29 settembre

Lemma Se $s > -\frac{d}{2}$ allora $C_c^\infty(\mathbb{R}^d)$ è un sottoinsieme denso di $\dot{H}^s(\mathbb{R}^d)$

<u>Osservazione</u> $C_c^\infty(\mathbb{R}^d)$ è denso anche $\dot{H}^{s_1}(\mathbb{R}^d) \cap \dots \cap \dot{H}^{s_k}(\mathbb{R}^d)$ per $s_1, \dots, s_k > -\frac{d}{2}$.

$u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^d)$ $\qquad$ $\mathcal{D}'(\mathbb{R}^d, \mathbb{R}^d)$

$$\operatorname{div} u = \nabla \cdot u = \sum_{j=1}^{d} \partial_j u_j$$

u ha div nullo se $\operatorname{div} u = 0$

$\Updownarrow$

$\xi_j \hat{u}_j = 0$

<u>Lemma</u> $u \in \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ allora

$$\Delta u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u)$$

Dim $\{e_j\}_{j=1}^{3}$ sia la base canonica

$$\Delta u = \Delta u_i \, e_i = \partial_j \partial_j u_i \, e_i =$$

$$= \underbrace{\partial_i (\underbrace{\partial_j u_j}_{\operatorname{div} u}) e_i}_{\nabla \operatorname{div} u} - (\partial_i \partial_j u_j \, e_i - \partial_j \partial_j u_i \, e_i)$$

$$\nabla \times (\nabla \times u) = \varepsilon_{ijk} \, \partial_j (\nabla \times u)_k \, e_i =$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{se } (i,j,k) = (1,2,3) \text{ o se è una permutazione pari di } (1,2,3) \\ -1 & \text{se è una permutazione dispari} \\ 0 & \end{cases}$$

$$= \varepsilon_{ijk} \, \varepsilon_{ki'j'} \, \partial_j \partial_{i'} u_{j'} \, e_i$$

$$= \varepsilon_{kij} \, \varepsilon_{ki'j'} \, \partial_j \partial_{i'} u_{j'} \, e_i$$

$$\varepsilon_{kij} \, \varepsilon_{ki'j'} = \delta_{ii'} \delta_{jj'} - \delta_{ij'} \delta_{ji'}$$

$$= \left(\delta_{ii'}\,\delta_{jj'} - \delta_{ij'}\,\delta_{ji'}\right)\partial_j\,\partial_{i'}\,u_{j'}\,e_i$$

$$= \partial_j\partial_i\,u_j\,e_i - \partial_j\partial_j\,u_i\,e_i$$

$$= \nabla \times (\nabla \times u)$$

<u>Def</u> Proiettore di Leray $L^2(\mathbb{R}^d, \mathbb{R}^d)_{\mathbb{R}}$

$$\widehat{(\mathbb{P}u)}_j = \hat{u}_j - \frac{1}{|\xi|^2}\,\xi_j\,\xi_k\,\hat{u}_k$$

<u>Osservazione</u> $\mathbb{P}$ è una proiezione in qualsiasi $\dot{H}^1(\mathbb{R}^d, \mathbb{R}^d)$

$$\xi_j\,\widehat{(\mathbb{P}u)}_j = \xi_j\left(\hat{u}_j - \frac{1}{|\xi|^2}\xi_j\,\xi_k\,\hat{u}_k\right)$$

$$= \xi_j\,\hat{u}_j - \frac{1}{\cancel{|\xi|^2}}\,\cancel{\xi_j}\,\xi_j\,\xi_k\,\hat{u}_k$$

$$\widehat{\operatorname{div} u} - \widehat{\operatorname{div} v} = 0$$

$$\operatorname{div} \mathbb{P} u = 0$$

$$\widehat{\left(\mathbb{P}^2 u\right)}_j = \widehat{\left(\mathbb{P} u\right)}_j - \frac{1}{|\xi|^2} \xi_j \underbrace{\xi_k \widehat{\left(\mathbb{P} u\right)}_k}_{0}$$

$\ker \mathbb{P} = ?$

$u \in \ker \mathbb{P}$ $\qquad L^2(\mathbb{R}^3, \mathbb{R}^3)$

$$0 = \widehat{\left(\mathbb{P} u\right)}_j = \hat{u}_j - \frac{1}{|\xi|^2} \xi_j \xi_k \hat{u}_k = 0$$

$$\hat{u}_j = \xi_j \frac{\xi_k \hat{u}_k}{|\xi|^2}$$

e se pongo $\quad \hat{V} = \dfrac{\xi_k \hat{u}_k}{|\xi|^2}$

$$V \in \dot{H}^{-1}(\mathbb{R}^3, \mathbb{R}^3)$$

$$u \approx \nabla V$$

$$H(\mathbb{R}^d) = \{ u \in L^2(\mathbb{R}^d, \mathbb{R}^d) : \operatorname{div} u = 0 \}$$

