



The role of algebraic thinking in dealing with negative numbers

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Accepted: 1 July 2022 / Published online: 28 September 2022
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Abstract

To date, of the many studies on early algebraic thinking, none, to our knowledge, has examined the relationships between algebraic thinking and negative numbers. Students encounter persistent difficulties in dealing with these numbers, and we believe that these could be addressed through the development of algebraic thinking. We are particularly interested in relational thinking, a form of algebraic thinking involved in generalised arithmetic, characterised by the ability to identify the structure of an expression as well as the relationships between numbers. The idea of the ‘subtractive number’ has been highlighted in this context. The aim of the study was to investigate the role of relational thinking in dealing with negative numbers. We submitted a paper-and-pencil test to 166 grade 6 students in order to analyse their skills in operations with integers, as well as their relational thinking in questions relating to the compensation strategy in subtraction. We then examined the extent to which the students who answered the compensation questions correctly performed the operations with integers better than those who answered them incorrectly. Our results showed that students’ ability to see the subtraction operation as a ‘transformation’ involving a unary use of the minus sign appears to be a factor in their success in operations with negatives. Few students demonstrated this ability, yet it can be seen as an essential stage on which to base the progressive development of relational thinking.

Keywords Early algebraic thinking · Negative numbers · Relational thinking · Subtractive number · Transformation · Compensation strategy in subtraction

1 Introduction

Over the past twenty years, there has been widespread interest among researchers in early algebra and algebraic thinking (e.g., Blanton et al., 2019; Irwin & Britt, 2005; Kaput, 2008; Kieran et al., 2016; Radford, 2018). The focus has been on dealing with the well-known difficulties experienced by students in order to smooth the transition to formal algebra, which is usually introduced at secondary level.

Negative numbers are an integral part of the algebra curriculum at the start of secondary education. However, while the difficulties experienced by early secondary school students in operations with these numbers have been highlighted for many years (Gallardo, 2002; Herscovics &

Linchevski, 1991; Vlassis, 2004, 2008), they have almost invariably been omitted from research studies concerning algebraic thinking. The question that arises, therefore, is whether the development of algebraic thinking would also lessen the difficulties encountered by students when dealing with negative numbers.

Previous research has focused on identifying models for supporting students’ understanding of operations with integers (Subramanian, 2019; Whitacre et al., 2015). Without calling into question the value of this approach, our study dealt with this issue from an original angle by examining how the development of a certain form of algebraic thinking, in this case relational thinking, would make it possible for students to make sense of operations with negatives.

To investigate this question, a paper-and-pencil test was used to compare grade 6 students’ performance on two types of problem, namely, addition and subtraction with integers, and subtraction with natural numbers. In Luxembourg, grade 6 students have not yet received any instruction in negative numbers. The operations with negatives used in this study

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therefore consisted of ‘elementary’ operations of the type $a \pm b = c$, where a , b and c are integers.

In the following, we first discuss algebraic thinking and relational thinking, then examine the main difficulties experienced by students in operations with negative numbers. Finally, we relate these two analyses in order to examine more precisely how relational thinking could support learning about operations with negatives.

2 Algebraic thinking and relational thinking

Despite some obvious overlap between different researchers’ conceptualisations of algebraic thinking, there is no clear consensus at present on what algebraic thinking is or what components it may have (Kiziltoprak & Köse, 2017; Stephens, 2008). A large body of authors regarded generalising as the key characteristic of early algebraic thinking (Blanton et al., 2019; Kaput, 2008; Kieran et al., 2016). In this context, Kaput (2008) defined two main content strands involving algebraic thinking, namely, algebra as generalised arithmetic, and algebra as the study of functions involving generalising relationships between co-varying quantities. However, according to Radford (2018) generalisation is a common attribute of human thinking and consequently cannot capture the specificity of algebraic thinking. For this author, algebraic generalization is characterised by analytical reasoning that involves handling indeterminate quantities as if they were known. This type of reasoning has been extensively investigated in generalisation activities based on patterns (e.g., Radford, 2018).

As we are interested in this question in the context of operations involving integers and natural numbers rather than functions or generalisation activities based on patterns, we focused on the strand of generalised arithmetic. This strand "includes generalized arithmetic operations and their properties and reasoning about more general relationships", leading students to "build the syntactic aspect of algebra from the structure of arithmetic" (Kaput, 2008, p. 12). Two aspects stand out in particular from this characterisation, namely, relationships and structure. These constitute relational thinking as defined by Molina and Castro (2021). According to these authors, such thinking implies, first, being able to consider expressions globally from a structural perspective instead of as a process to be carried out step by step (the ‘structure’ aspect), and second, recognising certain relations between the terms of the expressions and using them to design a strategy applying arithmetic properties (the ‘relationships’ aspect). Very recently, Kieran (2018) drew attention in particular to the importance of structuring activity in early algebra, which consists in identifying structure within numbers and numerical operations, with attention to the properties of numbers and operations. This structuring

allows students to consider expressions globally and to experience equivalence through decomposition, recomposition, and substitution (Pang & Kim, 2018). For Kieran (2018), generalizing and structuring are two complementary faces of algebraic thinking, one face looking towards generalizing and complementarily, the other face looking "towards ‘seeing through mathematical objects’ and drawing out relevant structural decompositions" (p. 101). This relational thinking allows students to restructure arithmetic operations in order to change the given calculation and carry it out more effectively (Kiziltoprak & Köse, 2017). For example, students can transform the expression $(85 + 69) + 15$ into $(85 + 15) + 69$ using the associative and commutative properties. For Irwin and Britt (2005), mental arithmetic strategies such as compensation also involve methods of a relational nature. For example, the strategy of ‘compensation of the subtrahend’ in subtraction leads students to decompose a calculation such as $45 - 29$ into $45 - 30 + 1$.

