchers and students, while new questions about their use in the classroom and the nature of the tasks to be designed to promote students' interaction and learning keep arising.

FINAL REFLECTIONS AND FUTURE DIRECTIONS

The study of how we teach and learn algebra has played a role in research within the PME community. However, some epistemological questions arise, as we need, as a community, to define what exactly we mean by algebra, algebraic thinking, algebraic reasoning, and algebraic problems. Researchers have worked so far as if those terms had the same meaning for all researchers. However, some positioning facing the difficulties in learning this subject make it clear that this is not the case. More effort to clarify these meanings is needed. This would not only add clarity to research results, but also point out to different research and teaching strategies tuned to possible different positions.

Even though algebraic thinking continues to be an important topic for the PME community of researchers, this review has evidenced that, while research in the domain of early algebra has increased and intensified, research pertaining to secondary and tertiary students has diminished as compared with previous decades. Juxtaposed against this trend is the continued difficulty that secondary and university students have in this domain. In spite of the progress we have made in our use of technology to teach algebra and our understanding of student difficulties, the problems that have been evidenced in past research still exist. Algebra remains an important domain of mathematics, and is fundamental for advanced mathematics learning. How can students learn university mathematics without a thorough understanding of algebra? Thus, not continuing research in this university and secondary sector would be a great mistake regarding the future of mathematics.

With respect to generalization, the majority of the research that has occurred has been in the early algebra area using qualitative methods with a focus on generalizing the structure of patterns. There is a call to reignite the focus on generalizing the structure of arithmetic, particularly in regard to its relation to the structure of algebra. This review evidences tentative findings that intervention focusing on the structure of arithmetic has positive pay offs for students successfully identifying the structure of algebra. However, this review shows, on the one hand, that many of the misconceptions that students have with regard to algebraic structure (e.g., the order of operations) are entrenched at an early age, and on the other hand, a lack of interest from researchers about the role that proofs play in generalization.

Thus important questions requiring further research are: When and how should this integration occur? When and how should we be engaging students in conversations and experiences that focus on arithmetic for algebraic purposes rather than arithmetic purely for arithmetic purposes? Should this occur somewhat simultaneously or consecutively? What is the role of algebraic proofs in this integration?

Research in the last decade has clearly put forward the importance of development of structure sense at the secondary and tertiary levels and its relation to the study of advanced algebra and advanced mathematics. Research efforts in this topic should be a priority in the years to come. Particularly, studies about intervention strategies, comparison and the influence of institutional constraints using sound methodological approaches need to be encouraged.

There is evidence throughout this review that how teachers teach has an impact on what students learn. Results of many studies also underline the importance of working both with students and teachers stressing the relevance of developing students' capability to generalize, to interpret symbols, and to express their thoughts and generalizations in correct mathematical (and algebraic) language, to fully understand functions, and to use modelling through all the school levels so the students can become acquainted with parameters and other elements of structure sense. Even though Kieran insisted ten years ago on the importance of doing more research about teachers' algebraic thinking development, this issue has not received enough attention in this decade. The knowledge that teachers have about mathematical structures and the beliefs they hold with regard to students' capability to learn algebra influences the decisions they make as they teach. In addition, it has been demonstrated in this review that coming to an understanding of algebraic concepts requires the use of a range of representations and modes of learning. Thus more research focusing on these topics is imperative to forwarding students' learning of algebra and new comparative studies may shed some light both in curricula and how it is enacted in different countries and school levels.

Finally, while research in the fields of use of technology in the classroom and its influence on students learning and on the use of algebraic language has been somewhat intense over the last 10 years, little research has occurred in the areas of generalization, cognition and students capability of understanding functions, developing structure sense and in how technology is used by teachers and students in real classroom contexts. We are thus calling for a balance of our knowledge across all these areas since it is important if our aim is to continue to enhance the development of students' algebraic thinking from elementary school up to the university.

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