16/09/21

28/06/22

16/09/21

Es 1: $f(z) = \dfrac{z^k}{1+z^n}$, $k, n \in \mathbb{N}$

$1 \leq k < n$

1) f ha polo di ordine 1 nelle n soluzioni di $z^n + 1 = 0$, $z_a^* = e^{i\frac{\pi}{n} + i\frac{2\pi(a-1)}{n}}$

$a = 1, \dots, n$

Ordine uno perché: $z^n + 1 = (z - z_1^*) \cdots (z - z_n^*)$

I fattori $(z - z_a^*)$ compaiono alla potenza 1 al denominatore.

Rimane da controllare $z = \infty$:

$$f(z) \underset{z\to\infty}{=} \frac{z^k}{z^n\left(1 + \frac{1}{z^n}\right)} = \underbrace{z^{k-n}}\left(1 - O\left(\frac{1}{z^n}\right)\right)$$

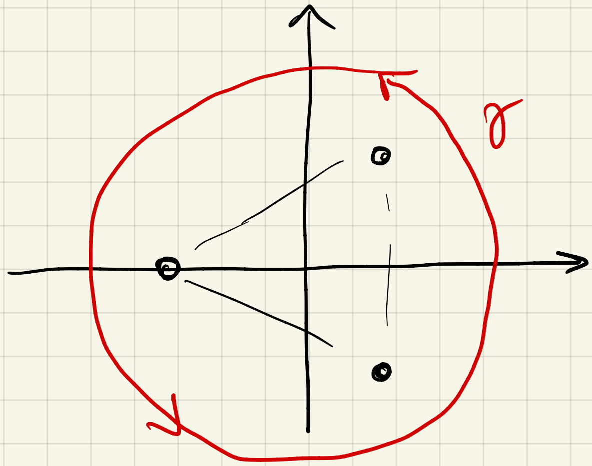
$k - n < 0 \Rightarrow f(z) \xrightarrow[z\to\infty]{} 0$

$\Rightarrow f$ è regolare a $z = \infty$.

2) Dal calcolo sopra vediamo che serve: $k - n = -1$

$\Rightarrow k_* = n - 1$

3)



$$\oint_\gamma f(z)\,dz \underset{\text{teo. interno}}{=} 2\pi i \sum_{a=1}^{n} \mathrm{Res}_f(z_a^*)$$

$$\underset{\text{teo. esterno}}{=} -2\pi i \, \mathrm{Res}_f(\infty)$$

$$\Rightarrow \sum_{a=1}^{n} \mathrm{Res}_f(z_a^*) = -\mathrm{Res}_f(\infty)$$

$$= \begin{cases} 0 & , \; k \neq k_* \\ -(-1) = 1 & , \; k = k_* . \end{cases}$$

Es 2:
$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0$$

1) $\phi(t,x) \rightarrow \hat{\phi}(t,k)$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\hat{\phi}(t,k) + k^2 \hat{\phi}(t,k) = 0$$

$$\Rightarrow \frac{\partial^2 \hat{\phi}(t,k)}{\partial t^2} = -c^2 k^2 \hat{\phi}(t,k)$$

Soluzione: $\hat{\phi}(t,k) = e^{ickt}\alpha(k) + e^{-ickt}\beta(k)$

$$\phi(0,x) = g(x) \Rightarrow \hat{\phi}(0,k) = \hat{g}(k)$$

$$\partial_t \phi(0,x) = h(x) \Rightarrow \partial_t \hat{\phi}(0,k) = \hat{h}(k)$$

$$\Rightarrow \quad \alpha(k) + \beta(k) = \hat{g}(k)$$

$$i C k (\alpha(k) - \beta(k)) = \hat{h}(k)$$

$$\hat{g}(k) + \frac{\hat{h}(k)}{i C k} = 2\alpha(k) \Rightarrow \alpha(k) = \frac{1}{2}\left(\hat{g} - i \frac{\hat{h}}{Ck}\right)$$

$$\hat{g}(k) - \frac{\hat{h}(k)}{i C k} = 2\beta(k) \Rightarrow \beta(k) = \frac{1}{2}\left(\hat{g} + i \frac{\hat{h}}{Ck}\right).$$