Irwin and Britt (2005) consider that arithmetic operations of this kind, which lead students to produce equivalent expressions, involve a process of generalisation in which the numbers themselves act as variables, and they highlight that, without recourse to literal symbolic forms, the strategy is generalisable. These authors refer to the concept of ‘quasi-variables’, which they define as "a number sentence or group of number sentences that indicate an underlying mathematical relationship which remains true whatever the numbers used are" (Irwin & Britt, 2005, p. 171). The ability involved in structuring the expression is intrinsically related to the ability to generalize strategies on the basis of quasi-variables. The idea is that, in order to perform operations such as $10 - 10 - 5 + 13$, it is relevant to understand that $+a - a$ will always be 0 whatever the value of a . Similarly, in mental computation strategies involving natural numbers such as the strategy of compensation in subtraction, which is applicable in problems such as $75 - 29$, it is useful to decompose the second term into a difference of two terms, and to bear in mind that, whatever numbers a , b and c stand for, the expression $a - (b - c) = a - b + c$; hence in this case $75 - 29 = 75 - (30 - 1) = 75 - 30 + 1$.

In conclusion, in the context of generalised arithmetic, we consider relational thinking as a form of algebraic thinking characterised by structuring expressions and identifying and using relationships, involving a generalisation process on the basis of the ‘quasi-variables’. For many authors (Blanton et al., 2019; Irwin & Britt, 2005; Stephens, 2008), early thinking of this kind in terms of holistic structures and number relationships leads to increased understanding and competence in algebra. Beyond general proficiency in algebra, we believe that the development of this form of algebraic thinking based on relational thinking may in particular help to reduce the difficulties with negative numbers that have been highlighted in the research literature.

3 Difficulties with negative numbers

The negative numbers are notable by their absence from reflections about algebraic thinking, but in fact it must be pointed out that little has been said about negative numbers so far in the research literature in general. Bishop et al. (2014), for example, noted that research on integers is relatively sparse compared to the literature on students' understanding of, for example, rational numbers or whole numbers. Young and Booth (2020) were still making the same observation six years later!

However, authors such as Lamb et al. (2016) or Thompson and Dreyfus (1988) showed that students' performance on integer addition and subtraction problems has major implications for their success in algebra. In addition, several researchers found that students already demonstrate the ability to reason about integers in relatively sophisticated ways from early grades, even before they have learnt about these numbers, and that this potential could constitute a basis for the teaching of negative numbers (Bofferding, 2014; Lamb et al., 2016). However, despite this potential, several authors (Gallardo, 2002; Herscovics & Linchevski, 1991; Vlassis, 2004, 2008) pointed to the many difficulties secondary school students experience when first learning about negative numbers, in terms of both these numbers themselves and the operations and equations involving them. It is therefore important to take a look at these difficulties in order to understand how developing algebraic thinking might help overcome them.

As regards the negative numbers themselves, some students, especially when first learning about integers, have difficulty in accepting a negative number as a solution to an equation (Gallardo, 2002; Vlassis, 2008). A negative solution seems impossible to them, as they are unable to give it a concrete interpretation. A classic example of this difficulty is students' assertion that a problem such as $3 - 5$ is equivalent to $5 - 3$ (Peled et al., 1989).

But the most significant and persistent difficulties have been found in performing operations with integers, and in particular addition and subtraction, regardless of the nature of the solution (positive or negative). Peled et al. (1989) showed in interviews with grade 6 students that, a year after they had learned addition and subtraction of negative numbers, their overall computational performance was low. Herscovics and Linchevski (1991) pointed out several mistakes made by many grade 7 students when required to simplify an algebraic expression. One of these, which they called 'Detachment From the Minus Sign' (DFMS), consisted of ignoring the minus sign before a number, and simplifying an expression such as $4 + n - 2 + 5$ by first calculating $2 + 5$ to obtain $4 + n - 7$, as if the minus sign was not attached to the 2. These authors were particularly surprised by the

frequency of this error. Vlassis (2004) also observed this type of error in polynomials such as $-12 - 7 - 7n + 4n$, which were simplified to $-12 - 7 - 11n$ rather than to $-12 - 7 - 3n$.

More surprisingly, DFMS has also been observed with natural numbers, in addition and subtraction operations consisting of more than two terms. For example, $237 + 89 - 89 + 67 - 92 + 92$ was simplified to $237 + 89 - 89 + 67 - 184$, as if the minus sign before 92 was not attached to the number (Herscovics & Linchevski, 1991). This observation is consistent with the findings of Livneh and Linchevski (2007), who showed empirically that the difficulties with algebraic structures stem from difficulties with similar numerical structures, that as a consequence, the obstacles identified in algebraic contexts, such as DFMS, also exist in the corresponding numerical contexts, and that these obstacles are widespread.

The DFMS error has since been reported on numerous occasions in the research literature (Banerjee & Subramaniam, 2012; Bishop et al., 2014; Linchevski & Livneh, 1999; Vlassis, 2004, 2008), to the point where DFMS can be regarded as a sign of a significant and persistent obstacle in the learning of both algebraic and numerical operations.

4 Relational thinking and operations with negative numbers

Two reasons may explain the difficulties discussed in the previous section, as follows.

