29 settembre

(0.7.3) $$\sum_{k=0}^{n} k^3 = \frac{1}{4} n^2 (n+1)^2$$

$n = 0$ $$\sum_{k=0}^{0} k^3 = 0^3 = 0$$

$$\frac{1}{4} 0^2 (0+1)^2 = 0$$

$n \Rightarrow n+1$ $$\sum_{k=0}^{n} k^3 = \frac{1}{4} n^2 (n+1)^2$$

$$\sum_{k=0}^{n+1} k^3 = \sum_{k=0}^{n} k^3 + (n+1)^3 = \frac{1}{4} n^2 (n+1)^2 + (n+1)^3 =$$

$$= (n+1)^2 \left(\frac{n^2}{4} + (n+1) \right) =$$

$$= (n+1)^2 \frac{n^2+4n+4}{4} = \frac{(n+1)^2 (n+2)^2}{4}$$

$$\sum_{k=1}^{n} (2k-1) = n^2$$

$$= \sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 1 = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$= \cancel{2} \frac{(n+1)\,n}{\cancel{2}} - n = (n+1)\,n - n =$$
$$= n^2 + n - n = n^2$$

$$\sum_{k=1}^{n} (2k-1) = n^2$$ Per induzione

$n = 0$ $$\sum_{k=1}^{0} (2k-1) = 0$$

$$0^2 = 0$$

$n = 1$ $$\sum_{k=1}^{1} (2k-1) = 2 - 1 = 1$$

$$1^2 = 1$$

$n \Rightarrow n+1$ $$\sum_{k=1}^{n} (2k-1) = n^2$$

$$\sum_{k=1}^{n+1} (2k-1) = \sum_{k=1}^{n} (2k-1) + 2(n+1) - 1$$

$$= n^2 + 2n + 2 - 1 = n^2 + 2n + 1 = (n+1)^2$$

$$x^2 - y^2 = (x-y)(x+y)$$
$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

$$x^n - y^n = (x-y) \sum_{j=1}^{n} x^{j-1} y^{n-j} \qquad \forall n$$
$$= (x-y) \sum_{j=1}^{n} x^{n-j} y^{j-1}$$

$\forall n \in \mathbb{N}^*, \ \forall x \in \mathbb{R} \quad x \geq 0$

$$* \quad (1+x)^n \geq 1 + nx + \frac{n(n-1)}{2} x^2 \qquad x \geq 0$$

$$(1+x)^n \geq 1 + nx \qquad x \geq -1$$

$n = 1$

$$(1+x)^n = 1+x$$

$$1 + x + 1 \frac{(1-1)}{2} x^2 = 1+x$$

$n \Rightarrow n+1$

$$(1+x)^n \geq 1 + nx + \frac{n(n-1)}{2} x^2 \quad \cdot (1+x)$$

$$(1+x)^{n+1} \geq \left(1 + nx + \frac{n(n-1)}{2} x^2\right)(1+x)$$

$$= 1 + x + nx(1+x) + \frac{n(n-1)x^2}{2} + \frac{n(n-1)}{2} x^3$$

$$= 1 + (n+1)x + x^2 \left[n + \frac{n(n-1)}{2}\right] + \frac{n(n-1)x^3}{2}$$

$$\geq 1 + (n+1)x + \frac{\cancel{2}n + n^2 - \cancel{n}}{2} x^2 \qquad \frac{n^2+n}{2}$$

$$= 1 + (n+1)x + \frac{n(n+1)}{2} x^2$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j = \qquad \binom{n}{j} > 0$$

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$$

$$= 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \sum_{j=3}^{n} \binom{n}{j} x^j$$

$$= 1 + n x + \frac{n!}{2!\,(n-2)!} x^2 + \sum_{j=3}^{n} \binom{n}{j} x^j$$

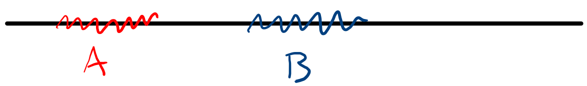
$$n! = n\,(n-1)! = n\,(n-1)\,(n-2)!$$

$$= 1 + nx + \frac{n\,(n-1)\,\cancel{(n-2)!}}{2\,\cancel{(n-2)!}} x^2 + \sum_{j=3}^{n} \binom{n}{j} x^j$$

$$\geq 1 + nx + \frac{n\,(n-1)}{2} x^2$$

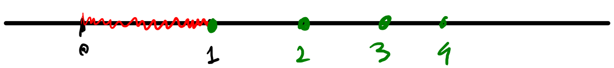
$\mathbb{R}$

Def Due insiemi $A \neq \emptyset$ e $B \neq \emptyset$ in $\mathbb{R}$ sono una coppia separata se $a \leq b \quad \forall\, a \in A$ e $b \in B$.