2)
$$\phi(t,x) = \int \frac{dk}{2\pi} \hat{\phi}(t,k) e^{-ikx}$$

$$= \underbrace{\int \frac{dk}{2\pi} \alpha(k) e^{-ik(x-ct)}}_{\underbrace{\mathcal{F}^{-1}[\alpha](x-ct)}_{\phi_1}} + \underbrace{\int \frac{dk}{2\pi} \beta(k) e^{-ik(x+ct)}}_{\underbrace{\mathcal{F}^{-1}[\beta](x+ct)}_{\phi_2}}$$

Es 3: $\left\{ \frac{e^{-in\theta}}{\sqrt{2\pi}} \right\}_{n \in \mathbb{Z}}$ s.o.c. su $L^2([0,2\pi])$

1) $$\left(\frac{e^{-in\theta}}{\sqrt{2\pi}}, F(\theta) \right) = \int_0^{2\pi} d\theta \, \frac{e^{in\theta}}{\sqrt{2\pi}} \, \frac{1}{5 + 4\cos\theta}$$

$n \geq 0$

$$e^{i\theta} = z, \quad dz = i e^{i\theta} d\theta = i z d\theta$$

$$= \oint_{|z|=1} \frac{dz}{iz} \frac{z^n}{\sqrt{2\pi}} \frac{1}{5+4\,\frac{z+1/z}{2}}$$

$$= \oint_{|z|=1} \frac{dz}{iz} \frac{z^n}{\sqrt{2\pi}} \frac{1}{5+2z+\frac{2}{z}}$$

$$= -\frac{i}{\sqrt{2\pi}} \oint_{|z|=1} dz \frac{z^n}{\underbrace{2z^2+5z+2}}$$

$$z_\pm = \frac{-5/2 \pm \sqrt{\frac{25}{4}-4}}{2} = \frac{-5/2 \pm 3/2}{2} \begin{cases} -1/2 \\ -2 \end{cases}$$

$$2\cdot 4 - 10 + 2 = 0$$

$$2\,1/4 - 5/2 + 2 = 0$$

$$= -\frac{i}{2\sqrt{2\pi}} \oint_{|z|=1} dz \frac{z^n}{(z+2)(z+1/2)}$$

$n \geq 0$: unica singolarità nel cerchio unitario è polo in $z = -1/2$

$$= -\frac{i}{2\sqrt{2\pi}}\, 2\pi i \frac{(-1/2)^n}{(-\frac{1}{2}+2)} = \frac{\sqrt{2\pi}}{2} \frac{(-1/2)^n}{3/2} = \frac{\sqrt{2\pi}}{3} \frac{(-1)^n}{2^n}$$

$$= \alpha_n$$

Per calcolare α_n con n negativo usiamo che:

$$\alpha_n^* = \left(\int_0^{2\pi} d\theta \, \frac{e^{in\theta}}{\sqrt{2\pi}} F(\theta)\right)^* = \int_0^{2\pi} d\theta \, \frac{e^{-in\theta}}{\sqrt{2\pi}} \underbrace{(F(\theta))^*}_{= F(\theta)}$$

$$= \alpha_{-n}$$

già calcolato

$$\Rightarrow \text{per } n<0, \quad \alpha_n = (\alpha_{-n})^* = \alpha_{-n} \quad \text{perché è reale}$$

$$\Rightarrow \boxed{\alpha_n = \frac{\sqrt{2\pi}}{3} \frac{(-1)^{|n|}}{2^{|n|}}, \quad \forall n \in \mathbb{Z}}$$

2) $$(G,F) = \sum_{n=-\infty}^{+\infty} b_n^* \alpha_n$$

$$= \sum_{n=0}^{\infty} 2^{-n} \frac{\sqrt{2\pi}}{3} \frac{(-1)^n}{2^n}$$

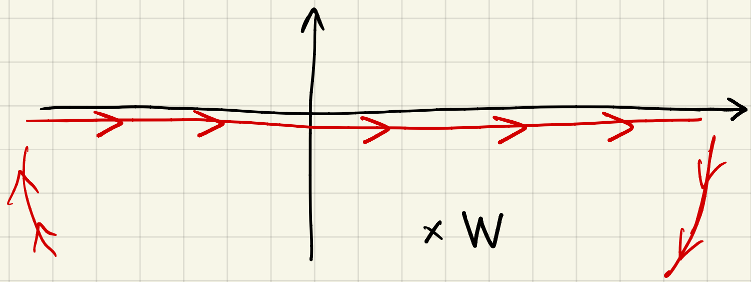
$$= \frac{\sqrt{2\pi}}{3} \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{\sqrt{2\pi}}{3} \frac{1}{1+\frac{1}{4}} = \sqrt{2\pi} \, \frac{4}{15}$$

28/06/22

Es 1 $\quad f(z)$ olomorfa su $\operatorname{Im} z \leq 0$

$$I(w) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dx \, \frac{f(x)}{(x-w)^2}$$

1)



Possiamo chiudere con arco a ∞ se:

$$\lim_{R\to\infty} 2\pi \cancel{R} \frac{1}{R^{\cancel{2}}} \max_{\gamma_R^-} |f| = 0$$

$\Rightarrow$ $\max_{\gamma_R^-} |f|$ cresce per $R\to\infty$ meno di R.

