$\mathcal{C} = \{A_1, A_2, A_3\}$

$\mathbb{P}_G(\mathcal{C})$

$A_1 \cap A_2 \cap A_3$ (1)

$A_1 \cap A_2 \cap \overline{A_3}$ (2)

$A_1 \cap \overline{A_2} \cap A_3$ (3)

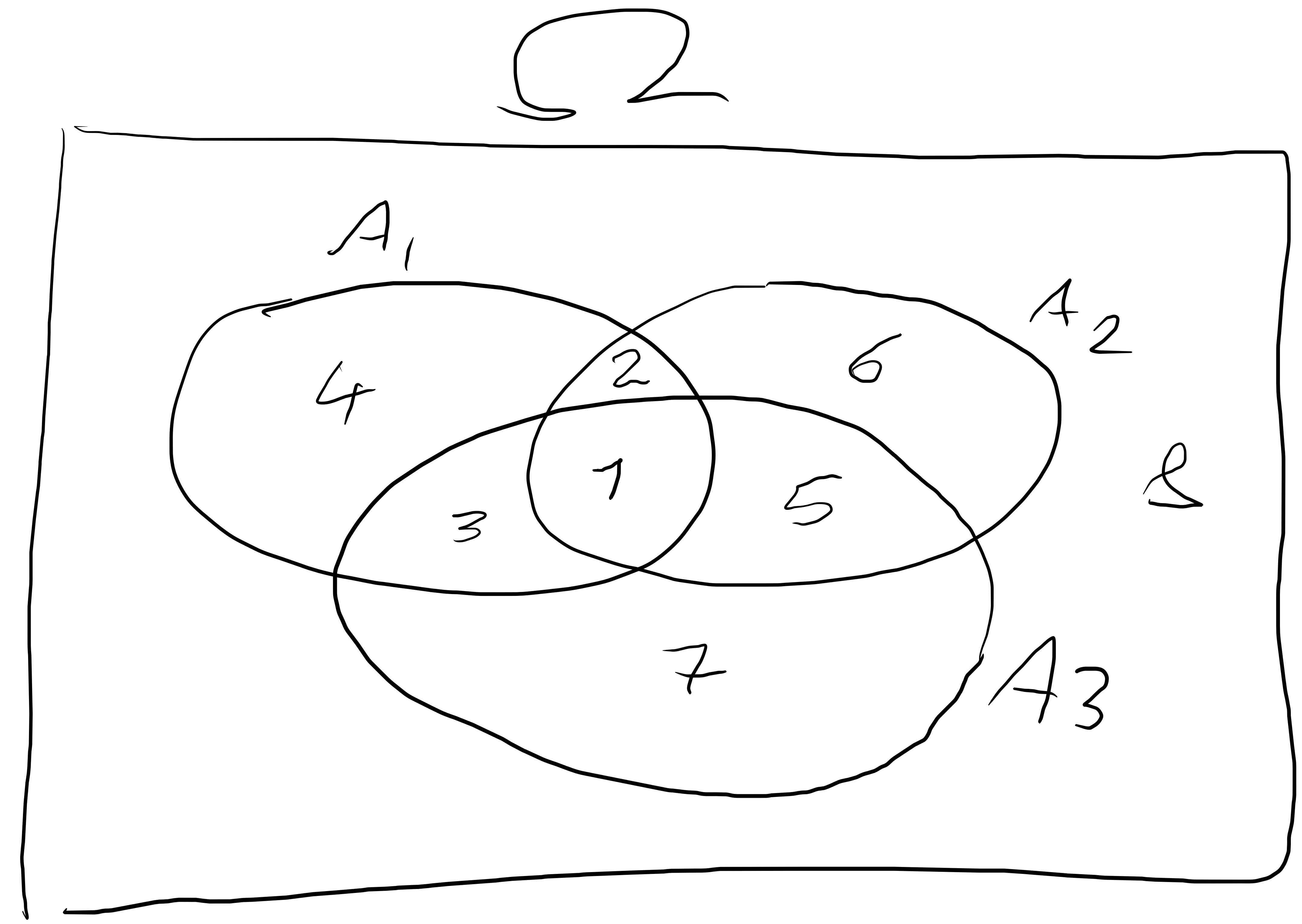
$A_1 \cap \overline{A_2} \cap \overline{A_3}$ (4)

$\overline{A_1} \cap A_2 \cap A_3$ (5)