4.1 A lack of structure sense

Linchevski and Livneh (1999) attributed the DFMS error to a misunderstanding relating to a lack of 'structure sense'. They defined structure sense as being able to identify all the equivalent forms of an expression and "the ability to discriminate between the forms relevant to the task—generally one or two forms—and all the others" (p. 175). In the example of the operation $237 + 89 - 89 + 67 - 92 + 92$, this consists of considering $237 + 89 - 89 + 67$ or $237 + 67$ as equivalent forms relevant to the task, whereas $237 + 89 - 89 + 67 - 184$ should not be recognised as an equivalent form.

4.2 A restrictive understanding of subtraction

The lack of structure sense referred to by Linchevski and Livneh (1999) is not unrelated to the restrictive understanding of subtraction as 'taking away', which is often observed in students (Selter et al., 2012). In this view, numbers represent 'unsigned' concrete quantities on which operations are performed in order to find the answer. Selter et al. (2012) believe that this computational view is too one-sided and probably leads to restricted mathematical thinking.

From our point of view, relational thinking, characterised by the notions of structures and relations, would help overcome both the lack of structure sense and the restrictive conception of subtraction. In relational thinking, detecting the structure of an expression means ‘seeing’ the expression holistically. In a numerical expression such as $237 + 89 - 89 + 67 - 92 + 92$, for example, this would involve ‘seeing’ the expression as follows:

$$\boxed{237 + 89 - 89 + 67 - 92 + 92}$$

This holistic view requires the expression to be considered as a sum of signed components separated by an implicit plus sign (Vlassis, 2009). This view is essential in algebra, for example when it is necessary to reduce polynomial expressions. It is not necessary in numerical operations, since it is always possible to proceed computationally, step by step, to find the answer. However, it is very useful for carrying out operations efficiently: first, it ensures that DFMS errors are avoided, and second, treating the expression as a sum allows properties of addition such as commutativity or associativity to be applied and makes clear the relationships between the numbers. In this case, this involves recognising the pairs of opposites $+92$ and -92 and $+89$ and -89 and using this relationship to cancel the opposites and rewrite the expression as $237 + 67$, which ultimately enables the answer to be found more easily.

By analogy, an expression involving negative numbers such as $2 - 5 - 3 + 5$ should be considered in the same way, that is:

$$\boxed{2 - 5 - 3 + 5}$$

After this, either mentally or in writing, students can put the opposites together and produce, for example, an equivalent expression such as $2 - 3 + 5 - 5$ by moving -5 after $+5$ by applying the commutativity principle, to finally obtain the expression $2 - 3$ and the answer, -1 .

These two examples illustrate the importance of the structuring activity mentioned by Kieran (2018) for performing this type of operation. However, this is a complex activity for students, as seeing an expression holistically requires unary use of the minus sign (as well as of the plus sign¹): in other words, the minus sign has to be treated as attached to the number that follows it (Vlassis, 2004). As Vlassis (2004) and McGowen and Tall (2013) have shown, the unary use of the minus sign represents a considerable obstacle for students, who have been accustomed throughout elementary school to think of this sign as binary (that is as an operation sign).

The unary aspect of the minus sign refers to the concept called ‘signed numbers’ by Thompson and Dreyfus (1988), who advocated the use of an arithmetic of signed numbers to facilitate algebraic operations in the elementary grades. To this end, they showed the effectiveness of teaching for grade 6 students based on thinking of integers as transformations of quantities, as defined by Vergnaud (1982). Thus, considering expressions holistically could involve, at the elementary level, getting students to think of operations in terms of a composition of transformations (Vergnaud, 1982). This requires a completely different conception of operations, which can no longer be thought of as processes applied to unsigned quantities.

This move from subtraction operation on unsigned quantities to signed numbers requires the subtraction operation to be reified as an object (Kumar et al., 2017). Subramaniam (2019) also contends that the extension of natural numbers to negative numbers is based on the reification of the subtraction operation. According to Bishop et al. (2014), historically, it recalls the idea of ‘subtractive number’ which was one of the earlier conceptions mathematicians had for negative numbers. However, it should be noted that this subtractive number does not yet have the status of a formal negative number (Gallardo, 2002): although it is a number, its meaning remains strongly contextualised in the idea of subtraction. According to Gallardo (2002), this interpretation represents the first level in the understanding of negative numbers, namely the ‘subtrahend level’, as it appeared in the history of these numbers. As a signed number, this subtractive number can be related to the idea of transformation of quantities (or composition of transformations, if there are two or more subtractive numbers) applied to an initial state (Thompson & Dreyfus, 1988), as in the following examples:

$$8 - 3 = \boxed{8} \xrightarrow{-3} \boxed{?} \quad \text{Transformation } (-3) \text{ applied to } 8$$

$$8 - 3 - 1 = \boxed{8} \xrightarrow{-3} \xrightarrow{-1} \boxed{?} \quad \text{Composition of two transformations } (-3 \text{ and } -1) \text{ applied to } 8$$

¹ However, inappropriate use of the plus sign goes unnoticed as it does not cause mistakes.

This analysis can lead to the holistic view of the expression that we described previously.

In conclusion, our hypothesis was that relational thinking, based on the one hand on the identification of structure and the emergence of the subtractive number, and on the other hand on the identification and generalisation of the relations and properties of operations, may help students to make sense of addition and subtraction of integers.

