4 ottobre

Th

$\gamma \in (0,d)$ $\qquad 1 < p < q < \infty$

$$\frac{1}{p} = \frac{1}{q} + \frac{d-\gamma}{d}$$

$$\| \, |x|^{-\gamma} * f \, \|_{L^q(\mathbb{R}^d)} \leq C \, |f|_{L^p(\mathbb{R}^d)}$$

for fixed C.

Stein Harmonic Analysis

Lieb - Loss

~~Lieb~~ Stein "Singular Integrals...." '70

Lemma $\quad \gamma \in (0,d)$

$$\mathcal{F}(|x|^{-\gamma})(\xi) = c_\gamma \, |\xi|^{\gamma - d}.$$

Pf $\quad \displaystyle\int_{\mathbb{R}^d} |x|^{-\gamma} \varphi(x)\, dx = c_\gamma \int_{\mathbb{R}^d} |\xi|^{\gamma-d} \hat{\varphi}(\xi)\, d\xi$

$$\int_{\mathbb{R}^d} \varepsilon^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \varphi(x)\, dx \qquad \varepsilon > 0$$

$$= c \int_{\mathbb{R}^d} e^{-\varepsilon \frac{|\xi|^2}{2}} \hat{\varphi}(\xi)\, dx$$

$$\int_0^{\infty} \frac{d\varepsilon}{\varepsilon} \varepsilon^{\frac{d-\gamma}{2}} \underline{\qquad\qquad}$$

$$* \int_{\mathbb{R}^d} dx\, \varphi(x) \int_0^{+\infty} \frac{d\varepsilon}{\varepsilon} \varepsilon^{\frac{\cancel{d}-\gamma}{2}} \varepsilon^{-\frac{\cancel{d}}{2}} e^{-\frac{|x|^2}{2\varepsilon}}$$

$$= c \int_{\mathbb{R}^d} d\xi\, \hat{\varphi}(\xi) \int_0^{+\infty} \frac{d\varepsilon}{\varepsilon} \varepsilon^{\frac{d-\gamma}{2}} e^{-\varepsilon \frac{|\xi|^2}{2}}$$

$$\int_0^{+\infty} \frac{d\varepsilon}{\varepsilon} \varepsilon^{-\frac{\gamma}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \qquad \varepsilon' = \frac{\varepsilon}{|x|^2}$$

$$= \int_0^{+\infty} \frac{d\varepsilon'}{\varepsilon'} \left(|x|^2 \varepsilon'\right)^{-\frac{\gamma}{2}} e^{-\frac{1}{2\varepsilon'}}$$

$$= a_\gamma \quad |x|^{-\gamma}$$

$$\int_0^{+\infty} \frac{d\varepsilon}{\varepsilon} \varepsilon^{\frac{d-\gamma}{2}} e^{-\varepsilon \frac{|\xi|^2}{2}} \qquad \varepsilon' = \varepsilon |\xi|^2$$

$$= \int_0^{+\infty} \frac{d\varepsilon'}{\varepsilon'} \left(\frac{\varepsilon'}{|\xi|^2}\right)^{\frac{d-\gamma}{2}} e^{-\frac{\varepsilon'}{2}}$$

$$= |\xi|^{\gamma-d}\, b_\gamma$$

$$* \int_{\mathbb{R}^d} dx\, \varphi(x) \overbrace{\int_0^{+\infty} \frac{d\varepsilon}{\varepsilon}\, \varepsilon^{\frac{\cancel{d}-\gamma}{2}}\, \varepsilon^{-\frac{\cancel{d}}{2}}\, e^{-\frac{|x|^2}{2\varepsilon}}}^{a_\gamma |x|^{-\gamma}}$$

$$= c \int_{\mathbb{R}^d} d\xi\, \hat{\varphi}(\xi) \underbrace{\int_0^{+\infty} \frac{d\varepsilon}{\varepsilon}\, \varepsilon^{\frac{d-\gamma}{2}}\, e^{-\frac{\varepsilon|\xi|^2}{2}}}_{b_\gamma |\xi|^{\gamma-d}}$$

$$c_\gamma = \frac{c\, b_\gamma}{a_\gamma}$$

Theorem Let $s \in (0, \frac{d}{2})$ and $\boxed{\frac{1}{q} = \frac{1}{2} - \frac{s}{d}}$. Then q^*_s

$$\|f\|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{\dot{H}^s}$$

$$\|f\|_{\dot{H}^s} = \left\| |\xi|^s \hat{f} \right\|_{L^2(\mathbb{R}^d)}$$

Pf

$$f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\xi x} \underbrace{\hat{f}(\xi)|\xi|^1}_{\hat{g}(\xi)} |\xi|^{-1} \, d\xi$$

$$= c \int_{\mathbb{R}^d} |x-y|^{1-d} \, g(y) \, dy$$

$$= |x|^{1-d} * g$$

$\gamma \in (0,d)$ $\quad 1 < 2 < q < \infty$

$$\frac{1}{2} = \frac{1}{q} + \frac{\cancel{d}-(\cancel{d}-1)}{d} = \frac{1}{q} + \frac{1}{d} = \frac{1}{2}$$

$$\| \underbrace{|x|^{-(d-1)} * g}_{f} \|_{L^q(\mathbb{R}^d)} \leq C \, |g|_{L^2(\mathbb{R}^d)}$$

