Developing Symbol Sense for the Minus Sign

Author(s): Lisa L. Lamb, Jessica Pierson Bishop, Randolph A. Philipp, Bonnie P. Schappelle, Ian Whitacre and Mindy Lewis

Source: *Mathematics Teaching in the Middle School*, Vol. 18, No. 1 (August 2012), pp. 5-9

Published by: National Council of Teachers of Mathematics

Stable URL: https://www.jstor.org/stable/10.5951/mathteacmiddscho.18.1.0005

Developing Symbol Sense for the Minus Sign

Lisa L. Lamb, Jessica Pierson Bishop, Randolph A. Philipp, Bonnie P. Schappelle, Ian Whitacre, and Mindy Lewis

Imagine that you have just asked your students to read this equation:

$$-x - 9 = 12$$

What would they say? If you expect, "Negative *x* minus 9 equals 12," then that statement would be consistent with how most students would respond. However, students could read this equation in a way that conveys a different, even deeper, understanding. We highlight three meanings of the minus sign and explore specific difficulties related to the minus sign that many students experience. We also suggest ways that you as a teacher can support your students in developing a robust understanding of the minus sign.

Research has shown that how students interpret and use the minus sign are facets of symbol sense, which Arcavi (1994) described as "making friends with symbols" (p. 25). "Making friends" with symbols includes an understanding of and feel for symbols and how to use them and read them. Research indicates that many students in middle school and even high school do not have fully developed symbol sense regarding the minus sign. In our research, we have learned that this limited view of the minus sign interferes with students' abilities to—

1. truly understand the process of solving equations, and
2. make sense of variables.

THREE MEANINGS OF THE MINUS SIGN

The minus sign is used in three common ways. The three problems in **table 1** use the symbol "–" that we refer to, in general, as the minus sign. However, each problem may elicit a different meaning for students.

The first meaning is shown in problem 1, in which the minus sign indicates subtraction, the original use of the symbol that young children encounter. In problem 2, the minus sign is part of the symbolic representation for a negative number, in this case, "negative 2." In problem 3, however, the first minus sign may be viewed as the *opposite of* so that one could read – –4 as "the opposite of negative 4" rather than students' more common reading of "negative negative 4" (see Bofferding 2010; Vlassis 2008).

Edited by **Debra I. Johanning**, debra.johanning@utoledo.edu. Readers are encouraged to submit manuscripts through **http://mtms.msubmit.net**.

Table 1 Exploring the three meanings of the minus sign will allow students to differentiate among them.

Problem	Meaning of the Minus Sign
1. 5 – 8 = □	Subtraction as a binary operation
2. □ + 5 = –2	A symbolic representation for a negative number
3. Which is larger, – –4 or –4?	The *opposite of*, a unary operation

Fig. 1 Three explanations offer justification for the equivalence of two expressions.

3 – (*x* + 5) = 3 – *x* – 5 because—

- 3 – 1(*x* + 5) = 3 + –1(*x* + 5). — Subtracting *x* is equivalent to adding –*x*.
- 3 + –1(*x* + 5) = 3 + –*x* + –5. — Using the distributive property of multiplication over addition removes the parentheses.
- 3 + –*x* + –5 = 3 – *x* – 5. — Adding –*x* is equivalent to subtracting *x*.

In problem 1, the minus sign functions as a *binary operator* in that two inputs are used to produce one output. Addition, subtraction, multiplication, and division are examples of binary operators. For example, subtraction is a binary operator because the inputs of 5 and 8 result in one output, –3.

In problem 3, in contrast, the minus sign is used as a *unary operator* in that it involves only one input and one output. When one thinks of –(–4) as "the opposite of negative 4," then one is viewing the first minus sign as the unary operator, the *opposite of*. However, in problem 2, some people may view the minus sign in –2 as a unary operator, not as part of the number but instead as the *opposite of* 2. One who can view –2 in both ways can be said to flexibly hold both meanings of the minus sign.

Although students may initially face difficulties because three meanings are assigned to the same symbol, having the same symbol represent several ideas is important.

Treating One Meaning as if It Were Another

We often treat one meaning of the symbol as if it were another. For example, mathematically proficient students and adults will often rewrite 3 – (*x* + 5) as 3 – *x* – 5 without pause. They implicitly invoke two mathematical principles:

a. The equivalence of subtraction and addition of the subtrahend's inverse
b. The distributive property of multiplication over addition (see **fig. 1**)

Those who reason in this way treat the subtraction symbol as if it were a negative sign, although they rarely identify it as such. This ability to treat one meaning of the sign as if it were another is efficient for making calculations.