5 Method

5.1 Research questions

1. To what extent can grade 6 students solve integer additions and subtractions before learning about negative numbers?
2. To what extent do grade 6 students develop relational thinking in mental computation strategies in arithmetic?
3. To what extent does the use of relational thinking in mental computation strategies go hand in hand with success in integer additions and subtractions?

5.2 Participants

A total of 171 grade 6 students in 13 classes at eight primary schools in the Grand Duchy of Luxembourg took part in the study by completing a paper-and-pencil test. Only five students had to be removed from the sample because they did not reply to more than 80% of the questions. The final sample was therefore composed of 166 students.

The students in our sample had not been taught about negative numbers, as this topic is not included in the Luxembourg elementary curriculum. In contrast, the meaning of equality and the relations between operations are explicitly included in the curriculum, as are the properties of addition (commutativity and associativity) for performing calculations effectively (MEN, 2011).

5.3 Test

The paper-and-pencil test was designed to be taken individually, and took approximately one hour to complete. The test had two parts: the first part consisted of decontextualised questions relating to integer additions and subtractions, the role of zero, the order of negative numbers, opposite numbers, and mental computations in addition and subtraction operations with more than two terms, and in subtractions with two terms where the compensation strategy would clearly be useful. The second part consisted of contextual problems, the solutions to which required operations similar to those in the first part. The contexts chosen were familiar contexts for calculations with integers and natural numbers,

such as temperatures, lifts, gains and losses, passengers getting in and out of a bus and so on.

For the purposes of this study, only decontextualised questions relating to (1) integer additions and subtractions, and (2) subtractions in which a compensation strategy would be useful, were analysed.

The items relating to these questions are presented in the following sections together with the analysis of the results.

Before the final version of the test was administered to the 13 experimental classes, a pilot test was carried out with five students. A second pilot test was then given to four grade 6 classes (i.e., 82 students). On the basis of these preliminary tests, the test instructions were adapted in order to be comprehensible to the students; the number of items was also adjusted so that the test time did not exceed one hour.

The first author of this paper administered the test in the different classes. After the general objectives of the test had been explained to the students, it was handed out to them without any specific instructions. As Luxembourg is a multilingual country, the test was provided either in German or in French, so that language would not be an obstacle to understanding the test instructions. It is in any case customary in Luxembourg to provide tests in both French and German, in both national and international assessments.² This is because, although the language of mathematics teaching at primary school is German, many students who speak Romance languages (mainly Portuguese, but also French, Italian, etc.) attend school in Luxembourg and understand French better. Each student was therefore able to choose the version he or she preferred. In the end, of the 171 tests distributed, 39 tests (23%) were taken in French.

6 Results

6.1 Integer additions and subtractions

Questions 1 and 2 in the test were designed to measure students' ability to perform additions and subtractions with integers. These two questions presented integer operations of the type $a \pm b = c$ where a , b and c are integers; the operations potentially corresponded to a concrete model from everyday life (temperature, lift, gains and losses, etc.). This type of operation was justified by the test's objective of investigating the knowledge of grade 6 students who had not received any instruction about negative numbers. The operations had to be accessible to the students. For this reason, none of the operations involved subtractions of negative numbers with two consecutive minus signs. Although these two questions

² French and German are two of the three national languages, along with Luxembourgish.

Table 1 Results for items concerning the acceptance of negative solutions (Question 1—Items 1 to 4)

	Items	Correct (%)	Incorrect (%)	Omission (%)	Total (N= 166)
1-1	$8 + \square = 5$	54	24	22	100
1-2	$\square + 9 = 6$	58	21	21	100
1-3	$4 - 6 = \square$	63	33	4	100
1-4	$5 - 10 = \square$	65	31	4	100
Mean score (Cronbach's alpha: 0.81)		60			

In this question, two additional items with natural numbers were added ($12 - \square = 8$ and $7 + \square = 12$) to prevent students from deducing that all the answers to this question were negative numbers and therefore automatically producing negative solutions without thinking

In Tables 1 and 2, the items have been presented in ascending order of percentage of correct answers for the sake of clarity

both presented integer operations, they pursued rather different objectives. The first question, containing items without any apparent negative numbers, essentially investigated students' ability to envisage a negative solution—in other words, their "acceptance of negative solutions" (Gallardo, 2002). The second question, involving operations with the same structure but using clearly identifiable negative numbers, was designed to test students' ability in integer operations, regardless of the positive or negative nature of the solution. Table 1 below shows students' results for the first question.

All of these items were answered correctly by more than 50% of students, with a mean score of 60%. More than half of these grade 6 students thus already showed some acceptance of negative solutions, without having received any teaching.

The two hardest items were Items 1-1 and 1-2, where the unknown negative number appeared as the first or second term in the operation. They were also characterised by the highest omission rates, namely, 22% and 21% respectively. The task was to find a number to add to the first term of an addition to obtain a smaller answer. The high omission rates observed for these items indicate some students' perplexity with these items. It is likely that students were confused

by these operations which contradicted their initial model of 'Addition makes bigger' stemming from their previous experience with natural numbers (Bofferding, 2014).

In the items that were answered correctly by more students, there were far fewer omissions. However, the possibility cannot be ruled out for these two items that even the correct answers did not indicate an understanding of negative numbers, but rather derived from superficial approaches. For example, some students may have solved Item 1-4 by calculating $10 - 5 = 5$, then adding a minus sign in front of their solution, 5.

Table 2 below presents the results of the students in items of Question 2.