$$V(\mathbb{R}^d) = H(\mathbb{R}^d) \cap H^1(\mathbb{R}^d, \mathbb{R}^d)$$

$$\nabla = -i$$

Lemma

$$\mathbb{P} u = -\Delta^{-1} \, \nabla \times (\nabla \times u)$$

$$\widehat{\nabla \cancel{f}} = -i \xi \hat{f}(\xi)$$

$$\widehat{\mathbb{P} u} = \frac{1}{|\xi|^2} \, -i\xi \times (-i\xi \times \hat{u})$$

$$= -\frac{1}{|\xi|^2} \, \xi \times (\xi \times \hat{u})$$

$$\widehat{(\mathbb{P} u)}_j = \hat{u}_j - \frac{1}{|\xi|^2} \xi_j \, \xi_k \hat{u}_k =$$

$$= \hat{u}_j + \frac{1}{|\xi|^2} (-i \xi_j) \, (-i \xi_k) \hat{u}_k)$$

$$\mathbb{P}u = u - \frac{1}{\Delta}\nabla \operatorname{div} u$$

$$\mathbb{P} = 1 - \frac{1}{\Delta}\nabla \operatorname{div} u$$

$$\Delta = \nabla \operatorname{div} - \nabla\times(\nabla\times \sqcup)$$

$$1 = \Delta^{-1}\nabla \operatorname{div} - \Delta^{-1}\nabla\times(\nabla\times \sqcup)$$

$$\underbrace{1 - \Delta^{-1}\nabla \operatorname{div}}_{\mathbb{P}} = -\Delta^{-1}\nabla\times(\nabla\times \sqcup)$$

$$C^\infty_{c\sigma}(\mathbb{R}^d, \mathbb{R}^d)$$

<u>Lemma</u> $C^\infty_{c\sigma}(\mathbb{R}^3, \mathbb{R}^3)$ è denso $H(\mathbb{R}^3)$

$V(\mathbb{R}^3)$

<u>Dim</u> $u \in H(\mathbb{R}^3)$

$$u = \nabla\times A$$

$$A = -\Delta^{-1}\nabla\times u \in \dot{H}^1(\mathbb{R}^3, \mathbb{R}^3)$$

$$\Delta u = \nabla\underbrace{(\nabla\cdot u)}_{0} - \nabla\times(\nabla\times u)$$

$C_c^\infty(\mathbb{R}^3,\mathbb{R}^3)$ è denso in $\dot{H}^1(\mathbb{R}^3,\mathbb{R}^3)$

$$\dot{H}^1(\mathbb{R}^3,\mathbb{R}^3) \xrightarrow{\nabla\times} L^2(\mathbb{R}^3,\mathbb{R}^3)$$

$$A_n \xrightarrow{n\to\infty} A$$

$$\underbrace{-\nabla\times A_n}_{u_n} \longrightarrow -\nabla\times A$$

$$u_n \longrightarrow u$$

$\dot{H}^1(\mathbb{R}^d)$ $\qquad \lambda > 0$

$$\widehat{P_\lambda u} = \chi_{D_{\mathbb{R}^d}(0,\lambda)}(\xi)\,\hat{u}(\xi) = \chi_{[0,\lambda]}(|\xi|)\,\hat{u}(\xi)$$

$$\chi_{[0,1]}\left(\frac{\sqrt{-\Delta}}{\lambda}\right) = \chi_{[0,\lambda]}\left(\sqrt{-\Delta}\right)$$

$$\boxed{P_\lambda \mathbb{P} = \mathbb{P} P_\lambda}$$

$$\chi_{[0,\lambda]}(|\xi|)\,\widehat{\mathbb{P}u}^j = \chi_{[0,\lambda]}(|\xi|)\left(\hat{u}^j - \frac{\xi_j\xi_k}{|\xi|^2}\hat{u}^k\right)$$

$$= \chi_{[0,\lambda]}(|\xi|)\,\hat{u}^j - \frac{\xi_j \xi_k}{|\xi|^2}\,\chi_{[0,\lambda]}(|\xi|)\,\hat{u}^k$$

$$= \widehat{P_\lambda u_j} - \frac{\xi_j \xi_k}{|\xi|^2}\,\widehat{P_\lambda u_k}$$