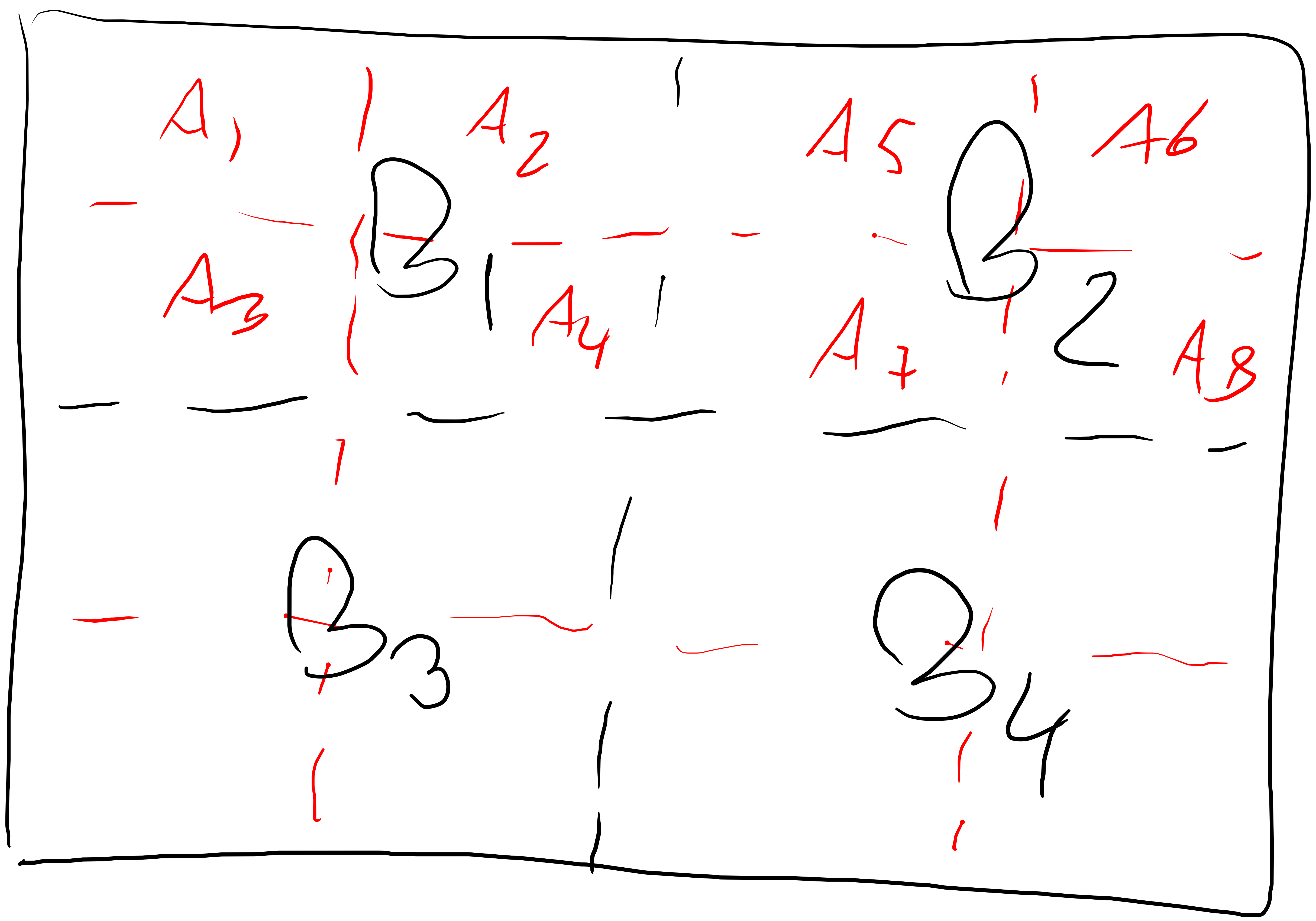
$\overline{A_1} \cap A_2 \cap \overline{A_3}$ (6)

$\overline{A_1} \cap \overline{A_2} \cap A_3$ (7)

$\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$ (8)



$A_1 = (1) \cup (2) \cup (3) \cup (4)$



$B_1 = A_1 \cup A_2 \cup A_3 \cup A_4$

$\{B_1, \text{———} B_4\}$

MENO FINE DI

$\{A_1 \text{———} A_{16}\}$

$$\mathbb{P} \doteq (A_n)_{n \geq 1}$$

$$\sigma(\mathbb{P}) = \underbrace{\left\{ \bigcup_{i \in I} A_i \;\middle|\; I \subset \mathbb{N}_+ \right\}}_{\mathcal{F}}$$

$\mathcal{F}$ È UNA σ-ALGEBRA

$I = \emptyset \longrightarrow \emptyset \in \mathcal{F}$

$I = \mathbb{N}_+ \longrightarrow A_1 \cup A_2 \cup A_3 \cup \ldots = \Omega \in \mathcal{F}$

$A \in \mathcal{F}$ QUINDI $A = \bigcup_{i \in I} A_i$ PER QUALCHE $I \subset \mathbb{N}_+$

AD ESEMPIO $A = A_1 \cup A_3 \cup A_5 \cup A_7 \ldots$

$\overline{A} = A_2 \cup A_4 \cup A_6 \cup A_8 \cup \ldots$

IN GENERALE $\overline{A} = \bigcup_{i \in \mathbb{N}_+ - I} A_i \in \mathcal{F}$

$B_1, B_2, \ldots, B_n, \ldots \in \mathcal{F}$

QUINDI

$B_n = \bigcup_{i \in I_n} A_i$ PER OGNI n, $I_n \subset \mathbb{N}_+$

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{i \in \bigcup_{n \geq 1} I_n} A_i$$

$\bigcup_{n \geq 1} I_n \subset \mathbb{N}_+$

$$\left.\begin{aligned} B_1 &= A_3 \cup A_7 \\ B_2 &= A_2 \cup A_7 \cup A_9 \end{aligned}\right\} B_1 \cup B_2 = A_2 \cup A_3 \cup A_7 \cup A_9$$

$\mathbb{P} \subset \mathcal{F}$ (BASTA PRENDERE I SINGLETON)

$\mathcal{F}$ È LA PIÙ PICCOLA σ-ALGEBRA CHE CONTIENE $\mathbb{P}$:

SE $\mathcal{G}$ È UNA σ-ALGEBRA CHE CONTIENE $\mathbb{P}$, ALLORA $\mathcal{F} \subset \mathcal{G}$