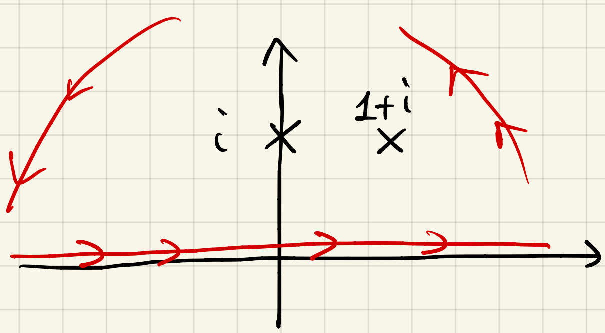
$$\Rightarrow I(w) = -2\pi i \frac{i}{2\pi} \operatorname{Res}_{\frac{f(z)}{(z-w)^2}} [w]$$

solo singolarità in w nel semipiano inferiore

$$= \lim_{z\to w} \frac{d}{dz}\left[\cancel{(z-w)^2} \frac{f(z)}{\cancel{(z-w)^2}} \right]\Bigg|_{z=w}$$

$$= f'(w).$$

2)



$$f(z) = \frac{1}{z-i} + \frac{1}{z-i-1}$$

$$I(w) = \frac{i}{2\pi} \times 2\pi i \times \left[\operatorname{Res}_{\frac{1}{(z-w)^2}\left(\frac{1}{z-i}+\frac{1}{z-i-1}\right)} [i] \right.$$

$$\left. + \operatorname{Res}_{\frac{1}{(z-w)^2}\left(\frac{1}{z-i}+\frac{1}{z-i-1}\right)} [i+1] \right]$$

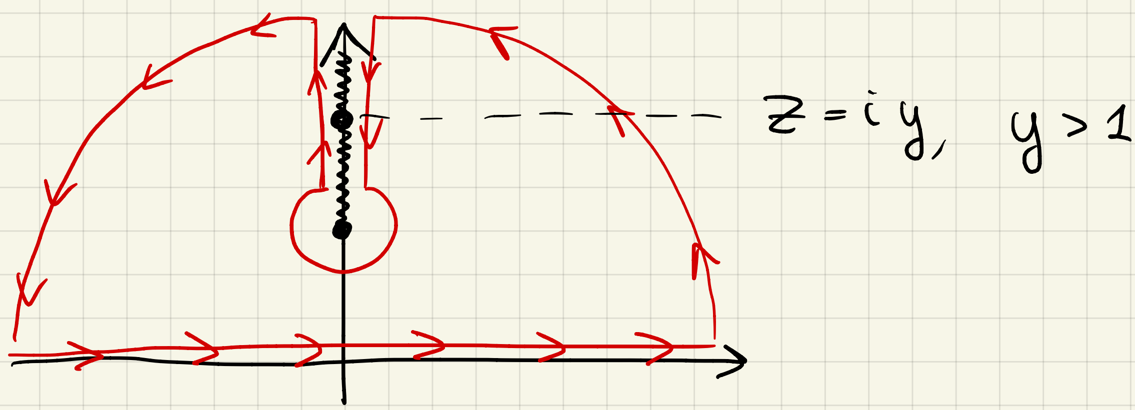
$$= - \lim_{z \to i} \left[(z-i) \frac{1}{(z-w)^2} \left(\frac{1}{z-i} + \frac{1}{z-i-1} \right) \right]$$

$$- \lim_{z \to i+1} \left[(z-i-1) \frac{1}{(z-w)^2} \left(\frac{1}{z-i} + \frac{1}{z-i-1} \right) \right]$$