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{d}$$

$$\| f \|_{L^q} \leq C \, | g |_{L^2} =$$

$$= C \, | \hat{g} |_{L^2} = C \, | \, |\xi|^1 \hat{f} \, |_{L^2}$$

$$= C \, |f|_{\dot{H}^1}$$

Lemma (Interpolation of Sobolev norms)

$s \in [0,1]$ $\qquad k = s k_1 + (1-s) k_2$

We have

$$\| f \|_{\dot{H}^k(\mathbb{R}^d)} \leq \| f \|_{\dot{H}^{k_1}}^{s} \, \| f \|_{\dot{H}^{k_2}}^{1-s}$$

Pf

$$\| f \|_{\dot{H}^k}^2 = \int_{\mathbb{R}^d} |\xi|^{2k} |\hat{f}(\xi)|^2 \, d\xi$$

$$= \int_{\mathbb{R}^d} |\xi|^{2(s k_1 + (1-s) k_2)} \, |\hat{f}(\xi)|^{2(s+1-s)} \, d\xi$$

$$= \int_{\mathbb{R}^d} \left(|\xi|^{2k_1} |\hat{f}|^2 \right)^s \left(|\xi|^{2k_2} |\hat{f}|^2 \right)^{(1-s)} d\xi$$

$$\leq \left| \left(|\xi|^{k_1} |\hat{f}| \right)^{2s} \right|_{L^{\frac{1}{s}}} \left| \left(|\xi|^{k_2} |\hat{f}| \right)^{2(1-s)} \right|_{L^{\frac{1}{1-s}}}$$

$s + (1-s) = 1$

$\frac{1}{p} + \frac{1}{q} = 1 \qquad p = \frac{1}{s}, \quad q = \frac{1}{1-s}$

$$\| f^a \|_{L^p} = \| f \|^a_{L^{pa}}$$

$$= \| |\xi|^{k_1} \hat{f} \|^{2s}_{L^2} \; \| |\xi|^{k_2} \hat{f} \|^{2(1-s)}_{L^2}$$

$$\| f \|^2_{\dot{H}^k} \leq \| f \|^2_{\dot{H}^{k_1}} \; \| f \|^2_{\dot{H}^{k_2}}$$

Then $\quad p \in [2, +\infty) \qquad \frac{1}{p} > \frac{1}{2} - \frac{1}{d}$

$$\| f \|_{L^p(\mathbb{R}^d)} \leq C \| f \|^{1-s}_{L^2(\mathbb{R}^d)} \; \| f \|^{s}_{\dot{H}^1}$$

$$s = d\left(\frac{1}{2} - \frac{1}{p}\right) \left(< \frac{d}{2} \right)$$

$$2 \leq p < q^*_s$$

Pf $\quad$ Here $\quad s < 1$

$$\frac{1}{d} > \frac{1}{2} - \frac{1}{p} \qquad 1 > d\left(\frac{1}{2} - \frac{1}{p}\right) = s$$

$$s < \frac{d}{2}$$

$$\frac{1}{p} = \frac{1}{2} - \frac{s}{d} = \frac{1}{2} - \frac{1}{\cancel{d}}\,\cancel{d}\left(\frac{1}{2} - \frac{1}{p}\right)$$

$$\|f\|_{L^p(\mathbb{R}^d)} \le C\, \|f\|_{\dot{H}^s(\mathbb{R}^d)}$$

$$s = (1-s)\,0 + s \cdot 1$$

$$\le C \;\|f\|_{L^2(\mathbb{R}^d)}^{(1-s)}\, \|f\|_{\dot{H}^1(\mathbb{R}^d)}^{s}$$

$$p = 4 \qquad d = 2, 3, \qquad s = \frac{d}{4}$$

$$s = d\left(\frac{1}{2} - \frac{1}{p}\right) = d\,\frac{1}{4}$$

Lemma (Gronwall) $T > 0$

$\lambda, \varphi \in L^1(0,T) \qquad \lambda, \varphi \ge 0$

$C_1, C_2 \ge 0 \qquad \lambda\varphi \in L^1(0,T)$

Then, if

$$\varphi(t) \le \overbrace{C_1 + C_2 \int_0^t \lambda(s)\varphi(s)\,ds}^{\psi(t)}$$