QUINDI $\mathcal{F} = \sigma(\mathbb{P})$

$\mathcal{C} = \{A_1 \ldots\ldots A_n\}$

$$\sigma(\mathcal{C}) = \sigma(\mathbb{P}_G(\mathcal{C}))$$

$$\mathcal{C} \subset \sigma(\mathbb{P}_G(\mathcal{C}))$$

OGNI INSIEME IN $\mathcal{C}$ È UNIONE (DISCRETA) DI INSIEMI DI $\mathbb{P}_G(\mathcal{C})$

$$\Rightarrow \sigma(\mathcal{C}) \subset \sigma(\mathbb{P}_G(\mathcal{C}))$$

ORA

$$\mathbb{P}_G(\mathcal{C}) \subset \sigma(\mathcal{C})$$

OGNI INSIEME IN $\mathbb{P}_G(\mathcal{C})$ È DEL TIPO $A_1' \cap A_2' \cap \ldots \cap A_n' \in \sigma(\mathcal{C})$

$$\Longrightarrow \sigma(\mathbb{P}_G(\mathcal{C})) \subset \sigma(\mathcal{C})$$

$\mathcal{C} = \{A, B\}$

$A \subset B$

$\sigma(\mathcal{C}) = ?$

$P_\sigma(\mathcal{C})$:

$A \cap B = A \quad (1)$

$A \cap \overline{B} = \emptyset$

$\overline{A} \cap B = B - A \quad (2)$

$\overline{A} \cap \overline{B} = \overline{B} \quad (3)$

$$\sigma(\mathcal{C}) = \sigma(\mathbb{P}_\sigma(\mathcal{C})) = \left\{ \bigcup_{i \in I} A_i \;\middle|\; I \subset \{1,2,3\} \right\}$$

$$= \{ \emptyset, A, \bar{A} \cap B, \overline{B}, \underbrace{A \cup (\bar{A} \cap B)}_{B}, A \cup \overline{B}, \underbrace{(\bar{A} \cap B) \cup \overline{B}}_{\overline{A}},$$

$$\Omega \}$$

$\mathcal{C} = \{ \text{INTERVALLI} \}$

INTERVALLI $\quad a \in \mathbb{R} \quad b \in \mathbb{R}$

$[a, b] \qquad (a, b) \qquad [a, b) \qquad (a, b]$

$\qquad\qquad\qquad\qquad\qquad\qquad *$

$(-\infty, a] \qquad (-\infty, a) \qquad [a, +\infty) \qquad (a, +\infty)$

$\quad * \qquad\qquad\qquad\qquad\qquad *$

$(-\infty, +\infty) = \mathbb{R}$

$$\mathcal{C}_1 = \{ (-\infty, a) \mid a \in \mathbb{R} \}$$

$$\sigma(\mathcal{C}_1) = \sigma(\{\text{INTERVALLI}\}) = \mathcal{B}$$

$\subset$ OVVIA

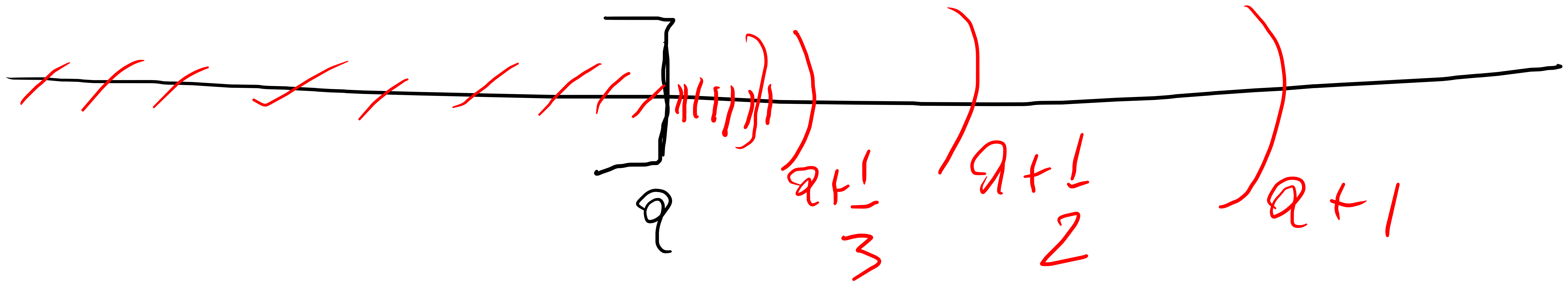
$\supset$?

$$\sigma(\mathcal{C}_1) \qquad \overline{(-\infty, a)} = [a, +\infty) \in \sigma(\mathcal{C}_1)$$