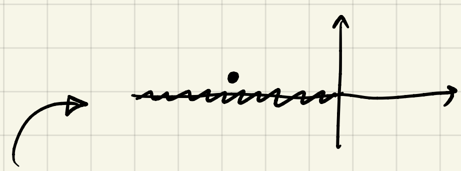
$$= - \frac{1}{(i-w)^2} - \frac{1}{(i+1-w)^2}$$

$$= - \frac{1}{(w-i)^2} - \frac{1}{(w-i-1)^2} = f'(w)$$

3)

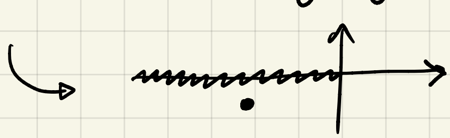


$$f(z) = \log(1+iz)$$



$$f_{destra}(iy+\delta) = \log(1-y+i\delta) = \log(y-1) + i\pi$$

$$f_{sinistra}(iy-\delta) = \log(1-y-i\delta) = \log(y-1) - i\pi$$



$$I(w) = \frac{i}{2\pi} \lim_{R \to \infty} \int_{-R}^{+R} dx \, \frac{f(x)}{(x-w)^2}$$

$$= \frac{i}{2\pi} \lim_{\substack{R\to\infty \\ \epsilon\to 0^+}} \Bigg[\oint dz \frac{f(z)}{(z-w)^2} - \int_{\gamma_R^+} dz \frac{f(z)}{(z-w)^2}$$

$$- \int_{\gamma_\epsilon} dz \frac{f(z)}{(z-w)^2} - \int_{1+\epsilon}^{R} dy \; i \; \frac{f_{sin} - f_{destra}}{(iy-w)^2} \Bigg]$$

$$(dz = i\,dy)$$

- $\displaystyle\int_{\gamma_R^+} dz \frac{f(z)}{(z-w)^2} \xrightarrow[R\to\infty]{} 0$ per le ragioni già viste

- $\displaystyle\left| \int_{\gamma_\epsilon} dz \frac{f(z)}{(z-w)^2} \right| = \left| \int_0^{2\pi} d\theta \frac{\log(\varepsilon e^{i\theta})}{(i - i\varepsilon e^{i\theta} - w)^2} \right|$

$$z = i - i\varepsilon e^{i\theta}$$

$$\theta \in [0, 2\pi]$$

$$\underset{\varepsilon\to 0^+}{\leq} 2\pi\varepsilon \max_{\theta\in[0,2\pi]} \underbrace{|\log(\varepsilon e^{i\theta})|}_{\sqrt{(\log\varepsilon)^2 + \theta^2} = |\log\varepsilon|(1 + O(1/|\log\varepsilon|^2))} (1 + O(\varepsilon)) \xrightarrow[\varepsilon\to 0^+]{} 0.$$

- $\displaystyle\oint dz \frac{f(z)}{(z-w)^2} = 0$ perché non c'è singolarità

$$\Rightarrow I(w) = \frac{i}{\cancel{2\pi}} \left(-\int_1^\infty dy \frac{\cancel{2\pi}}{(iy-w)^2} \right)$$

$$= -i \int_1^\infty dy \frac{1}{(iy-w)^2} = -\cancel{i} \frac{1}{\cancel{i}} \left(-\frac{1}{iy-w} \right) \Big|_1^\infty = -\frac{1}{i-w}$$

$$= -\frac{1}{i}\frac{1}{1+iW} = i\frac{1}{1+iw} = f'(w).$$

Es 2

$$T_G[\phi](t) = \int_{-\infty}^{+\infty} dt'\, G(t-t')\phi(t')$$

$$T_G[\phi](t) = \lambda\phi(t)$$

1) $G \in L^1(\mathbb{R})$

Applicando $\mathcal{F}$ all'eq. agli autovalori

$$\hat{G}(\omega)\hat{\phi}(\omega) = \lambda\hat{\phi}(\omega)$$

2) $$G(t) = P\left(\frac{1}{t}\right) = \lim_{\epsilon\to 0^+} \frac{1}{2}\frac{1}{t+i\varepsilon} + \frac{1}{2}\frac{1}{t-i\varepsilon}$$

$$= \lim_{\epsilon\to 0^+} \frac{t}{t^2+\varepsilon^2}$$

$$\hat{G}(\omega) = \lim_{\epsilon\to 0^+} \int dt\, \frac{t}{t^2+\varepsilon^2} e^{it\omega}$$

$$= \lim_{\epsilon\to 0^+}\Big[\Theta(\omega)\, 2\pi i\, \mathrm{Res}_{\frac{t}{t^2+\varepsilon^2}e^{it\omega}}[i\varepsilon]$$

