27 settembre

Analisi Armonico (Teoria di Calderon Zygmund)

Caffarelli - Kohn - Nirenberg

Robinson - Rodrigo - Sadosky

Teorema esistenza soluzioni deboli di Leray

Teoria delle mild solutions Kato

Equazioni nonlineari di Schrödinger

$\mathcal{F}f(\xi) =$

$$\hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\xi} f(x)\, dx$$

$$f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\xi} \hat{f}(\xi)\, d\xi$$

$$\widehat{f * g}(\xi) = (2\pi)^{\frac{d}{2}} \hat{f}(\xi)\, \hat{g}(\xi)$$

$$L^2(\mathbb{R}^d, \mathbb{C}) \overset{\mathcal{F}}{\circlearrowleft}$$

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_d(x) \end{pmatrix} \in L^2(\mathbb{R}^d, \mathbb{C}^d) \; \underset{\mathcal{F}}{\circlearrowleft}$$

$$\mathcal{F}f = \begin{pmatrix} \mathcal{F}f_1 \\ \vdots \\ \mathcal{F}f_d \end{pmatrix}$$

$u \in \mathcal{D}'(\mathbb{R}^d, \mathbb{C})$ $\qquad$ $\Delta u = \sum_{j=1}^{d} \partial_j^2 u$

$f \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C})$

$$\begin{cases} \partial_t u - \Delta u = 0 & t > 0 \\ u(0) = f \end{cases}$$

$$\begin{cases} \partial_t \hat{u}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) = 0 & \xi = (\xi_1, \dots, \xi_d) \\ \hat{u}(0,\xi) = \hat{f}(\xi) \end{cases}$$

$$\hat{u}(t,\xi) = e^{-t|\xi|^2} \hat{u}(0,\xi) = e^{-t|\xi|^2} \hat{f}(\xi)$$

$$e^{-t|\xi|^2} = \hat{G}(t,\xi)$$

$$G(t,x) = (2t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$$

$$(2\pi)^{\frac{d}{2}} \hat{u}(t,\xi) = \hat{G}(t,\xi) \hat{f}(\xi) (2\pi)^{\frac{d}{2}} = \widehat{G(t) * f}$$

$$u(t,x) = (2\pi)^{-\frac{d}{2}} G(t,\cdot) * f =$$

$$\left(\widehat{f * g} = (2\pi)^{\frac{d}{2}} \hat{f} \hat{g} \right)$$

$$= (2\pi)^{-\frac{d}{2}} \int G(t, x-y) f(y) =$$

$$= (2\pi)^{-\frac{d}{2}} (2t)^{-\frac{d}{2}} \int e^{-\frac{|x-y|^2}{4t}} f(y)\,dy$$

$$u(t,x) = (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)\,dy$$

$$K(t, x-y)$$

Lieb - Loss

$$u = K(t,\cdot) * f$$

$$K_t(\cdot)$$

<u>Lemma</u> $\forall\ q \geq p \geq 1 \quad \exists\ C_{pq}$ t.c.

$$|K_t * f|_{L^q(\mathbb{R}^d)} \leq C_{pq}\, t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} |f|_{L^p(\mathbb{R}^d)}$$

<u>Dim</u>

$$|K_t * f|_{L^q(\mathbb{R}^d)} \leq |K_t|_{L^a}\ |f|_{L^p}$$

$$\frac{1}{q} + 1 = \frac{1}{a} + \frac{1}{p} \qquad \text{Young}$$

$$| K_t |_{L^a} = (4\pi t)^{-\frac{d}{2}} \left| e^{-\frac{|x|^2}{4t}} \right|_{L^a(\mathbb{R}^d)} =$$

$$= (4\pi t)^{-\frac{d}{2}} \left| e^{-\frac{\left|\frac{x}{\sqrt{t}}\right|^2}{4}} \right|_{L^a(\mathbb{R}^d)}$$

$$= (4\pi t)^{-\frac{d}{2}} \quad t^{\frac{1}{2}\frac{d}{a}} \quad \left| e^{-\frac{|x|^2}{4}} \right|_{L^a(\mathbb{R}^d)}$$

$$y = \frac{x}{\sqrt{t}} \qquad dx = t^{\frac{d}{2}} dy$$

$$t^{\frac{d}{2}\left(\frac{1}{a} - 1\right)}$$

$$= \underbrace{(4\pi)^{-\frac{d}{2}} \left| e^{-\frac{|x|^2}{4}} \right|_{L^a(\mathbb{R}^d)}}_{C_{pq}}$$

$$\frac{1}{q} + 1 = \frac{1}{a} + \frac{1}{p}$$

$$\frac{1}{a} - 1 = \frac{1}{q} - \frac{1}{p}$$

$$= C_{pq} \quad t^{\frac{d}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} \qquad \square$$

$$C_{pp} = 1$$

Spazi di Sobolev basati su $L^2(\mathbb{R}^d)$

$\xi \in \mathbb{R}^d \qquad \langle \xi \rangle = \sqrt{1 + |\xi|^2}$

$$H^n(\mathbb{R}^d) = W^{n,2}(\mathbb{R}^d)$$

$$|u|^2_{H^n(\mathbb{R}^d)} = \sum_{|\alpha| \leq n} |\partial^\alpha_x u|^2_{L^2(\mathbb{R}^d)}$$