$$(-\infty, a) \cap [b, +\infty) = [b, a) \in \sigma(\mathcal{C}_\ell)$$

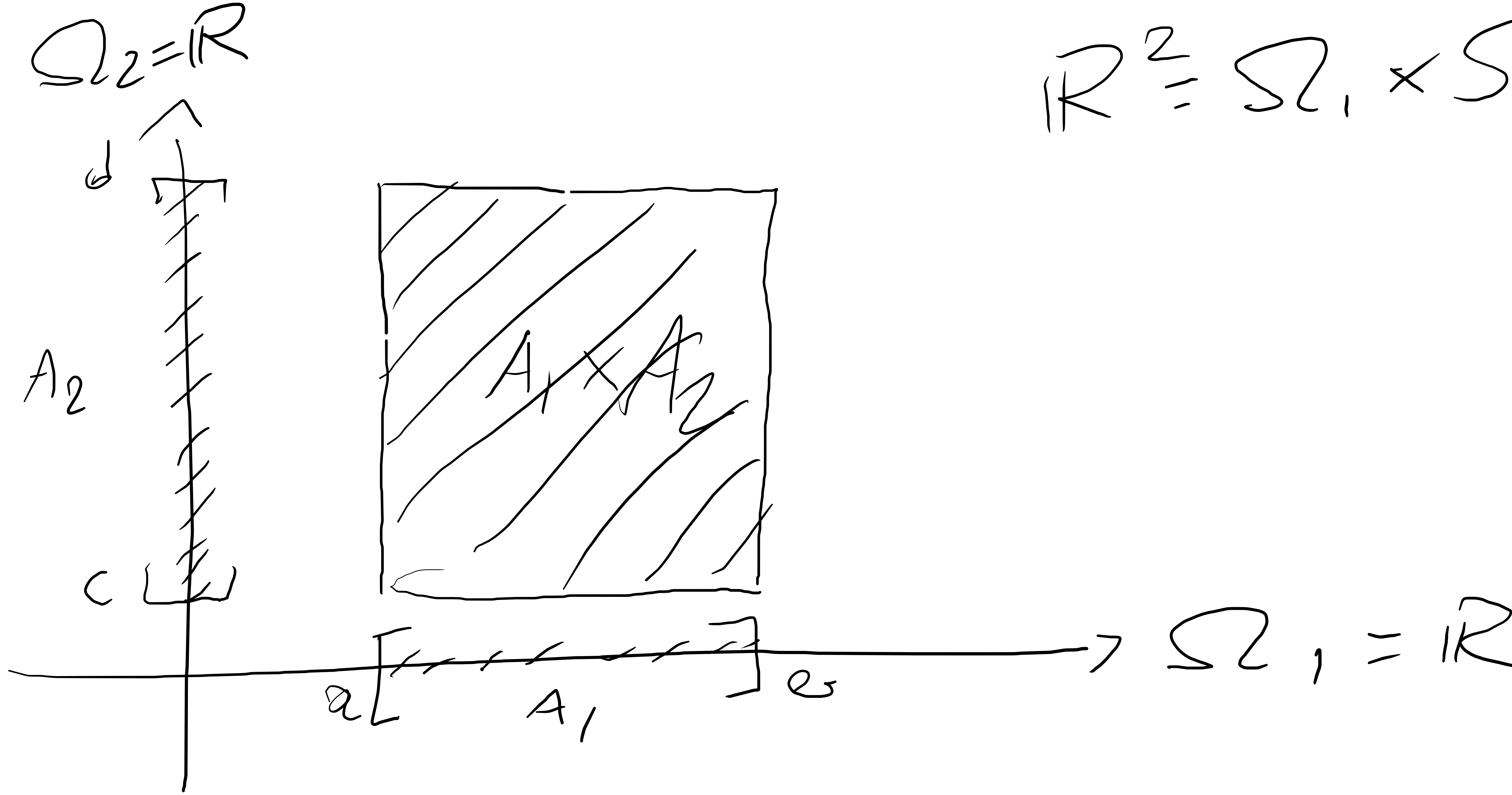
$$(-\infty, a] = (-\infty, a+1) \cap \left(-\infty, a+\frac{1}{2}\right) \cap \left(-\infty, a+\frac{1}{3}\right)$$

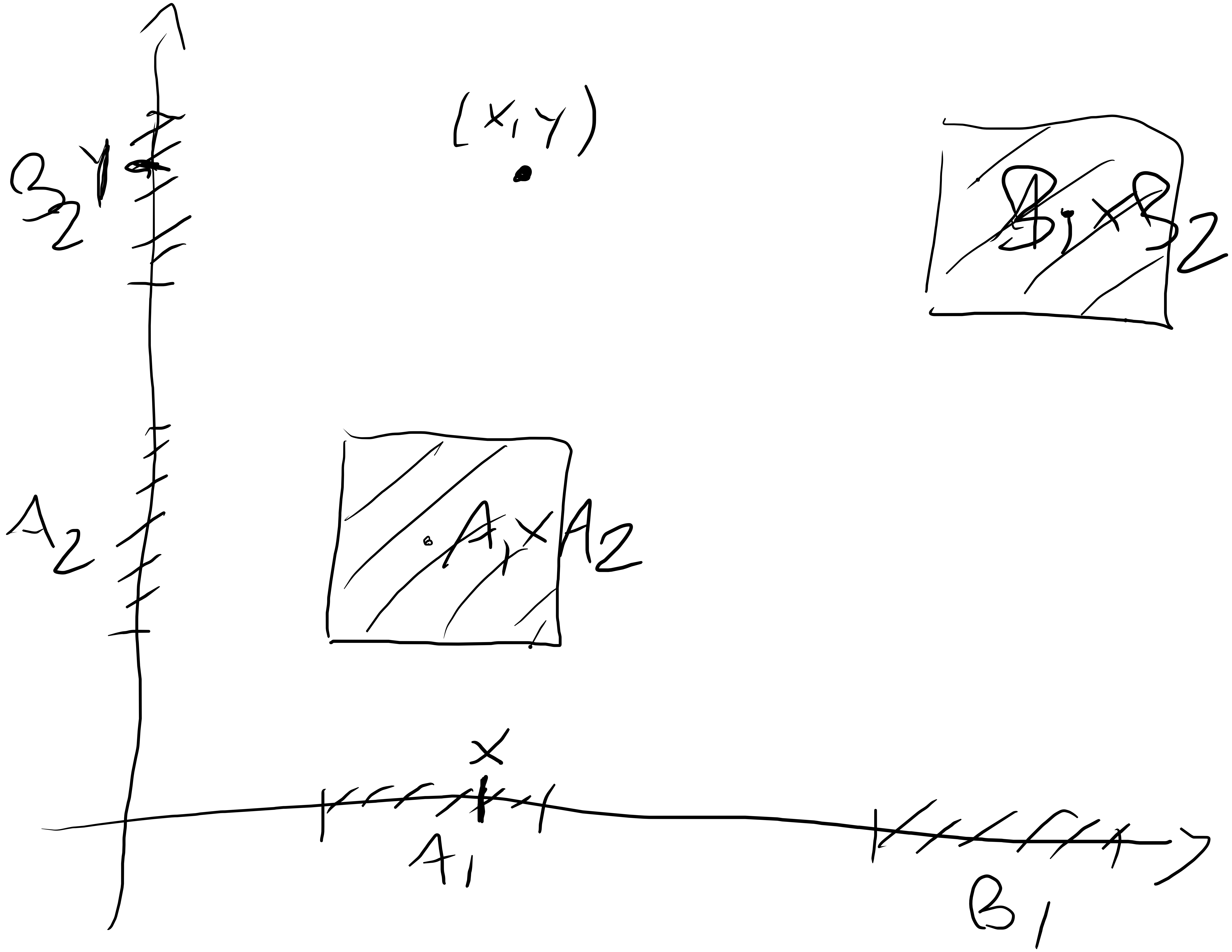
$$\cap \dots \cap \left(-\infty, a+\frac{1}{n}\right) \cap \dots$$