a.e. in $[0,T]$

then

$$\varphi(t) \le C_1 \, e^{c_2 \int_0^t \lambda(s)\, ds} \qquad a.e.$$

<u>Pf</u>

$$\psi(t) = C_1 + C_2 \int_0^t \lambda(s)\, \varphi(s)\, ds$$

$$\psi'(t) = C_2 \lambda(t) \varphi(t) \le C_2 \lambda(t) \psi(t)$$

$$\psi(t) \ge \varphi(t)$$

$$e^{-C_2 \int_0^t \lambda\, ds} \left(\psi'(t) - C_2 \lambda(t) \psi(t) \right) \le 0$$

$$\left(\psi(t) \, e^{-C_2 \int_0^t \lambda\, ds} \right)' \le 0$$

$$\varphi \, e^{-C_2 \int_0^t \lambda\, ds} \le \psi(t) \, e^{-C_2 \int_0^t \lambda\, ds} \le \underbrace{\psi(0)}_{= C_1} + \cancel{(\cdots)}$$

$$\varphi(t) \le C_1 \, e^{C_2 \int_0^t \lambda(s)\, ds} .$$

$T > 0 \qquad f : [0,T] \to \underbrace{\dot H^{s-1}(\mathbb{R}^d, \mathbb{R}^d)}_{\dot H^{s-1}}$

$d = 2, 3 \qquad f = \mathbb{P} f$

$$\begin{cases} \dot u - \Delta u = f \\ \nabla \cdot u = 0 \\ u(0) = u_0 \in \mathbb{P}\, \underbrace{\dot H^s(\mathbb{R}^d, \mathbb{R}^d)}_{\dot H^s} \end{cases} \qquad \langle \cdot, \psi \rangle$$

Def For $s < \frac{d}{2}$ and $f \in L^2([0,T], \dot H^{s-1})$

$$\left(\int_0^T |f(t)|^2_{\dot H^{s-1}} \, dt \right)^{\frac12}$$

$f = \mathbb{P} f$. Then u is a solution

$u \in L^\infty([0,T], \dot H^s)$

$\cancel{\nabla} u \in L^2([0,T], \dot H^{s+1})$

u is $C^0_w([0,T], \dot H^s)$

$\langle u(t), \phi \rangle \in C^0([0,T], \mathbb{R}) \quad \forall \phi \in \dot H^{-s}$

$\dot H^s \times \dot H^{-s} \longrightarrow \mathbb{R}$

if for any $\psi \in C^\infty_c([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$

$$\boxed{\begin{aligned} \langle u(t), \psi(t) \rangle &= \langle u_0, \psi(0) \rangle + \qquad \psi(t) \in \dot H^{-s} \\ &+ \int_0^t \left(-\langle \nabla u(t'), \nabla \psi(t') \rangle + \langle u(t'), \partial_t \psi(t') \rangle \right. \end{aligned}}$$

$$+ \langle f(t'), \psi(t') \rangle \Big) \, dt'$$

$$\begin{cases} \dot u - \Delta u = f \\ \nabla \cdot u = 0 \\ u(0) = u_0 \in L \end{cases} \qquad \psi \in L^2([0,T], \dot H^{-(s-1)}) \quad \langle \cdot , \psi \rangle$$

$$s - 1 < \frac{d}{2} \qquad -(s-1) > -\frac{d}{2}$$

$$\int_0^t \left(\langle \partial_t u, \psi \rangle - \langle u, \Delta \psi \rangle \right) = \int_0^t \langle f, \psi \rangle$$

$$\int_0^t \left(\partial_t \langle u, \psi \rangle - \langle u, \partial_t \psi \rangle \right) =$$

$$= \int_0^t \left(\langle f, \psi \rangle + \langle u, \Delta \psi \rangle \right)$$

$$\langle u(t), \psi(t) \rangle - \langle u_0, \psi(0) \rangle =$$

$$= \int_0^t \left(\langle f, \psi \rangle + \langle u, \Delta \psi \rangle + \langle u, \partial_t \psi \rangle \right)$$

Theor There exists and is unique a weak solution. Furthermore u satisfies the energy equality

$$|u(t)|^2_{\dot H^s} + 2 \int_0^t |\nabla u|^2_{\dot H^s} \, dt'$$

$$= |u_0|^2_{\dot H^s} + 2 \int_0^t \langle f, u \rangle_{\dot H^s} \, dt'$$

($\langle f, g \rangle_{\dot{H}^1} = \langle |\xi|^1 \hat{f}, |\xi|^1 \hat{g} \rangle_{L^2}$)

Fur [illegible] $u \in C^0([0,T], \dot{H}^1)$

$$\hat{u}(t) = e^{-t|\xi|^2} \hat{u}_0 + \int_0^t e^{-(t-t')|\xi|^2} \hat{f}(t',\xi)\, dt'$$

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-t')\Delta} f(t')\, dt'$$