$$\mathcal{H}^n(\mathbb{R}^d)$$

$$|u|^2_{\mathcal{H}^n(\mathbb{R}^d)} = |\langle \xi \rangle^n \hat{u}|_{L^2(\mathbb{R}^d)}$$

$$\langle \langle \xi \rangle^n \hat{u}, \varphi \rangle = \langle \hat{u}, \langle \xi \rangle^n \varphi \rangle$$

$$H^n(\mathbb{R}^d, \mathbb{C}) = \mathcal{H}^n(\mathbb{R}^d, \mathbb{C})$$

$$|u|_{H^1(\mathbb{R}^d)} = |\langle \xi \rangle^1 \hat{u}|_{L^2(\mathbb{R}^d)}$$

$$W^{n,p}(\mathbb{R}^d)$$

$1 < p < \infty$

$W^{s,p}(\mathbb{R}^d)$

$$\| u \|_{W^{s,p}(\mathbb{R}^d)} = \| \left(\langle \xi \rangle^s \hat{u} \right)^\vee \|_{L^p(\mathbb{R}^d)}$$

$L^2(\mathbb{R}^d) \qquad H^1(\mathbb{R}^d)$

$s \in \mathbb{R}$

$\dot{H}^s(\mathbb{R}^d)$ è lo spazio formato dalle distribuzioni temperate u t.c. $\hat{u} \in L^1_{loc}(\mathbb{R}^d)$ e

$$\| u \|_{\dot{H}(\mathbb{R}^d)} = \| \, |\xi|^s \hat{u} \, \|_{L^2(\mathbb{R}^d)} < +\infty$$

$d \geq 3$

$2^* \qquad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{d}$

$$\| u \|_{L^{2^*}(\mathbb{R}^d)} \leq C \, \| u \|_{\dot{H}^1(\mathbb{R}^d)}$$

$s \in \mathbb{R}$

$\dot{H}^s(\mathbb{R}^d)$ è lo spazio formato dalle distribuzioni temperate u t.c.

$\hat{u} \in L^1_{loc}(\mathbb{R}^d)$ e

Bahouri, Chemin, Desjardins

<u>Lemma</u> $\mathcal{S}(\mathbb{R}^d) \subseteq \dot{H}^s(\mathbb{R}^d)$

$\iff \quad s > -\frac{d}{2}$

$$|u|_{\dot{H}^s} = \| \, |\xi|^s \hat{u} \, \|_{L^2}$$

$$\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}|^2 \, d\xi < +\infty$$

$\hat{u}(0) \neq 0$ e $2s \leq -d$ non è vero.

$\mathcal{S}(\mathbb{R}^d)$ è denso in $\dot{H}^s(\mathbb{R}^d)$

se $s > -\frac{d}{2}$

Prop $\dot{H}^s(\mathbb{R}^d)$ è completo per $s < \frac{d}{2}$

e $\mathcal{F} : \dot{H}^s(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d \setminus \{0\}, |\xi|^{2s} d\xi)$

è una isometria ed un isomorfismo

Lemma $s < \frac{d}{2}$

- $L^2(\mathbb{R}^d \setminus \{0\}, |\xi|^{2s} d\xi) \subset L^1_{loc}(\mathbb{R}^d, d\xi)$

- $\underbrace{L^2(\mathbb{R}^d \setminus \{0\}, |\xi|^{2s} d\xi)}_{g} \subset \mathcal{S}'(\mathbb{R}^d)$

$L^2(\mathbb{R}^d \setminus \{0\}, |\xi|^{2s} d\xi) \subseteq L^1_{loc}(\mathbb{R}^d \setminus \{0\}, d\xi)$

$B = B_{\mathbb{R}^d}(0,1)$

$$\int_B |g| \, d\xi = \int_B |\xi|^{-s} |\xi|^s |g| \, d\xi$$

$$\leq \underbrace{\left(\int_B |\xi|^{-2s} d\xi \right)^{\frac{1}{2}}}_{< +\infty \atop s < \frac{d}{2}} \left(\int_{\mathbb{R}^d} |\xi|^{2s} |g|^2 d\xi \right)^{\frac{1}{2}}$$

$$\Rightarrow -2s > -d$$

$$g \in L^2(\mathbb{R}^d \setminus \{0\}, |\xi|^{2s}\, d\xi)$$

$$g\chi_B + g(1-\chi_B)$$

$$\Uparrow$$

$$L^1(\mathbb{R}^d, d\xi)$$

$$g(1-\chi_B) \in L^2(\mathbb{R}^d, \langle \xi \rangle^{2s}\, d\xi)$$

$$\left| \int g(1-\chi_B)\, \varphi\, d\xi \right| =$$

$< +\infty$

$$\leq \left(\int \langle \xi \rangle^{2s} g^2 (1-\chi_B)^2\, d\xi \right)^{\frac{1}{2}}$$

$$\left(\int \langle \xi \rangle^{-2s} \varphi^2\, d\xi \right)^{\frac{1}{2}} =$$

$$\int \langle \xi \rangle^{-2s} \langle \xi \rangle^{-N} \langle \xi \rangle^{N} \varphi^2\, d\xi$$

$$\leq \sup \langle \xi \rangle^{2N} \varphi^2 \qquad \int_{\mathbb{R}^d} \langle \xi \rangle^{-2s-2N} d\xi$$

$$-2s - N < -d$$