a
a+1/3
a+1/2
a+1

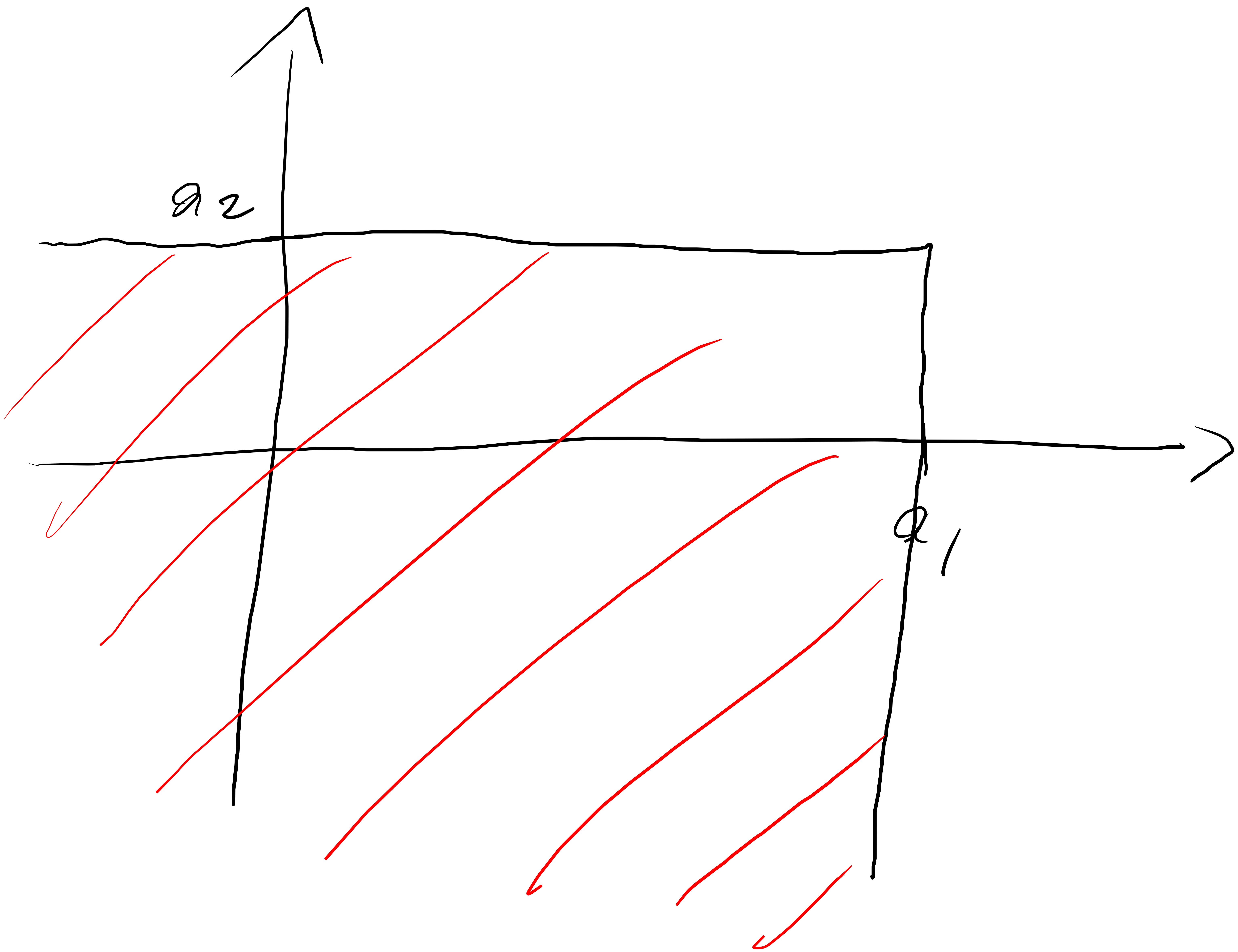
$\mathbb{R}^2 = \Omega_1 \times \Omega_2 =$