By contrast with the items of Question 1, most of the four items with the lowest correct response rates, i.e. answered correctly by less than 50% of the students (Items 2-1, 2-2, 2-3 and 2-4, shaded in Table 2), are characterised by a solution representing the final state. Only Item 2-3 (among these four items) required a solution corresponding to the initial state.

In an attempt to understand the difficulties encountered by the students in these operations, in Table 3 below we present the incorrect solutions which were the most common (Error 1) and the second most common (Error 2).

Table 2 Results for items concerning skills in integer additions and subtractions (Question 2—Items 1 to 8)

Items	Solution	Correct (%)	Incorrect (%)	Omission (%)	Total (N = 166)
2-1	$-12 - 8 = \square$	36	60	4	100
2-2	$4 + (-9) = \square$	44	34	22	100
2-3	$\square + 4 = -6$	45	47	8	100
2-4	$-3 - 5 = \square$	47	43	10	100
2-5	$-7 + 4 = \square$	52	45	3	100
2-6	$7 + \square = -3$	52	36	22	100
2-7	$\square - 3 = -8$	62	34	4	100
2-8	$-5 + 8 = \square$	68	30	2	100
Mean score (Cronbach's alpha: 0.88)		48			

Table 3 Most common errors made by the students in the items of Question 2

Item	Error 1	%	Error 2	%	Total % (N = 166)	
2-1	$-12 - 8 = \square$	-4	36	4	16	52
2-2	$4 + (-9) = \square$	-13	11	5	7	18
2-3	$\square + 4 = -6$	-2	24	2	13	37
2-4	$-3 - 5 = \square$	2	19	-2	10	29
2-5	$-7 + 4 = \square$	-11	13	11	12	25
2-6	$7 + \square = -3$	-4	9	4	3	12
2-7	$\square - 3 = -8$	-11	21	11	20	41
2-8	$-5 + 8 = \square$	13	15	-13	4	19

Table 4 Percentage of correct answers in compensation questions

	Correct (%)	Incorrect (%)	Omission (%)	Total (%) (N = 166)
Question 1	25	66	9	100
Question 2	33	66	1	100

The first point that is noticeable in Table 3 is that the incorrect solutions produced by the students always represent numbers with the same absolute value, except in Item 2–2. Students seem to have performed the operations in order to find a value, which in some cases they prefixed with a minus sign. Secondly, the results in Table 3 seem to show that the difficulty this time lay not necessarily in accepting negative solutions, since the students produced both positive and negative solutions, but in the operations themselves.

The errors made in Items 2–1, 2–4, 2–5 and 2–8 appear to reflect a DFMS approach. For example, in Item 1, to perform $-12 - 8$ students detached the minus sign in front of 12, calculated $12 - 8$ and ended up with -4 (Error 1) or 4 (Error 2), depending on whether or not they attached the minus sign to their solution after performing the calculation. For this

item, which also had the lowest correct answer rate, 52% of the students made this mistake. The solutions that may reflect DFMS have been shaded in Table 3 in order to show the significance of this erroneous approach, which appeared quite systematically when the first term was negative. Other types of errors are listed in Table 3, but occur less frequently than those relating to DFMS.

Analysis of Tables 1 and 2 shows that the items presented in Table 2, with a mean score of 48%, were answered correctly less often than those in Table 1 (mean score of 60%). This difference is statistically significant (ANOVA: $F = 8.96$; $p = 0.003$). This result shows that students seem to have more difficulties in performing operations with negatives than in accepting a negative solution. In the operations of Table 2, DFMS consisted of detaching the minus sign from the first term of the operation, whether or not this represented the unknown. It thus appeared to be a major obstacle in this type of operation. Peled et al. (1989) previously also observed this problem in the same type of operations in grade 6 students, but as far as we know did not demonstrate the extent of the phenomenon.

6.2 Algebraic thinking in mental computation strategies

Among the questions concerning mental computation strategies, we chose two questions about the strategy of compensation in subtraction. We decided to analyse these two questions in particular because both the format of the question and the nature of this strategy require the development of relational thinking. Regarding the format of the questions, students were not asked to solve operations using this strategy of compensation in subtraction, as it is known that students rarely use it spontaneously (Hickendorf, 2018). They were rather asked to choose the correct equivalent three-term expression (Question 1) or to decide on a given equivalent

Table 5 Types of arguments used by students in compensation questions

Arguments	Abbreviation	Description
Computational	Comp	The student claims—after actually performing the calculations or not—that the operations lead to the same result.
DFMS	DFMS	The student notes that $219 = 220 - 1$ and $\neq 220 + 1$ (in Question 1), or claims that Lydia is wrong because $79 \neq 80 + 1$ (in Question 2). He/she thus detaches the minus sign from 220 or 79 to justify his/her choice
+1 -1 rule	+1 -1 rule	The student seems to overgeneralize a compensation rule that works with addition but not with subtraction. He/she explains that you have to apply +1 -1, and that therefore, for example, $3648 - 220 - 1$ (in Question 1) is correct because to get 220, you have to add 1 to 219, so then you have to subtract 1 afterwards. This strategy is also based on DFMS, as the student is thinking in terms not of $-220 - 1$ but $220 - 1$.
Relational	RTh	The student uses the relationships between the numbers to explain his/her correct choice of operation 2 (that is $3648 - 219 = 3648 - 220 + 1$) in Question 1. The student says that with -220 , 1 too much has been taken away, so you have to add 1 afterwards. In Question 2, he/she decides that Lydia is right on the basis of the same type of reasoning. Where this type of justification is given, the student is thinking in terms of -220 or -79 , attaching the minus sign to 220 or 79.

three-term expression (Question 2). The operations in the two items were $3648 - 219 =$ or $843 - 79 =$, inviting the use of compensation. The two questions were as follows:

- Question 1: Choose the correct equivalent expression and justify it (Could you reduce the figure below?)