$$+\ \Theta(-\omega)(-2\pi i)\mathrm{Res}_{\frac{t}{t^2+\varepsilon^2}e^{it\omega}}[-i\varepsilon]\Big]$$

$$= \lim_{\epsilon\to 0^+}\Big[\Theta(\omega)\, 2\pi i \,\frac{\cancel{i}\varepsilon}{2\cancel{i}\varepsilon}\, e^{-\epsilon\omega}$$

$$+ \Theta(-\omega)(-2\pi i)\,\frac{\cancel{-i}\varepsilon}{\cancel{-}2\cancel{i}\varepsilon}\, e^{+\epsilon\omega}\Big]$$

$$= \lim_{\epsilon\to 0^+}\left[\pi i\, \Theta(\omega)\, e^{-\varepsilon\omega} - \pi i\, \Theta(-\omega)\, e^{\varepsilon\omega}\right]$$

$$= \pi i \operatorname{sgn}(\omega)$$

$$\Rightarrow \quad \pi i \operatorname{sgn}(\omega)\, \hat{\phi}(\omega) = \lambda\, \hat{\phi}(\omega)$$

3) Se $\hat{\phi}(\omega) \neq 0$ solo per $\omega > 0$:

$$\pi i\, \hat{\phi}(\omega) = \lambda\, \hat{\phi}(\omega) \Rightarrow \text{soddisfatta con } \lambda = \pi i$$

Se $\hat{\phi}(\omega) \neq 0$ solo per $\omega < 0$:

$$-\pi i\, \hat{\phi}(\omega) = \lambda\, \hat{\phi}(\omega) \Rightarrow \text{soddisfatta con } \lambda = -\pi i.$$

Es 3 $\quad \{e^{(n)}\}_{n\geq 1}$ s.o.c. su H Hilbert

1) $f^{(1)} = e^{(1)} + e^{(2)}$

$f^{(2)} = e^{(1)} - e^{(2)} + e^{(3)}$

$$f^{(3)} = e^{(1)} - e^{(2)} - e^{(3)} + e^{(4)}$$

$\vdots$

$$f^{(n)} = e^{(1)} - e^{(2)} - \dots - e^{(n)} + e^{(n+1)}$$

Per vedere se è completo, assumiamo che $\exists\, v$ con $(f^{(n)}, v) = 0$, $\forall n$. Dobbiamo mostrare che $v = 0$.

$$v = \sum_{n=1}^{\infty} \alpha_n e^{(n)}, \quad 0 = (f^{(1)}, v) = \alpha_1 + \alpha_2$$

$$\Rightarrow \alpha_2 = -\alpha_1$$

$$0 = (f^{(2)}, v) = \alpha_1 - \alpha_2 + \alpha_3 = 2\alpha_1 + \alpha_3 \Rightarrow \alpha_3 = -2\alpha_1$$

$$0 = (f^{(3)}, v) = \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = \alpha_1 + \alpha_1 + 2\alpha_1 + \alpha_4$$

$$\Rightarrow \alpha_4 = -4\alpha_1$$

Proviamo per induzione: $\alpha_n = -2^{n-2}\alpha_1$

Assumiamo sia vero per tutti gli $n \leq k$. Allora:

$$0 = (f^{(k)}, v) = \alpha_1 - \alpha_2 - \alpha_3 - \dots - \alpha_k + \alpha_{k+1}$$

$$\Rightarrow \quad = \alpha_1 + \left(\sum_{n=2}^{k} 2^{n-2}\right)\alpha_1 + \alpha_{k+1}$$

$$= \alpha_1 + \left(\sum_{i=0}^{k-2} 2^i \right) \alpha_1 + \alpha_{k+1}$$

$$= \alpha_1 + \frac{1 - 2^{k-1}}{1 - 2} \alpha_1 + \alpha_{k+1}$$

$$= \cancel{\alpha_1} + (2^{k-1} \cancel{-1}) \alpha_1 + \alpha_{k+1} \Rightarrow \boxed{\alpha_{k+1} = -2^{k-1} \alpha_1.}$$

$\alpha_n = 2^{n-2} \alpha_1$ deve essere in ℓ^2

$\Rightarrow \alpha_1 = 0 \Rightarrow \alpha_n = 0 \ \forall n \geq 1 \Rightarrow V = 0.$

<u>16/11/21</u> <u>02/09/21</u>