$$(A_1 \times A_2) \cup (B_1 \times B_2)$$

$(-\infty, a_1] \times (-\infty, a_2]$



FINITA ADDITIVITÀ

$$P(A_1 \cup A_2 \cup \dots \cup A_n) =$$

$$= P(\underbrace{A_1 \cup A_2 \cup \dots \cup A_n \cup \phi \cup \phi \cup \dots \cup \phi \dots}_{\text{A DUE A DUE DISGIUNTI}})$$

σ-ADDITIVITÀ

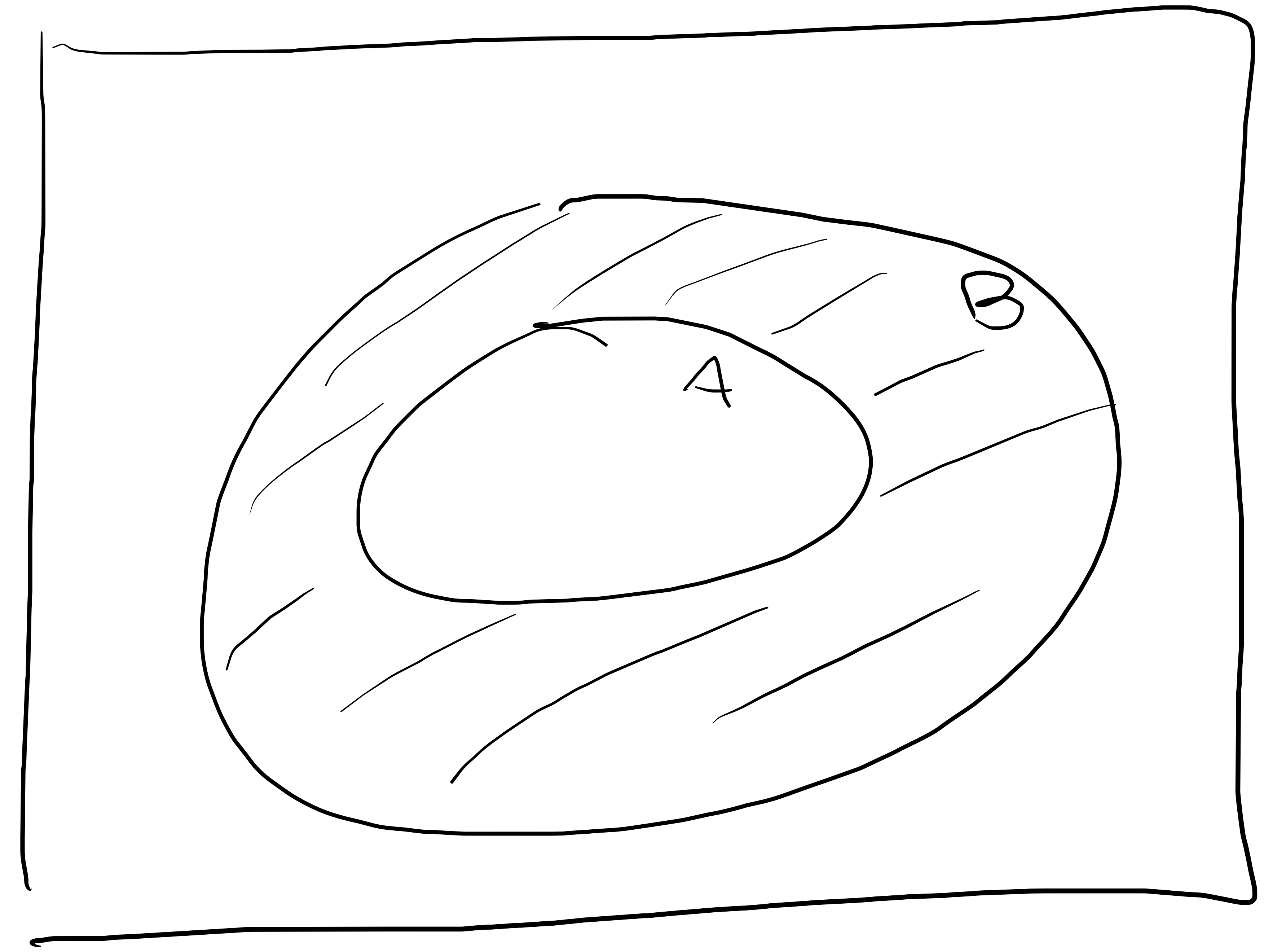
$$= P(A_1) + P(A_2) + \dots + P(A_n) + \underbrace{P(\phi)}_{=0} + \underbrace{P(\phi)}_{=0} + \dots \underbrace{\dots}_{=0}$$