Is $3648 - 219$ equal to: $3648 - 220 - 1$?
 $3648 - 220 + 1$?

Explain

- Question 2: Decide on an equivalent expression and justify it (Can you reduce the figure below and make only one box with the two?)

Emin would like to find the answer to the following subtraction problem: $843 - 79 =$

He gets out his calculator, but there's a problem: The 9 key doesn't work!
 Lydia suggests that he put in $843 - 80 + 1$.
 Do you agree with her? Yes No .
 Explain why

Table 4 shows the percentage of correct answers for the two compensation questions. The answer was counted as correct when the student not only made the right choice but also gave an explanation consistent with his or her choice and clearly justified it. The aim was to avoid answers picked correctly by chance.

The results in Table 4 show that only a small proportion of students gave correct answers to these items involving compensation in subtraction: 25% in Question 1 and slightly more (33%) in Question 2. However, the correlation between the two items was strong (0.53). It cannot therefore be said that the format of the questions has a significant impact on students' performance. The omission percentages relate to students who omitted the whole question (both choice and justification).

To explore the students' difficulties in more depth, Table 5 presents the four main arguments used by students to justify their choice.

What fundamentally differentiates DFMS-type reasoning from reasoning based on relational thinking is that in the first case, students base their reasoning on the *numbers* ($219 = 220 - 1$ and $\neq 220 + 1$), whereas in the second case, they take the *operation* as their basis. This involves

reasoning that 'taking away 220' means taking away 1 too many, so that 1 must be added, or that 'something minus 220' will give you 1 less than 'something minus 219', and so 1 needs to be added back. With this type of reasoning, the minus sign is attached to 220.

Table 6 presents the percentages of each type of argument provided by the students, according to whether they answered the questions correctly ('correct') or not ('incorrect').

The results presented in Table 6 clearly show that the most frequent response profile in these questions used the erroneous strategy of DFMS. In both questions, the percentages were high: in Question 1, 42% of the students made this error if we add the +1 -1 strategy, also based on DFMS, (33% + 9%), while in Question 2, as many as 49% of students committed DFMS errors (46% + 3%).

Figures 1 and 2 present examples of this DFMS approach in each of the two questions.

Where the correct answer was given, the results in Table 6 show that the methods used were divided between computational reasoning based on the result (12% in Question 1 and 17% in Question 2), and relational thinking (13% in Question 1 and 15% in Question 2). These results seem to show that the successful completion of the compensation items did not necessarily imply a particular type of reasoning, whether computational or relational.

The example presented in Fig. 3 below demonstrates a computational type of reasoning.

Figure 4, by contrast, shows a relational type of reasoning.

It should be noted that the argument in Fig. 4, while relational in nature, is based on the 'taking away' meaning of subtraction.

Again, to avoid the effects of chance, we regarded only those students who produced a relational type of reasoning in the two compensation items as having developed genuine relational thinking: they seemed to be able to treat the expressions holistically, as, for example, in Question 1:

$$3648 - 219 = \boxed{3648} \boxed{- 220} \boxed{+ 1}$$

This led them to recognise the correct expression and to deduce the relation 'by doing -220, 1 too much has been

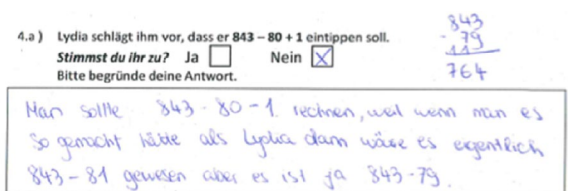
Table 6 Percentages of each type of argument used in the compensation questions according to whether the question was answered correctly or not

	Correct (%)		Incorrect (%)					Om. ^b (%)	Total (N=166)
	Comp	RTh	Comp	DFMS	+1-1 rule	Oth ^a	Om. ^b		
Q. 1	12	13	4	33	9	4	16	9	100
Q. 2	17	15	5	46	3	2	11	1	100

^aOthers

^bOmission

Fig. 1 Justification based on DFMS in the Compensation Question 2



‘One should calculate $843 - 80 - 1$, because if we had done as Lydia did, it would have been $843 - 81$, but it is $843 - 79$ ’

Table 7 Relationship between scores on compensation items and integer operations

Scores for compensation items	Scores for integer operations (%)
0/2	48
1/2	43
2/2	72

taken away, so you have to add 1’. Moreover, they could identify and use this relationship in both compensation questions. This could be a sign that these students seem to be able to generalise this relationship in other operations of the same type.

It is likely that with these criteria, both for treating the question as answered correctly and for considering genuine relational thinking to have been used, we underestimated the number of students who had actually developed this kind of thinking. According to Pang and Kim (2018), students' prior experience with algorithmic arithmetic leads them to have confidence in computation. So, it may be easy or even ‘desirable’ for them to calculate in solving a given operation or to use such computational ability when asked to justify their answers.

In Table 6, the omission rates shown under incorrect answers relate to students who chose the wrong answer without giving an explanation. There were no omissions under correct answers, as a correct answer was defined as a correct choice accompanied by a clear and consistent justification. It is worth noting that the omission rates were relatively low, even though students often found it difficult to justify and present arguments for their answer. Table 6 shows that a total of 25% of students (16% + 9%) did not answer the whole of Question 1 or did not justify their incorrect choice, while the corresponding proportion of students for Question 2 was smaller, at 12% (11% + 1%).