Changing the Meaning When Solving Equations

The meaning of the minus sign can change during the process of solving equations (Vlassis 2008). For example, in the equation

$$5 - x = 12,$$

the initial meaning of the minus sign is subtraction. When one solves for *x* by subtracting 5 from both sides, writing

$$-x = 7,$$

that action necessitates a change in the meaning of the minus sign from a binary operator (*subtraction*) to a unary operator (*the opposite of*). Students who recognize the minus sign as sometimes meaning the *opposite of* could reason that because the *opposite of x* is 7, then *x* must be equal to –7. For students to truly understand the process of solving equations, they need to be able to flexibly move among the different interpretations of the minus sign and be aware of when and why this sign can change interpretations.

Meaning Opposite Of

The notion of the minus sign as the unary operator *opposite of* is critical in supporting a sophisticated understanding of variables, algebraic expressions, and symbolically represented definitions. However, research indicates that middle school and high school students have difficulty with problems that require them to (at least implicitly) conceive of the minus sign as the *opposite of*. For example, in our research, we found that only about one-fourth of middle school students who were interviewed correctly identified – –4 as larger than –4. Further, they did not recognize that they had insufficient information to determine the larger of –*x* or *x*. Having a conception of the minus sign representing a negative number or representing subtraction is not particularly helpful when users are faced with these types of problems.

Table 2 These questions highlight the different meanings of the minus sign.

Student Work	Teacher Questions
A student remarks that "–3 – 5 has two negatives so my answer is –8."	"I see one negative [points to –3] and a subtraction sign [points to the subtraction sign]. Can you explain how you see two negatives?"
A student has written "–5" and "–*x*" on the board.	After a teacher sees this student's work, he or she asks: 1. "What would you call this symbol [points to –5]? Can you place this on the number line?" 2. "What would you call this symbol [points to –*x*]? Can you place this on the number line?" 3. "Is there another place we could locate –5? Is there another place we could locate –*x*? In what ways do these two symbols [points to the minus sign in each expression] have the same meaning? In what ways do they have different meanings?" 4. "Could we place the *opposite of x* [–*x*] on the positive side of the number line? Why, or why not?"

We view the responses to these problems as evidence that students not only struggle to respond correctly but also, by and large, have not engaged with the meaning of the minus sign as *opposite of*. Students who conceive of the expression –*x* as the *opposite of x* may be poised to understand that –*x* may represent a positive number. But students who lack a strong conception of –*x* and *opposite of* can, understandably, struggle.

When students use the language "negative *x*" to read or say "–*x*," they may think that "–*x*" should represent a negative number. *Negative* is used in the term, so why would one expect that "negative *x*" could represent a positive number?

Although we have found that students can move between the meanings of the minus sign as subtraction and as the sign of the number, students often implicitly make these shifts and struggle to explain their thinking. Further, very few students invoke the meaning of the symbol as *opposite of*. This limited view of the minus sign hinders students' abilities to understand expressions related to symbolically represented definitions (such as the definition of absolute value) and appropriately make sense of algebraic expressions (such as –*x*).

SUPPORTING MINUS-SIGN SYMBOL SENSE

We believe that supporting students' sense making of the three meanings of the minus sign and the students' ability to identify when each meaning might be appropriately invoked are important. Time and attention need to be paid for students to—

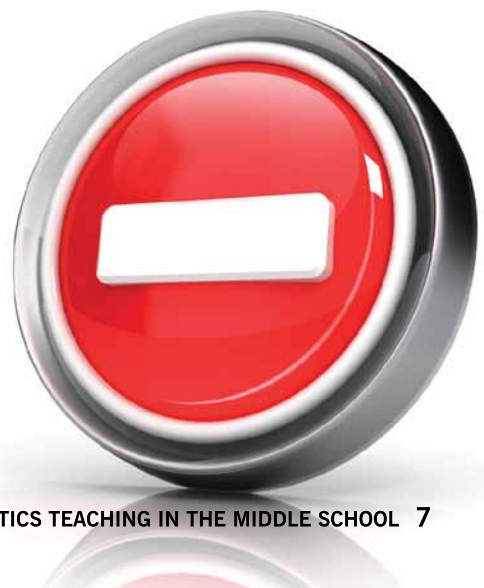
- learn the different meanings of the minus sign;
- recognize the appropriate meaning in a problem;
- understand when the meaning shifts during problem solving; and
- flexibly move among the meanings.

In the following sections, we share three ideas to help students (1) better understand the three meanings of the minus sign and (2) become aware of the shifting meanings.

Discuss and Make Explicit the Meanings of the Minus Sign

Table 2 presents two examples of problems that may arise naturally in your classroom and how you might leverage those problems to explicitly support students' understanding and awareness of the multiple meanings of the minus sign. A student might state that "–3 – 5 has two negatives" (see the first row), even though this problem as stated contains one negative number and subtraction of a positive number. Pressing students to explain their strategies will draw out both interpretations for discussion.

Similarly, when –*x* appears in a problem or as part of a solution (see the second row of **table 2**), stop and ask students to read the expression out loud. Many students will likely refer to it as "negative *x*." You might



PICTAFOLIO/ISTOCKPHOTO

Table 3 Various examples can help students become aware of language use in relation to the minus sign.