$$= P(A_1) + \dots + P(A_n)$$

$A \subset B$

$P(B - A) = P(B) - P(A)$

$\underbrace{B - A}_{B \cap \overline{A}}$



Ω

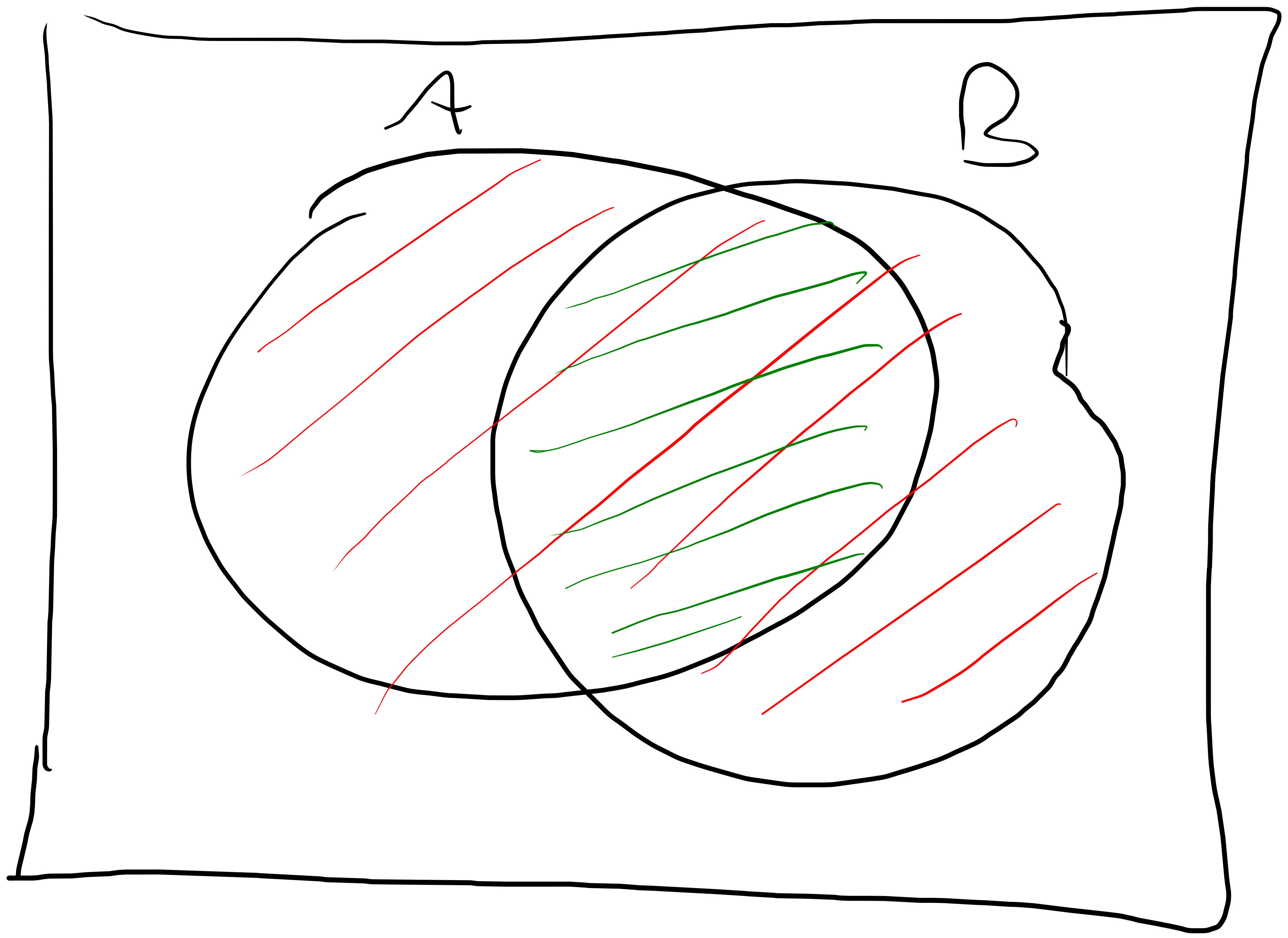
$B = A \cup (B - A)$

DISGIUNTI

$P(B \cap \overline{A}) =$

$1 - P(\overline{B} \cup A) = 1 - P(\overline{B}) - P(A) = P(B) - P(A)$

$P(B) = P(A \cup (B - A)) = P(A) + P(B - A)$



Ω

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$A \cup B = A \cup (B - A)$ DISGIUNTI

$$P(A \cup B) = P(A) + \underbrace{P(B - A)} - P(A \cap B) + \underbrace{P(A \cap B)}$$