6.3 Relations between relational thinking and negative numbers

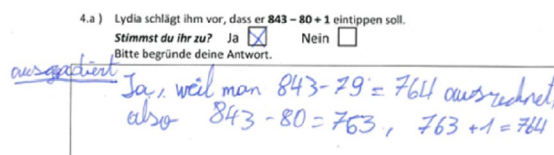
To examine the relationships between relational thinking and negative numbers, we compared the scores for the two compensation questions (Sect. 6.2) with the scores for the 12 items relating to negative numbers (Sect. 6.1). For this last score, the items of Questions 1 (4 items) and 2 (8 items) were combined. More specifically, we wondered if the students who used relational type of reasoning in the two compensation questions performed better on the 12 items with integers.

Fig. 2 Justification based on the +1 - 1 rule in the Compensation Question 1



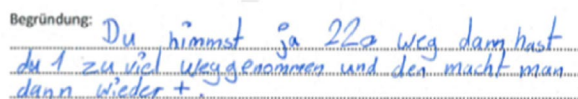
‘If you do +1 to the first number, you have to do - 1 to the second’

Fig. 3 Computational reasoning in the Compensation Question 2



‘Yes, because you calculate $843 - 79 = 764$, so $843 - 80 = 763$, $763 + 1 = 764$ ’

Fig. 4 Relational reasoning in the Compensation Question 1



‘You’re taking away 220 so you have taken away an extra one and you add it back again’

Table 8 Relationship between type of justification given in the two correctly answered compensation items and success in integers operations

	Computational justification in the 2 items	Relational justification in the 2 items	Mixed profile	Total (N=166)
Correct answers in compensation items	8% (N=14)	9% (N=15)	2% (N=3)	19% (N=32)
Correct answers in integer operations	76%	70%	61%	72%

In Table 7, we compare the scores for compensation items with those for integer operations. The two types of items show good internal consistency (for compensation items, Cronbach's alpha was 0.70 and for the 12 items with negatives, Cronbach's alpha was 0.90).

The performance on the compensation items was scored according to the 'correct answer' criteria stated above. Thus a score of 2/2 refers to students who answered both items correctly (i.e., made the correct choice in both items accompanied by a clear and consistent justification), and a score of 1/2 refers to students who made the correct choice accompanied by a justification in one of the two items. A score of 0/2 means that the students failed to answer either item correctly.

Table 7 shows that the students who made the correct choice and justified it in both items (2/2) were clearly more likely to answer the items with integers correctly (72%). This result was significantly different ($F=14.79$; $p=0.0002$) from those for profiles 0/2 and 1/2. Where a student had correctly answered just one compensation item, this seemed to have little connection with the score on items with integers (43%). The same was true for students who failed to answer either compensation item correctly (48%). Surprisingly, these students had a (slightly) better score on the items with integers than the students who answered one of the two compensation items correctly.

In order to explore the question further, we looked at the type of justification used—whether computational or relational—by the students who had answered the compensation questions correctly.

Table 8 presents the results of the integer items for the students who succeeded in the two compensation questions. These results were analysed according to the type of justification given.

Table 8 shows that the type of justification did not seem to have an influence on the correct answering of items with integers: the differences between the scores of 76% and 70% is not statistically significant (ANOVA: $F=0.28$; $p=0.5984$). It should be noted that very few students ($N=3$) had a mixed profile: they used relational arguments in one question and computational arguments in the other.

In the end, the main effect on integer operations seemed to come more from the fact of having answered both compensation items correctly than from the type of reasoning

used to justify a correct choice (computational or relational). What fundamentally distinguished correct from incorrect answers to the compensation questions was students' ability to focus on the *operation* of subtraction involving a unary use of the minus sign (-219 and -220), rather than on *numbers* detached from this sign (219 and 220).

7 Discussion

In our study we examined the extent to which algebraic thinking might be associated with better performance in addition and subtraction with negative numbers. The form of algebraic thinking relevant to such operations belongs to the generalised arithmetic strand (Kaput, 2008). It is characterised by relational thinking involving a holistic view of expressions as well as the use of relationships between numbers and properties of operations (Molina & Castro, 2021).

In order to investigate this issue, we first examined the students' correct and incorrect answers in operations with integers (RQ1). The results showed that while the students showed some acceptance of negative solutions, they experienced greater difficulty when they had to perform operations with negative numbers. These results confirm those of previous research (Gallardo, 2002; Vlassis, 2004, 2008). In particular, the analyses revealed that the most frequent mistakes occurred when the operation began with a negative number. In these situations, a large proportion of students detached the minus sign from the first term before performing the operation (DFMS error).

The students' relational thinking (RQ2) was then analysed on the basis of two questions relating to the mental computation strategy of compensation in subtraction. The rate of correct answers was low. But why did these questions on compensation prove so difficult? Analysis of the justifications produced by the students who made mistakes sheds light on this issue. Most of the explanations given by these students were based on DFMS. These students had focused on the number detached from the minus sign, rather than on the subtraction operation, involving a unary use of the minus sign. This represents a significant difficulty for students (Vlassis, 2004), and in this sense, notably, the compensation strategy is complex.