Equation	Teacher Actions
$6 \ominus x = 9$	Points to the minus sign and reminds students that the minus sign means subtraction.
$\ominus x = 3$	Points to the minus sign and states, "The *opposite of* x is equal to 3."
$x = \ominus 3$	Points to the minus sign and reminds students that the minus sign can also mean the sign of the number.

Table 4 The meaning of the minus sign as an opposite can be illustrated with several permutations.

Tasks		Directions for Each Task
a. $-(-4)$	4	Ask students to circle the larger quantity or write an equal sign if the values are equal. Have students write a question mark if too little information is given to determine the larger quantity. After each task, ask students how they determined the larger quantity and which meaning of the minus sign they used and why.
b. -6	$-(-6)$	
c. -4	x	
d. $x + x$	x	
e. x	$-x$	
f. $5 = -$_______		Ask, "Is there anything that you can put in the blank to make this equation true?"

then ask students how they would read the expression "–5." We expect that students will respond, "negative 5." If students think of and read $-x$ as "negative x," we suspect that when asked to place $-x$ and –5 on the number line, they will place $-x$ on the same side of the number line as –5. Using the line of questioning suggested in the second row of **table 2**, the-opposite-of interpretation can be discussed in conjunction with the negative-number interpretation.

Use Opposite of, Subtraction, and Negative Language

For students to truly understand the process of solving equations, they need to flexibly move among the different interpretations of the minus sign and be aware of when and why they can change. Being aware of the language you use can support students' understanding and help them recognize that the meaning of the symbol can change during the equation-solving process. **Table 3** provides suggestions for explicitly helping students become aware of language use in relation to the minus sign. In $6 - x = 9$, the initial meaning of – is subtraction. If one subtracts 6 from both sides, the resulting equation is $-x = 3$, necessitating a change in meaning from subtraction to *opposite of.* In the final answer, $x = -3$, the minus sign represents the sign of a number.

Comparison Tasks That Support the Opposite-of Interpretation

The tasks in **table 4** can be used to support students in making meaning of the minus sign as the *opposite of.* In discussing problems (a) and (b), students can reason about the minus sign as the *opposite of* by comparing numerical expressions. By using problems (c) and (d), students can expand their conceptions so that the possible values for x include negative numbers. Students can discuss how, depending on the value of x, either expression may be larger. After students have reasoned through several problems similar to (a)–(d), pose a task such as (e) to give students opportunities to expand their ideas about the symbol by combining ideas about possible values for x with the meaning of the minus sign as the *opposite of.*

CONCLUSION

In closing, we believe that students, given opportunities to engage in discussions like those suggested in **tables 2–4**, can develop a sophisticated understanding of the three meanings of the minus sign. Explicitly discussing each meaning of the symbol with a particular emphasis on the *opposite of* concept will enable students to robustly reason about operations and algebraic expressions. We also believe that these experiences will support students in developing symbol sense in relation to the minus sign that will foster their future learning as they move from middle school into high school and beyond.

This material is based on work supported by the National Science Foundation (NSF) under grant number DRL-0918780. Any opinions, findings, conclusions, and recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NSF. The authors thank Debra Johanning for her thoughtful comments on drafts of this article.

REFERENCES

Arcavi, Abraham. 1994. "Symbol Sense: Informal Sense-Making in Formal Mathematics." *For the Learning of Mathematics* 14:24–35.

Bofferding, Laura. 2010. "Addition and Subtraction with Negatives: Acknowledging the Multiple Meanings of the Minus Sign." In *Proceedings of the 32nd Annual Conference of the North American Chapter of the Psychology of Mathematics Education*, edited by Patricia Brosnan, Diana B. Erchick, and Lucia Flevares, pp. 703–10. Columbus, OH: The Ohio State University. http://pmena.org/2010/

Vlassis, Joelle. 2008. "The Role of Mathematical Symbols in the Development of Number Conceptualization: The Case of the Minus Sign." *Philosophical Psychology* 21:555–57.

(*Ed. note.* For more on this topic, read "A Proposed Instructional Theory for Integer Addition and Subtraction," by Michelle Stephan and Didem Akyuz, in the July 2012 issue of the *Journal for Research in Mathematics Education*.)






Lisa L. Lamb, Lisa.Lamb@sdsu.edu; **Jessica Pierson Bishop**, jpierson@mail.sdsu.edu; **Randolph A. Philipp**, rphilipp@mail.sdsu.edu; **Bonnie P. Schappelle**, bschappe@sunstroke.sdsu.edu; **Ian Whitacre**, ianwhitacre@yahoo.com; and **Mindy Lewis**, mindydanenhauer@hotmail.com, work together at San Diego State University to investigate grades K–12 students' conceptions of negative numbers.