$+$

$= P(B)$

$B = (B - A) \cup (A \cap B)$ DISGIUNTI

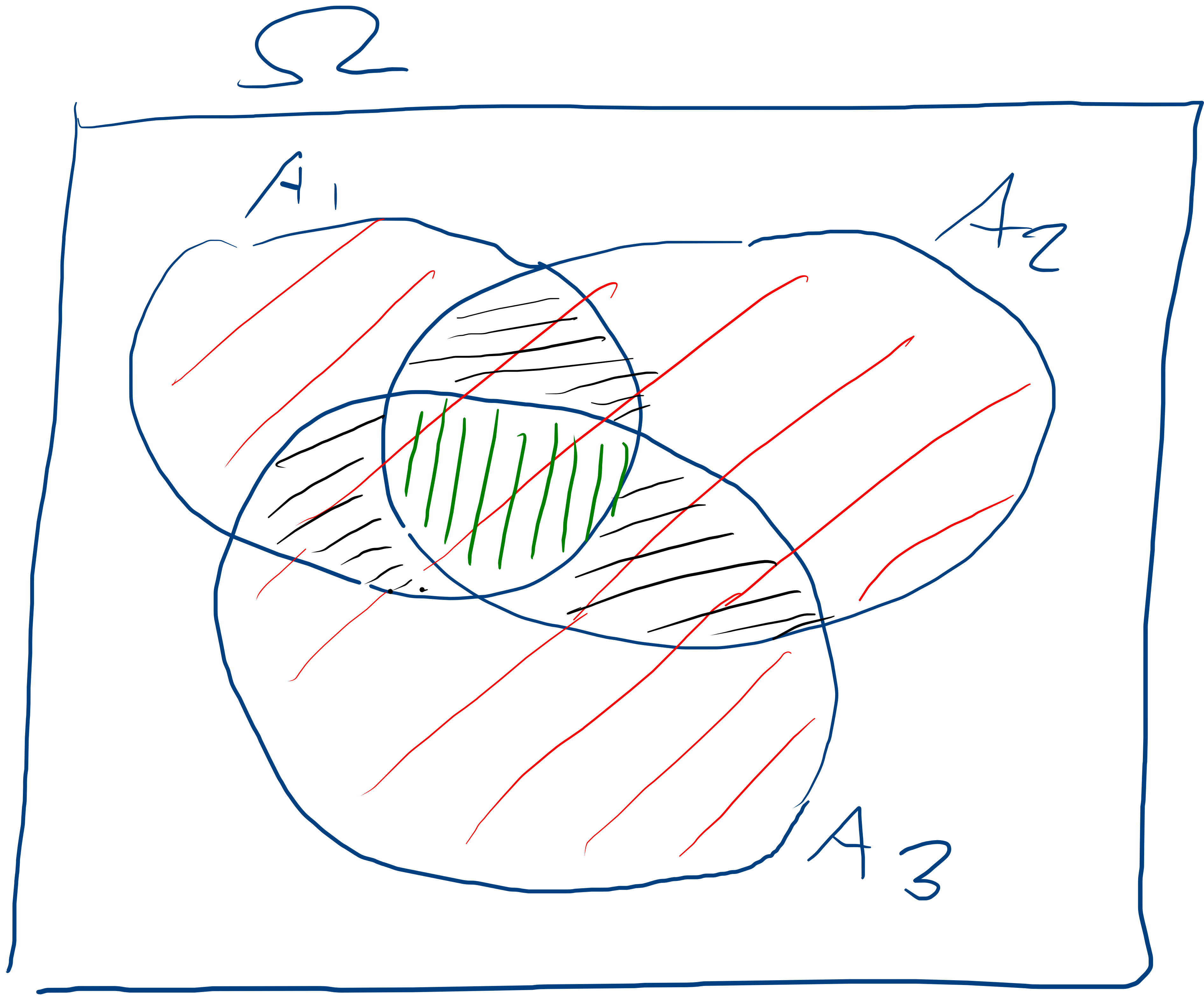
$$P(A_1 \cup A_2 \cup A_3) =$$

$$= P(A_1) + P(A_2) + P(A_3)$$

$$- P(A_1 \cap A_2) - P(A_1 \cap A_3)$$

$$- P(A_2 \cap A_3)$$

$$+ P(A_1 \cap A_2 \cap A_3)$$



$$P(A_1 \cup A_2 \cup \dots \cup A_n) \leq P(A_1) + \dots + P(A_n)$$

$n = 2$ VERA

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - \underbrace{P(A_1 \cap A_2)}_{\geq 0}$$

$$\leq P(A_1) + P(A_2)$$

PASSO INDUTTIVO: SE LA PROPRIETÀ È VERA PER n EVENTI, MOSTRO CHE È VERA PER $n+1$ EVENTI

$$P(\underbrace{A_1 \cup A_2 \cup \ldots \cup A_n}_{A} \cup \underbrace{A_{n+1}}_{B}) \leq$$

$$\leq P(A) + P(B)$$

$$= P(\underbrace{A_1 \cup \ldots \cup A_n}) + P(A_{n+1})$$

$$\leq P(A_1) + \ldots + P(A_n) \quad \text{IPOTESI INDUTTIVA}$$

$$\leq P(A_1) + \ldots + P(A_n) + P(A_{n+1})$$

$$A_1 \subset A_2 \subset A_3 \ldots \ldots \subset A_n \subset A_{n+1} \subset \ldots \ldots$$

$A_n \in \mathcal{F} \quad \forall n$

$$\lim_n A_n = \bigcup_{n=1}^{\infty} A_n = A \qquad A_n \uparrow A$$

ESEMPIO : $\Omega = \mathbb{R} \quad \mathcal{F} = \mathcal{B}$

$$A_n = [0, n] \qquad \lim A_n = [0, +\infty)$$

$A_1 \supset A_2 \supset A_3 \supset \dots A_n \supset A_{n+1} \supset \dots$

$\lim A_n = \bigcap_{n=1}^{\infty} A_n = A \qquad A_n \downarrow A$

$\Omega = \mathbb{R} \qquad \mathcal{F} = \mathcal{B}$

$A_n = \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$

$[0,2) \supset \left[\frac{1}{2}, \frac{3}{2}\right) \supset \left[\frac{2}{3}, \frac{4}{3}\right) \supset \dots$

$\lim A_n = \{1\}$

f CONTINUA IN x $\qquad f: \mathbb{R}^n \to \mathbb{R}$

SE E SOLO SE

PER OGNI SUCCESSIONE DI (x_n)

TALI CHE $x_n \to x$ ALLORA $f(x_n) \to$

$\to f(x)$