26 settembre

Tutore dott. Graffeo

Venerdì 16 - 18

Giovedì niente lezione

Sommatorie e prodotti

Definizione Dati i numeri $a_1, \dots, a_N$ la somma $a_1 + \dots + a_N$ viene denotata con

$$\sum_{j=1}^{N} a_j := a_1 + \dots + a_N$$

(ed il prodotto $a_1 \cdot \dots \cdot a_N$ viene denotato con

$$\prod_{j=1}^{N} a_j = a_1 \cdot \dots \cdot a_N$$)

$a_1, \dots, a_N$

Proprietà

$$\sum_{j=1}^{N} a_j = \sum_{k=1}^{N} a_k$$

$$\sum_{j=1}^{1} a_j = a_1$$

$$\sum_{j=1}^{10} a_j = \sum_{j=1}^{7} a_j + \sum_{j=8}^{10} a_j$$

$$\sum_{j=1}^{N} (a_j + b_j) = \sum_{j=1}^{N} a_j + \sum_{j=1}^{N} b_j$$

$$\sum_{j=1}^{N} c\, a_j = c \sum_{j=1}^{N} a_j$$

Lemma
$$\sum_{j=1}^{N} j = \frac{(N+1)N}{2} \qquad (P_N)$$

per ogni $N = 1, 2, 3, \dots\dots$.

Dim Per induzione

1) Verificare che (P_1) è vera

$$\sum_{j=1}^{1} j = 1$$

$$\frac{(1+1)\,1}{2} = 1$$

2) Verificare che $(P_N) \Rightarrow (P_{N+1})$

$$\sum_{j=1}^{N} j = \frac{(N+1)N}{2} \qquad (P_N)$$

$$\boxed{\sum_{j=1}^{N+1} j} = \boxed{\frac{(N+2)(N+1)}{2}} \qquad (P_{N+1})$$

$$\boxed{\sum_{j=1}^{N+1} j} = \sum_{j=1}^{N} j + N + 1 = \frac{(N+1)N}{2} + (N+1) =$$

$$= (N+1)\left(\frac{N}{2} + 1\right) = \boxed{(N+1)\,\frac{N+2}{2}}$$

Concludiamo che (P_N) è vera $\forall N = 1, \dots$

$\square$

(Notare che ad esempio $\sum_{j=N}^{N-1} a_j = 0$)

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2}$$

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2}$$

$$= 1+2+3+4+\dots\dots+(N-3)+(N-2)+(N-1)+N$$

$N+1$ $N+1$ $N+1$ $N+1$

Teorema (somma geometrica di ragione x)

Sia $x \neq 1$. Allora

$$\sum_{j=0}^{N} x^j = \frac{1-x^{N+1}}{1-x}$$

$(x^0 = 1$, $0^0 = 1)$

$$\left(\text{se } x=1 \quad \sum_{j=0}^{N} 1^j = \sum_{j=0}^{N} 1 = N+1 \right)$$

$$\sum_{j=0}^{2} 1 = 1+1+1 = 3$$

$$\sum_{j=0}^{N} x^j = \frac{1-x^{N+1}}{1-x}$$

Dim

1) Dimostrare caso $N=0$

$$\sum_{j=0}^{N} x^j = x^0 = 1$$

$$\frac{1-x}{1-x} = 1$$

2) Dimostrare $N \Rightarrow N+1$

Assumiamo $\displaystyle\sum_{j=0}^{N} x^j = \frac{1-x^{N+1}}{1-x}$

dimostriamo $\displaystyle\sum_{j=0}^{N+1} x^j = \frac{1-x^{N+2}}{1-x}$

$$\sum_{j=0}^{N+1} x^j = \sum_{j=0}^{N} x^j + x^{N+1} = \frac{1-x^{N+1}}{1-x} + x^{N+1}$$

$$= \frac{(1-x^{N+1}) + x^{N+1}(1-x)}{(1-x)} = \frac{1-\cancel{x^{N+1}}+\cancel{x^{N+1}}-x^{N+2}}{1-x}$$

$$= \frac{1-x^{N+2}}{1-x}$$

$\square$

$$\sum_{j=0}^{N} x^j = \frac{1-x^{N+1}}{1-x} \qquad \cdot(1-x)$$

$$= (1-x)\sum_{j=0}^{N} x^j = 1-x^{N+1}$$

$$(1-x)\left(1+x^1+x^2+\dots+x^{N-2}+x^{N-1}+x^N\right)$$

$$= 1+\cancel{x^1}+\cancel{x^2}+\dots+\cancel{x^{N-2}}+\cancel{x^{N-1}}+\cancel{x^N}-\cancel{x}-\cancel{x^2}-\cancel{x^3}-\cancel{\dots}-\cancel{x^{N-1}}-\cancel{x^N}-x^{N+1}$$

$$= 1-x^{N+1}$$

0.7.3 0.7.6 0.7.4

0.7.5 0.7.7

Def per $n \in \mathbb{N}$ denotiamo con $n!$ (n fattoriale) il seguente numero

$$n! = \begin{cases} 1 & \text{se } n = 0 \\ \prod_{j=1}^{n} j \end{cases}$$

Es $\quad 3! = 1 \cdot 2 \cdot 3 = 6 \qquad 4! = 24$

__Esercizio__ Dimostrate che per

$$(n+1)! = n! \, (n+1)$$

__Def__ (Coefficienti binomiali) Dati $N \geq k \geq 0$

si definisce il "coefficiente binomiale"

$$\binom{N}{k} = \frac{N!}{k! \, (N-k)!}$$

__Osservazione__ $\binom{N}{k} = \binom{N}{N-k}$ sempre :

$$\binom{N}{N-k} = \frac{N!}{(N-k)! \, (N-(N-k))!} = \frac{N!}{(N-k)! \, k!}$$

$$\binom{N}{0} = \frac{N!}{0! \, N!} = 1 \qquad \forall N \geq 0$$

$$\binom{N}{N} = 1 \qquad \forall N \geq 0$$

$$\binom{N}{1} = \binom{N}{N-1} = N \qquad \forall N$$

$$\binom{N}{1} = \frac{N!}{\underbrace{1!}_{1} \, (N-1)!} = \frac{N!}{(N-1)!} = \frac{\cancel{(N-1)!} \; N}{\cancel{(N-1)!}}$$

Teorema (Formula di Newton del binomio) $\quad N \geq 1$

$$(a+b)^N = \sum_{j=0}^{N} \binom{N}{j} a^j \, b^{N-j}$$

$$= \sum_{j=0}^{N} \binom{N}{j} a^{N-j} \, b^j$$

__Osservazione__

$$\binom{N}{j} = \frac{N!}{j! \, (N-j)!} = \frac{\cancel{1 \cdot 2 \cdots (N-j)} \, (N-j+1) \cdots N}{j! \, \cancel{(N-j)!}}$$

$$= \frac{(N-j+1) \cdots N}{j!} = \frac{\prod_{k=1}^{j} (N-k+1)}{j!}$$