Disug. di Young per convoluzioni

$$\|f * g\|_{L^r} \le \|f\|_{L^p}\,\|g\|_{L^q}$$

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

<u>Def</u> se $g : \mathbb{R}^d \to \mathbb{R}$

$$d_g(\alpha) := \left|\{x \in \mathbb{R}^d : |g(x)| > \alpha\}\right|$$

d_g è decrescente $[0, +\infty) \to \mathbb{R}$

è misurabile

$g \in L^p(\mathbb{R}^d)$ $\quad 1 \le p < +\infty$

$$\int_{\mathbb{R}^d} |g(x)|^p \, dx = \int_{\mathbb{R}^d} dx \int_0^{|g(x)|} p\, \alpha^{p-1} \, d\alpha$$

$$= p \int_0^{+\infty} d\alpha \; \alpha^{p-1} \int_{\{x \in \mathbb{R}^d : |g(x)| > \alpha\}} dx$$

$$= p \int_0^{+\infty} d\alpha \; \alpha^{p-1} \, d_g(\alpha)$$

$$F(x,\alpha) = \alpha^{p-1} \chi_{\mathbb{R}_+}(|g(x)| - \alpha) \, \chi_{\mathbb{R}_+}(\alpha)$$

Def.

$$\|f\|_{L^1 + L^\infty} = \inf \{ (\|a\|_{L^1} + \|b\|_{L^\infty}) : f = a + b \}$$

$$L^{p,\infty}(\mathbb{R}^d) = \{ f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) : \|f\|_{L^{p,\infty}(\mathbb{R}^d)} := \sup \alpha \, d_f^{\frac{1}{p}}(\alpha) < +\infty \}$$

$p < +\infty$

$$\| f \|_{L^{p,\infty}} \leq \| f \|_{L^p} \quad \text{Chebyshev}$$

$$d_f(\alpha) = |\{x : |f(x)| > \alpha\}| \leq \frac{\| f \|_{L^p}^p}{\alpha^p}$$

$\forall \alpha > 0$

$$\int_{\{x : |f(x)| > \alpha\}} dx \leq \int_{\mathbb{R}^d} \frac{|f(x)|^p}{\alpha^p} dx$$

$\alpha > 0$

$$\| f \|_{L^p}^p = \int_{\mathbb{R}^d} |f|^p dx \geq$$

$$\geq \int_{\{x : |f(x)| > \alpha\}} |f|^p dx \geq \alpha^p d_f(\alpha)$$

$$\underbrace{\sup_{\alpha > 0} \alpha^p d_f^{\frac{1}{p}}(\alpha)}_{\| f \|_{L^{p,\infty}}} \leq \| f \|_{L^p}$$

Esempio $\quad a \in (0, d)$

$$|x|^{-a} \in L^{p,\infty}(\mathbb{R}^d) \iff ap = d$$

$$a = \frac{d}{p}$$

$$p = \frac{d}{a}$$

$$\alpha \, |\{x \in \mathbb{R}^d : |x|^{-a} > \alpha\}|^{\frac{1}{p}} =$$

$$\frac{1}{\alpha} > |x|^{a}$$

$$= \alpha \, |\{x : |x| < \left(\frac{1}{\alpha}\right)^{a}\}|^{\frac{1}{p}}$$

$$= C_d^{\frac{1}{p}} \, \alpha \, \frac{1}{\alpha^{\frac{ad}{p}}} = C_d^{\frac{1}{p}} \, \alpha^{\overbrace{1 - \frac{ad}{p}}^{0}}$$

Teor $\quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$

$1 < p, q, r < \infty$. Allora $\exists\, C$

t.c. $\quad \|g * f\|_{L^r} \leq C \, \|g\|_{L^{q,\infty}} \, \|f\|_{L^p}$

In Bahouri et al.
si usa l'argomentazione di Keel-Tao
nella dimostrazione delle
end point Strichartz estimates.

<u>Teor</u> (Hardy-Littlewood-Sobolev)

$\forall\, \gamma \in (0,d)$ e $\forall$

$$1 < p < q < \infty \quad \text{con} \quad \frac{1}{p} = \frac{1}{q} + \frac{d-\gamma}{d}$$

si ha $\exists\, C > 0$ t.c.

$$\left\| \int_{\mathbb{R}^d} f(x-y)\, |y|^{-\gamma}\, dy \right\|_{L^q(\mathbb{R}^d)} \leq$$

$$\leq C \quad |f|_{L^p(\mathbb{R}^d)} .$$

<u>Dim</u> $\| f * |x|^{-\gamma} \|_{L^q(\mathbb{R}^d)} \leq$

$$\left\{ \begin{aligned} & |x|^{-\gamma} \in L^{\frac{d}{\gamma},\infty}(\mathbb{R}^d) \\ & \\ & \leq \left\| |x|^{-\gamma} \right\|_{L^{\frac{d}{\gamma},\infty}} \| f \|_{L^p} \end{aligned} \right.$$

$$\frac{1}{q} + 1 = \frac{\gamma}{d} + \frac{1}{p}$$

$$\Updownarrow$$

$$\frac{1}{p} = \frac{1}{q} + \frac{d-\gamma}{d}$$