The final goal of our study was to investigate the relationships between relational thinking and negative numbers (RQ3). Our analyses revealed a close link between success in operations with natural numbers (in compensation questions) and operations with negative numbers: students who answered both compensation questions correctly were significantly more likely to complete the integer operations accurately. Again, the analysis of the arguments put forward by these students gave us an interesting insight. Students who successfully answered these questions provided justifications based on either computational or relational reasoning. We observed that this latter type of reasoning could rely on the ‘taking away’ meaning of subtraction, often considered as leading to a computational view of mental arithmetic (Selter et al., 2012). In this sense of subtraction, numbers are usually seen as *unsigned* quantities separated by a minus sign (Vlassis, 2004). It seems, however, that a unary use of this sign may exist in the ‘taking away’ conception of subtraction. That is what we call a ‘signed taking-away’ meaning of subtraction. Half of the students who performed well on the compensation items—and therefore had a higher probability of also performing well on the integer operations—relied on computational reasoning. This result seemed surprising to us, as we had hypothesised that it was relational thinking that could help students to make sense of integer operations, and not computational reasoning.

What the students who answered correctly had in common was the ability to consider the operation as a signed ‘entity’, that is a ‘transformation’ (Vergnaud, 1982) involving a unary use of the minus sign. This ability seems to have been decisive in the successful completion of the integer operations. We refer to signed ‘entities’ rather than signed ‘numbers’ because we believe that they had not yet fully acquired the status of number: computational reasoning, conducted step by step, did not allow these students to see expressions holistically and to apply the properties of addition. The subtraction operation therefore could not be completely reified. In this sense, we believe that these students had not fully developed the concept of the subtractive number (Bishop et al., 2014). Genuine relational thinking was probably not necessary to answer these questions on compensation correctly. However, it becomes essential for efficiently performing complex numerical operations consisting of more than two terms such as $147 + 10 - 27 - 10 + 15$. This requires a holistic view of the expression, where the terms -27 and -10 are considered as subtractive numbers that can be moved, like objects, by applying commutativity. It is this holistic view that allows the restructuring of the expression $147 - 27 + 10 - 10 + 15$, to obtain the simplified expression, $120 + 15$. To the extent that the difficulties with algebraic structures stem from difficulties with similar numerical structures (Banerjee & Subramaniam, 2012; Livneh & Linchevski, 2007), working with this kind of expression

could help reduce the well-known phenomenon of DFMS, observed in operations with natural numbers and integers as well as in algebraic expressions.

8 Conclusion

Our study was a first approach to the question of the relationship between negative numbers and algebraic thinking. The data were collected by means of a paper-and-pencil test. This format represents a limitation of the study, in that the nature of students’ thinking was deduced from their answers or explanations rather than being observed; as Hickendorff (2018) stressed, previous studies have shown that what students do on a standard task may not fully reflect what they know. Thus this first approach could be taken further through student interviews which would provide a better understanding of students’ actual thinking.

Ultimately, our results, and in particular the analysis of the arguments put forward by the students, throw interesting light on a potential developmental path for students. Rather than dichotomising a computational versus relational conception of expressions (Subramaniam, 2019), it is interesting to consider how the students made use of their ‘met-before’—the mental structure resulting from previous experiences (McGowen & Tall, 2013). On the one hand, the students’ previous experience with computational arithmetic based on unsigned quantities led some of them to maintain a computational perspective, yet at the same time they showed themselves capable of evolving towards an understanding of subtraction as transformation. On the other hand, students who gave relational arguments, while demonstrating a holistic view of operations and analysis of relationships, based them on the ‘taking away’ meaning of subtraction.

These analyses suggest the potential value of a ‘teaching/learning trajectory’ (Warren & Cooper, 2009) starting from the idea of transformations as well as a ‘signed taking away’ meaning of subtraction, and moving towards the development of fully relational thinking. In secondary school, this thinking is essential to avoid, in particular, the many errors due to erroneous use of the minus sign in polynomial reductions and equation-solving (Vlassis, 2004, 2009). In elementary school, while students often have to perform operations consisting of two terms, they should be presented with expressions with multiple operations that can be interpreted as compositions of transformations. Various activities could serve this purpose, including problems such as ‘The Piggy Bank problem’ (Carraher et al., 2006), in which students compare the movements of money entering and leaving the piggy bank of two children. ‘Think of a number’-type problems could also be useful. These consist of applying a sequence of operations to any number and analysing the link between the starting number and the final result. However,

these activities alone are not enough to develop full relational thinking as defined in Sect. 2. The teacher must play a proactive role, initiating discussions with students about the structure of expressions, leading them to see these expressions as a whole and focusing particularly on the unary use of the minus sign, which, as our results have shown, constitutes a major difficulty for students. These discussions will also address the relationships between numbers and the properties of operations in order to make them explicit. The activity can only be termed algebraic when "we are stating these properties explicitly and examining their generality" (Kaput, 2008, p. 13).

It is clear from the above reflections that it is possible to offer activities in primary school that will prepare students to deal with negative numbers, even when these numbers are not included in the curriculum. As well as improving efficiency in operations with natural numbers, these activities will help reduce the difficulties in operations with negative numbers such as DFMS. Our results have shown that it is possible to envisage a gradual development in such activities of relational thinking and the idea of a subtractive number which, it is worth stressing again, represents the first stage in the acceptance of negative numbers (Gallardo, 2002). In further investigations, it would be of major interest to examine whether such activities offered to primary school students would indeed reduce the difficulties often observed in secondary school students in operations with negative numbers. This would also be an important challenge for the implementation of professional development programmes, because many primary school teachers still tend to view arithmetic not in a relational way, but as a set of rules and procedures to be memorised and applied (Stephens, 2008).

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