$A = [0,1] = \{ x \in \mathbb{R} : \ 0 \leq x \leq 1 \}$

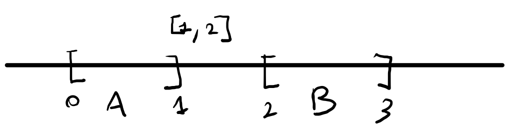
$B = \{ n \in \mathbb{N} : \ n \geq 1 \} = \mathbb{N}^*$



Assioma (di Dedekind) Data una coppia separata A e B in $\mathbb{R}$ $\exists\, c \in \mathbb{R}$ t.c.

$$a \leq c \leq b \qquad \forall\, a \in A \ \text{e} \ \forall\, b \in B.$$

Questi "c" sono detti elementi di separazione per A e B.



$[0,1]$ e $\mathbb{N}^*$ sono una coppia contigua perché 1 è l'unico elemento di separazione

Def Sia $A \neq \emptyset$ $\quad A \subseteq \mathbb{R}$. Un numero $x_0 \in \mathbb{R}$ è un (minorante) maggiorante di A se

$$a \leq x_0 \quad (a \geq x_0) \qquad \forall a \in A$$

Es $\quad A = \mathbb{R}$, questo non ammette alcun maggiorante (minorante)

Def Diremo che se $A \subseteq \mathbb{R}$ non ha alcun maggiorante (minorante) allora A è illimitato superiormente (inferiormente)

e che $\quad$ e diremo che $+\infty$ $(-\infty)$ è l'estremo superiore (inferiore) di A

$$\sup A = +\infty \qquad (\inf A = -\infty)$$

$$\sup \mathbb{R} = +\infty$$

$$\boxed{\sup \mathbb{N} = +\infty} \qquad \sup \mathbb{Z} = +\infty$$

$$\sup \mathbb{Q} = +\infty$$

Def Sia $A \neq \emptyset$ $\quad A \subseteq \mathbb{R}$. Diremo che un punto $a_0 \in A$ è il massimo di A se

$$a \le a_0 \quad \forall a \in A$$

e scriviamo

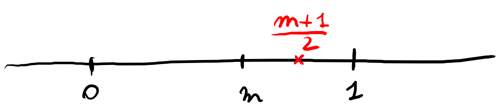
$$a_0 = \max A$$

Es $[0,1] = \{x \in \mathbb{R} : 0 \le x \le 1\}$

$\max [0,1] = 1$

$(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$ non ammette massimo

Se avesse massimo $m \in (0,1)$, cioè $0 < m < 1$



$$\left.\begin{array}{l} m < 1 \\ 1 \le 1 \end{array}\right\} \iff \left.\begin{array}{l} \frac{m}{2} < \frac{1}{2} \\ \frac{1}{2} \le \frac{1}{2} \end{array}\right\} \text{sommo} \implies \frac{m}{2} + \frac{1}{2} < 1$$

$$\frac{m+1}{2} < 1$$

$$\left.\begin{array}{l} m \le m \\ m < 1 \end{array}\right\} \iff \left.\begin{array}{l} \frac{m}{2} \le \frac{m}{2} \\ \frac{m}{2} < \frac{1}{2} \end{array}\right\} \text{sommo} \implies \frac{m}{2} + \frac{m}{2} < \frac{m}{2} + \frac{1}{2}$$

$$m < \frac{m+1}{2}$$

Abbiamo trovato $\frac{m+1}{2} \in (0,1)$

con $\frac{m+1}{2} > m \ge \frac{m+1}{2} \implies \frac{m+1}{2} > \frac{m+1}{2}$